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BULLETIN
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THE OCTOBER MEETING IN NEW YORK

The three hundred sixty-second meeting of the American Mathematical Society was held at Columbia University on Saturday, October 28, 1939. The attendance included the following one hundred seventy-nine members of the Society:

H. W. Alexander, C. B. Allendoerfer, Warren Ambrose, R. L. Anderson, R. G. Archibald, L. A. Aroian, R. P. Bailey, M. F. Becker, D. L. Bernstein, A. C. Berry, E. E. Betz, J. H. Blau, Salomon Bochner, Samuel Borofsky, Joseph Bowden, C. B. Boyer, A. B. Brown, J. H. Bushey, Jewell H. Bushey, S. S. Cairns, A. D. Campbell, Claude Chevalley, Alonzo Church, G. A. Coon, T. F. Cope, Byron Cosby, Richard Courant, A. P. Cowgill, H. B. Curry, J. H. Curtiss, M. D. Darkow, Norman Davidowitz, M. M. Day, R. P. Dilworth, W. J. Dixon, Jesse Douglas, Arnold Dresden, F. G. Dressel, Nelson Dunford, W. D. Duthie, J. E. Eaton, M. L. Elveback, Paul Erdös, Aaron Fialkow, W. B. Fite, Edward Fleisher, W. W. Flexner, M. C. Foster, R. M. Foster, J. S. Frame, K. O. Friedrichs, G. N. Garrison, Abe Gelbart, J. J. Gergen, B. P. Gill, Michael Goldberg, Saul Gorn, C. H. Graves, M. C. Gray, S. L. Greitzer, H. M. Griffin, M. B. Haberzetle, D. W. Hall, Marshall Hall, N. A. Hall, P. R. Halmos, J. A. Hamilton, D. C. Harkin, Philip Hartman, Maximilian Herzberger, E. H. C. Hildebrandt, T. H. Hildebrandt, L. S. Hill, Einar Hille, Banesh Hoffmann, T. R. Hollcroft, Harold Hotelling, S. E. Hotelling, E. M. Hull, Witold Hurewicz, R. P. Isaacs, Nathan Jacobson, S. A. Joffe, R. A. Johnson, Wilfred Kaplan, L. S. Kennison, S. C. Kleene, E. R. Kochin, B. O. Koopman, Jack Laderman, A. W. Landers, M. K. Landers, Madeline Levin, Marie Litzinger, E. R. Lorch, A. N. Lowan, N. H. McCoy, Brockway McMillan, L. A. MacColl, Saunders MacLane, H. F. MacNeish, A. J. Maria, W. T. Martin, Walther Mayer, H. L. Mintzer, Deane Montgomery, D. C. Murdoch, F. J. Murray, Tadasi Nakayama, C. A. Nelson, C. O. Oakley, Rufus Oldenburger, J. M. H. Olmsted, Oystein Ore, F. W. Owens, H. B. Owens, J. C. Oxtoby, R. S. Phillips, E. L. Post, Walter Prenowitz, J. F. Randolph, H. W. Raudenbush, H. W. Reddick, C. J. Rees, M. S. Rees, H. J. Riblet, Moses Richardson, J. F. Ritt, H. E. Robbins, J. H. Roberts, S. L. Robinson, Benjamin Rosenbaum, J. E. Rosenthal, J. B. Rosser, Arthur Sard, Abraham Schwartz, Hyman Serbin, Stephan Serghiesco, I. M. Sheffer, L. W. Sheridan, Seymour Sherman, Max Shiffman, J. A. Shohat, L. P. Siceloff, L. G. Simons, James Singer, Abraham Sinkov, L. L. Smail, M. F. Smiley, R. C. Stephens, Wolfgang Sternberg, S. W. Stewart, R. R. Stoll, M. H. Stone, W. C. Strodt, M. M. Sullivan, J. D. Tamarkin, J. H. Taylor, E. W. Titt, C. B. Tompkins, A. W. Tucker, J. W. Tukey, Annita Tuller, Henry Wallman, J. L. Walsh, J. V. Wehausen, Louis Weisner, M. E. Wells, J. S. de Wet, A. H. Wheeler, A. P. Wheeler, O. L. Wheeler, A. L. Whiteman, S. S. Wilks, W. A. Wilson, H. P. Wirth, Audrey Wishard, H. N. Wright, Leo Zippin.

The meeting opened Saturday morning with two sessions: Analysis and Geometry, Professor J. H. Bushey presiding; and Algebra and Topology, Professor J. S. Frame presiding. At the opening of the afternoon session, at which Vice President J. F. Ritt presided, the members of the Society voted, on recommendation of the Council, to adopt the following amendment to the by-laws as Article 2, Section 3:

Section 3. The Board of Trustees shall have the power to appoint a manager and such assistants and agents as may be necessary or convenient to facilitate the conduct of the affairs of the Society, and to fix the terms and conditions of their employment. The Board may delegate to the officers of the Society duties and powers normally inhering in their respective corporate offices, subject to supervision by the Board. The Board of Trustees may appoint committees to facilitate the conduct of the business of the Society and delegate to such committees such powers as may be necessary or convenient for the proper exercise of those powers. Agents appointed, or members of committees designated, by the Board of Trustees need not be members of the Board.

Nothing herein contained shall be construed to empower the Board of Trustees to divest itself of responsibility for, or legal control of, the investments, properties and contracts of the Society.

After the short business meeting, Professor J. J. Gergen gave an address entitled *Double Fourier series*. The general session was followed by two sectional sections: Analysis, Professor Anna Pell Wheeler presiding; and Algebra and Geometry, Professor Saunders MacLane presiding.

Titles and cross references to the abstracts of the papers read follow below. Papers whose abstract numbers are followed by the letter *t* were read by title. The papers numbered 1 to 9 were read before the section for Analysis and Geometry; those numbered 10 to 18 before the section for Algebra and Topology; those numbered 19 to 23 before the section for Analysis; those numbered 24 to 28 before the section for Algebra and Geometry; and papers numbered 29 to 50 were read by title. Paper 16 was read by Professor MacLane, paper 17 by Dr. Zippin, paper 20 by Dr. Maria, and paper 24 by Dr. Becker. Dr. S. A. Jennings was introduced by Professor Oystein Ore and Dr. Alfred Brauer by Professor Hermann Weyl.

1. L. A. Aroian: *New continued fractions for the incomplete beta function.* (Abstract 45-11-408.)

2. Aaron Fialkow: *The conformal theory of curves.* (Abstract 45-11-415.)

3. Abraham Schwartz: *The Gauss-Codazzi-Ricci equations in Riemannian manifolds.* (Abstract 45-9-374.)

4. M. C. Foster: *Note on autopolar curves.* (Abstract 45-11-416.)

5. A. H. Wheeler: *A one-sided polyhedron having one hundred twenty faces.* (Abstract 45-11-439.)

6. W. T. Martin: *On a minimal problem in the theory of analytic functions of several variables.* (Abstract 45-11-425.)

7. Rufus Oldenburger: *Exponent trajectories in symbolic dynamics.* (Abstract 45-11-428.)

8. Wolfgang Sternberg: *The general limit theorem in the theory of probability.* (Abstract 45-9-380.)

9. C. B. Allendoerfer: *The Euler number of an imbedded Riemann manifold.* (Abstract 45-11-406.)

10. J. E. Eaton: *Theory of cogroups.* (Abstract 45-9-310.)

11. S. A. Jennings: *The structure of the group ring of a p-group over a modular field.* (Abstract 45-11-422.)

12. Benjamin Rosenbaum: *Ideals of a quadratic field in canonical form.* (Abstract 45-11-434.)

13. D. C. Murdoch: *Further theory of quasi-groups.* (Abstract 45-11-427.)

14. J. H. Roberts: *A theorem on two-to-one transformations.* (Abstract 45-11-433.)

15. A. W. Tucker: *The algebraic structure of complexes.* Preliminary report. (Abstract 45-11-437.)

16. V. W. Adkisson and Saunders MacLane: *On extending maps of plane Peano continua.* (Abstract 45-11-404.)

17. Deane Montgomery and Leo Zippin: *Topological group foundations of "rigid" space geometries.* (Abstract 45-11-402.)

18. H. J. Riblet: *Symmetric differential expressions and applications.* (Abstract 45-11-432.)

19. C. J. Rees: *Differential equation of elliptic orthogonal polynomials.* (Abstract 45-11-431.)

20. T.S. George and A. J. Maria: *Equilibrium point of Green's function for an n-dimensional spherical shell.* (Abstract 45-11-419.)

21. Einar Hille: *Contributions to the theory of Hermitian series. II. The representation problem.* (Abstract 46-1-456.)

22. Abe Gelbart: *On functions of two complex variables given by power series expansion.* (Abstract 45-11-418.)

23. J. F. Randolph: *Metric separability.* (Abstract 45-11-430.)

24. Miriam F. Becker and Saunders MacLane: *The minimum number of generators for inseparable algebraic extensions.* (Abstract 45-9-300.)

25. Alfred Brauer: *On the non-existence of the euclidean algorithm in certain quadratic number fields.* (Abstract 45-11-411.)

26. R. P. Dilworth: *The arithmetical theory of Birkhoff lattices.* (Abstract 45-11-413.)

27. Walter Prenowitz: *Projective geometry as a group-like system.* (Abstract 45-11-429.)

28. Michael Goldberg: *New five-bar and six-bar linkages in three dimensions.* (Abstract 45-11-315.)

29. R. P. Agnew: *On continuity and periodicity of measurable functions.* (Abstract 45-11-405-*t*.)

30. R. P. Agnew: *On rearrangements of series.* (Abstract 46-1-441-*t*.)

31. Salomon Bochner: *Harmonic surfaces in Riemann metric.* (Abstract 45-11-410-*t*.)

32. S. S. Cairns: *Homeomorphisms between topological manifolds and analytic manifolds.* (Abstract 45-9-306-*t*.)

33. J. J. DeCicco: *The circular group in an infinite plane of the Kasner space.* (Abstract 45-11-398-*t*.)

34. R. P. Dilworth, *Lattices with unique irreducible decompositions.* (Abstract 45-11-412-*t*.)

35. P. W. Gilbert: *Two-to-one transformations on acyclic continuous curves.* (Abstract 45-11-420-*t*.)

36. Nathan Jacobson: *A note on hermitian forms.* (Abstract 45-11-421-*t*.)

37. Edward Kasner and J. J. DeCicco: *Characterization of the conformal group by horn angles of second order.* (Abstract 45-11-400-*t*.)

38. J. L. Doob and Warren Ambrose: *On continuous stochastic processes.* (Abstract 45-11-414-*t*.)

39. E. S. Kennedy: *Exponential analogues of the Lambert series.* (Abstract 45-11-423-*t*.)

40. A. N. Lowan and Jack Laderman: *On the distribution of errors in nth tabular differences.* (Abstract 45-9-292-*t*.)

41. A. N. Lowan: *On the problem of wave-motion for the wedge of an angle.* (Abstract 45-11-291-*t*.)

42. Deane Montgomery and Leo Zippin: *Note on rotation-group of the two-sphere.* (Abstract 45-11-426-*t*.)

43. G. E. Raynor: *On Serret's integral formula.* (Abstract 46-1-471-*t*.)

44. O. F. G. Schilling: *Regular normal extensions over complete fields.* (Abstract 45-7-284-*t*.)

45. A. R. Schweitzer: *On a classification of "configurational" sets of ordered dyads.* (Abstract 45-9-375-*t*.)

46. A. R. Schweitzer: *On a classification of systems of "configurational" sets of ordered dyads.* (Abstract 45-9-376-*t*.)

47. W. T. Scott and H. S. Wall: *A convergence theorem for continued fractions.* III. (Abstract 45-11-435-*t*.)

48. W. E. Sewell: *Continuity and degree of approximation by rational functions.* (Abstract 45-7-286-*t*.)

49. W. E. Sewell: *Continuity and integral approximation to an analytic function by polynomials in z and $1/z$.* (Abstract 45-7-287-*t*.)

50. W. E. Sewell: *The derivative of a polynomial on further arcs of the complex domain.* (Abstract 45-7-288-*t*.)

T. R. HOLLCROFT,
Associate Secretary

THE NOVEMBER MEETING IN COLUMBIA

The three hundred sixty-third meeting of the Society was held at the University of Missouri, Columbia, Missouri, on Friday and Saturday, December 1–2, 1939, in connection with the centennial celebration of the University of Missouri. Seventy-three persons registered including the following forty-nine members of the Society:

V. W. Adkisson, H. A. Arnold, J. V. Atanasoff, W. L. Ayres, R. W. Babcock, Wealthy Babcock, P. O. Bell, Herman Betz, F. L. Black, L. M. Blumenthal, C. C. Camp, H. H. Campaigne, E. W. Chittenden, J. S. Creane, R. D. Daugherty, G. M. Ewing, B. F. Finkel, L. R. Ford, W. L. Graves, O. H. Hamilton, N. A. Haynes, J. J. L. Hinrichsen, H. E. Jordan, I. F. Keeler, J. R. F. Kent, J. W. Kitchens, J. V. McKelvey, R. J. Michel, U. G. Mitchell, G. C. Munro, G. D. Nichols, E. N. Oberg, O. J. Peterson, G. B. Price, W. C. Randels, Robert Rawhouser, P. R. Rider, E. H. Rothe, D. T. Sigley, E. R. Smith, G. W. Smith, Julia Spalding, E. B. Stouffer, W. T. Stratton, C. B. Tucker, G. E. Wahlin, L. E. Ward, W. D. A. Westfall, J. J. Wheeler.

The meeting opened Friday afternoon with a session devoted to short papers, followed by an address by Professor L. R. Ford of Armour Institute of Technology entitled *Projective transformations in two complex variables.* On Saturday morning Professor E. W. Chittenden of the University of Iowa delivered an address entitled *Topological functions.* These two addresses were given under the joint auspices of the Society and the University of Missouri. The meeting closed Saturday morning with a second session of short papers.

On Friday evening a dinner was held at the Tiger Hotel with seventy-nine persons attending. Professor W. D. A. Westfall presided at the dinner and Dean Henry Bent of the Graduate School of the University of Missouri welcomed the visitors on behalf of the host institution. Professor L. R. Ford gave reminiscences of student days at the University of Missouri. Professor W. L. Ayres spoke of the founding of a new mathematical journal, MATHEMATICAL REVIEWS.

Professor E. R. Smith and Dean E. B. Stouffer presided at the two sessions for short papers. Professor W. D. A. Westfall presided at the two invited addresses. At the close of the Saturday morning session Professor V. W. Adkisson presented a resolution of thanks to the hosts for their generous hospitality. The titles of papers read at this meeting follow, papers 1–6 being read Friday afternoon and 7–11 on Saturday morning. Papers 12–17, whose abstract numbers are followed by the letter *t*, were read by title. Mr. Fred Robertson was introduced by the Associate Secretary, Mr. R. H. Tripp by Professor D. L. Holl.

1. L. M. Blumenthal and C. V. Robinson: *A new metric characterization of the straight line.* (Abstract 45-11-409.)

2. W. L. Ayres: *A note on the definition of arc-sets.* (Abstract 46-1-6.)

3. O. H. Hamilton: *A fixed point theorem for a cell in a Hilbert space.* (Abstract 46-1-15.)

4. W. C. Randels: *On the absolute summability of Fourier series.* III. (Abstract 46-1-31.)

5. G. B. Price: *Convex operators.* (Abstract 46-1-28.)

6. Fred Robertson: *The general differential operator.* (Abstract 46-1-33.)

7. E. R. Smith: *Generating functions for certain confluent hypergeometric functions.* (Abstract 46-1-35.)

8. P. O. Bell: *Projective analogues of the congruence of normals.* (Abstract 46-1-8.)

9. D. T. Sigley: *Sets of conjugate subgroups in direct product groups.* (Abstract 46-1-34.)

10. R. H. Tripp: *Stresses in an orthotropic elastic layer.* (Abstract 46-1-36.)

11. J. V. Atanasoff: *Generalized Taylor expansions.* Preliminary report. (Abstract 46-1-4.)

12. D. G. Bourgin: *Closure of products of functions.* (Abstract 46-1-12-*t*).

13. E. P. Lane: *A theorem on surfaces.* (Abstract 46-1-20-*t*.)

14. A. A. Albert: *On ordered algebras.* (Abstract 46-1-3-*t*.)

15. A. A. Aucoin: *Diophantine equations of degree n.* (Abstract 46-1-5-*t*.)

16. W. H. Myers: *Finite linear groups with abelian subgroups of variety* 3. (Abstract 46-1-27-*t*.)

17. D. G. Bourgin and R. J. Duffin: *The Dirichlet problem for the vibrating string.* (Abstract 46-1-13-*t*.)

W. L. AYRES,
Associate Secretary

THE DECEMBER MEETING IN PASADENA

The three hundred sixty-fourth meeting of the Society was held at the California Institute of Technology, Pasadena, California, on Saturday, December 2, 1939. The attendance was about seventy-five including the following forty-seven members of the Society:

C. R. Adams, R. B. Adams, O. W. Albert, L. D. Ames, H. M. Bacon, Harry Bateman, Clifford Bell, H. F. Blichfeldt, H. F. Bohnenblust, C. P. Brady, J. L. Brenner, F. A. Butter, W. D. Cairns, A. B. Coble, P. H. Daus, A. L. Foster, W. H. Gage, H. E. Glazier, H. J. Hamilton, W. L. Hart, E. R. Hedrick, P. G. Hoel, D. G. Humm, C. G. Jaeger, G. F. McEwen, Rhoda Manning, W. A. Manning, A. D. Michal, C. B. Morrey, W. T. Puckett, E. J. Purcell, T. M. Putnam, R. W. Rector, A. C. Schaeffer, G. E. F. Sherwood, Gabor Szegö, A. E. Taylor, B. P. Taylor, S. E. Urner, F. A. Valentine, Morgan Ward, L. E. Wear, D. E. Whelan, M. B. White, W. M. Whyburn, E. R. Worthington, Max Zorn.

A general session was held Saturday morning and another in the afternoon. Professors Michal, Adams, Coble and Szegö presided at various times. At the morning session Professor C. B. Morrey of the University of California gave an address on *Existence and differentiability theorems for the solutions of variational problems for multiple integrals*. At the close of the afternoon session Professor and Mrs. Michal entertained at tea for members of the Society and their friends.

The titles and cross references to abstracts of papers read at the meeting are given below. Papers whose abstract numbers are followed by the letter *t* were read by title. Mr. Max Wyman, Mr. A. B. Mewborn and Mr. J. P. LaSalle were introduced by Professor Michal.

1. Rhoda Manning: *On the derivatives of the sections of bounded power series*. (Abstract 46-1-22.)

2. E. J. Purcell: *A multiple null-correspondence and a space Cremona involution of order* $2n-1$. (Abstract 46-1-29.)

3. P. G. Hoel: *The errors involved in evaluating correlation determinants*. (Abstract 46-1-18.)

4. C. R. Adams and A. P. Morse: *Continuous additive functionals on the space* (BV) *and certain subspaces*. (Abstract 46-1-1.)

5. Max Wyman: *Curl differential equations in abstract spaces*. (Abstract 46-1-40.)

6. A. L. Foster: *The structure of regularly ordered natural systems with 3 primes*. (Abstract 46-1-14.)

7. H. F. Bohnenblust: *Axiomatic characterization of* l_p *spaces*. (Abstract 46-1-11.)

8. A. B. Mewborn: *Abstract local geometry of paths*. Preliminary report. (Abstract 46-1-24.)

9. J. P. LaSalle: *Linear functions and functionals in linear topological spaces.* Preliminary report. (Abstract 46-1-21.)

10. Max Zorn: *Topological studies in the theory of analytic functions. II. Potential functions.* Preliminary report. (Abstract 46-1-42).

11. Morgan Ward: *Residuated distributive lattices.* (Abstract 46-1-38).

12. A. C. Schaeffer and Gabor Szegö: *Polynomials whose real part is bounded on a Jordan curve in the complex plane.* (Abstract 46-1-118.)

13. E. P. Vance: *On locally non-separating and locally non-alternating transformations.* (Abstract 46-1-37-*t*).

14. W. F. Whitmore: *Convergence theorems for functions of two complex variables.* (Abstract 46-1-39-*t*.)

15. A. D. Michal: *Differentials in abstract abelian topological groups.* (Abstract 46-1-25-*t*.)

16. Max Wyman: *Projective non-holonomic tensor analysis.* (Abstract 46-1-41-*t*.)

17. E. T. Bell: *Transformed multiplicative diophantine equations.* (Abstract 46-1-7-*t*.)

18. R. M. Robinson: *On the mean values of an analytic function.* (Abstract 46-1-116-*t*.)

T. M. PUTNAM,
Associate Secretary

BOOK REVIEWS

General Analysis. Part 2. *The Fundamental Notions of General Analysis.* By Eliakim Hastings Moore. (Memoirs of the American Philosophical Society, vol. 1, part 2.) Philadelphia, American Philosophical Society, 1939. 6+255 pp.

The first part of this work, embodying the contributions of the late E. H. Moore on general analysis, was reviewed in this Bulletin, vol. 42 (1936), pp. 465–468, by C. C. MacDuffee.[1] The first part was devoted in the main to considerations of an algebraical character, no notion of continuity or limit being involved. While a general variable appears in the considerations, its generality is only incidental in that it is usually permitted to assume at most a finite number of distinct values. This second part begins a study from the point of view of analysis, in that limits notions play a substantial role. It is general analysis in the sense that a general unconditioned variable is involved.

The basis of this part includes a number system $\mathfrak{A}$ of the type used at the end of the first part, with a continuity axiom added, namely, the existence of a greatest lower bound for nonvacuous sets of positive numbers. The resulting number system is then shown to be isomorphic to either the real number system, the complex number system, or the system of quaternions. There is assumed a general range $\mathfrak{P}$ unconditioned. Vectors enter as functions on $\mathfrak{P}$ to $\mathfrak{A}$, matrices as functions on $\mathfrak{P}^1\mathfrak{P}^2$ to $\mathfrak{A}$, and so on. While recent developments in linear functional analysis usually postulate a vector without considering it as a function on some range, the assumption made here, aside from its inherent advantages, has additional justification in that virtually all examples of the notion of vector have the character of a function.

The first chapter of this part (Chapter IV of the complete work) is devoted to an exposition of the notion of general limit. The general limit is a natural generalization of the notion of limit of a sequence, the positive integers being replaced by a general set $\mathfrak{L}$ of elements l, on which there is given a relation R on pairs of elements which is transitive, that is, such that $l_1 R l_2$ and $l_2 R l_3$ imply $l_1 R l_3$, and compositive or semiordered, that is, l_1 and l_2 imply the existence of an l

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[1] A review of this same part by the present reviewer appeared in Zentralblatt für Mathematik, vol. 13 (1936), pp. 116–117. We take this opportunity to correct a misprint appearing in MacDuffee's review, namely, the definition of general reciprocal λ of a matrix κ in the second last line of page 467 should read $SS\kappa\lambda\kappa=\kappa$ instead of $SS\lambda\kappa=\kappa$.

such that $l\ R\ l_1$ and $l\ R\ l_2$. Then the definition $\lim_l a_l = a$ follows the usual formula: for every positive e, there exists an l_e such that $l\ R\ l_e$ implies $|a_l - a| < e$. All the well known limits of analysis are special cases of this limit, provided in the continuous limit $\lim_{x \to x_0} f(x) = b$ it is assumed that $x \neq x_0$. In this particular work, the special case most frequently used is that in which the element l of $\mathfrak{L}$ consists of a finite number of points σ of a set $\mathfrak{P}$, and the relation $\sigma_1\ R\ \sigma_2$ means that σ_1 contains all of the points of σ_2. In the theory of integration this limit has demonstrated its usefulness in its interpretation of l as a partition of the fundamental set into permissible subsets, the relation R being the notion "finer than." A detailed exposition of the principal properties of this limit as well as a theory of double and iterated limits is given. An interesting notion presented in this chapter is that of general sum: $\sum_l a_l$. Essentially $\sum_l$ is $\lim_\sigma \sum_\sigma a_l$, where $\sigma = l_1, \cdots, l_k$. As might be expected, the general sum exists only if at most a denumerable number of the elements a_l are different from zero, but the interesting thing is that the general sum can exist if and only if the sum of the absolute values exists, which applied to sequences says that an infinite series converges absolutely if and only if it converges in the sense that σ spreads.

The next chapter (Chapter V) gives an exposition of the basic feature of this contribution to general analysis, namely, the important theorem that any positive hermitian matrix (that is, a function ϵ on $\mathfrak{P}\mathfrak{P}$ to $\mathfrak{A}$ for which $\epsilon(p_1 p_2) = \bar{\epsilon}(p_2 p_1)$ and such that if $p_1, \cdots, p_k$ are any elements of $\mathfrak{P}$, and $a_1, \cdots, a_k$, any numbers of $\mathfrak{A}$, then $\sum_i \sum_j \bar{a}_i \epsilon(p_i p_j) a_j \equiv S_\sigma S_\sigma \bar{\alpha} \epsilon \alpha \geqq 0$) defines a normed linear vector space, which has an inner product, or hermitian form, that is, is a general unitary space. The process of definition involves carrying over the notion of limited or modular matrix to limited or modular vector, and replacing the Kronecker δ by the positive hermitian matrix ϵ; that is, we have that the vector ξ on $\mathfrak{P}$ to $\mathfrak{A}$ is modular relative to ϵ if and only if the $|S_\sigma \bar{\alpha} \xi|$ is bounded for all σ of $\mathfrak{P}$, and all α such that $S_\sigma S_\sigma \bar{\alpha} \epsilon \alpha \leqq 1$. The least upper bound of these values when it exists is called the modulus of ξ, and the vectors for which this number is finite form the class $\mathfrak{M}$ of modular vectors μ, which is a linear normed vector space. It is possible to obtain other forms of this notion of modularity. A vector ξ is modular if there exist scalars s, such that the matrix $\epsilon s^2 - \xi \bar{\xi} \equiv \epsilon(p_1 p_2) s^2 - \xi(p_1) \bar{\xi}(p_2)$ is positive, the smallest s for which this is so being the modulus of ξ; or if γ_σ is the general reciprocal of $\epsilon_\sigma = \epsilon(p_i p_j)$, $(i, j = 1, \cdots, n)$, then ξ is modular if the expression $S_\sigma S_\sigma \bar{\xi} \gamma_\sigma \xi$ is bounded as a function of σ, and the least upper bound or limit in the σ sense of these scalars gives the square of the modulus.

If the matrix ϵ is positive definite, then this latter expression is well known as the quotient of two determinants, the denominator being that of $\epsilon(p_i p_j)$, $(i, j=1, \cdots, n)$, and the numerator being obtained by bordering this matrix with 0, $\xi(p_i)$ and $\bar{\xi}(p_j)$. It should be mentioned that the general reciprocal defined in Part I shows its usefulness here, in that it need not be assumed that the columns or rows of ϵ are linearly independent vectors. Moreover in addition to the boundedness property, the definition requires of any vector which is modular an accordance property; that is, if any set of rows (or columns) is linearly dependent, the corresponding places in the accordant vector satisfy the same dependence relation, that is, the rank of any submatrix of ϵ finite by rows is not changed by adding the accordant vector ξ to the columns.

The matrix ϵ gives rise to a bilinear integration process, namely, $\lim_\sigma S_\sigma S_\sigma \bar{\xi}\gamma_\sigma\eta = J\xi\eta$ (γ_σ the general reciprocal of ϵ_σ), effective between any two vectors which are modular, and resulting in an hermitian form, which gives us then a general not necessarily separable Hilbert space.

Special cases of the notion of modularity are interesting. It is obvious that when $\mathfrak{P}$ is the set of positive integers, and ϵ is the Kronecker δ, then we obtain ordinary Hilbert space as the space of modular vectors. There are obvious generalizations of this to the case when $\mathfrak{P}$ is any. If $\mathfrak{P}$ is the continuous range $0 \leqq p \leqq 1$, and $\epsilon(p_1 p_2)$ is the greater of p_1 and p_2, then modular vectors are absolutely continuous functions ξ such that $\xi(p) = \int_0^p \phi(q)dq$ where ϕ is of L^2. If $\beta(E)$ is a positive additive set function, then the positive hermitian matrix $\beta(E_1 E_2)$ gives rise to Hellinger Radon integrals, for $p=2$. If a_{pn} are complex-valued vectors in ordinary Hilbert space for each p, then the positive hermitian matrix $\alpha_{pq} = \sum_n \bar{a}_{pn} a_{qn}$ gives rise to a modular space which plays an important role in the Schmidt theory of solutions of a system of infinitely many equations in Hilbert space.[2] If we think of $\mathfrak{P}$ as consisting of the elements $f, g, \cdots$ of a Hilbert space, with hermitian form (f, g), then the positive hermitian matrix (f, g) on $\mathfrak{P}\mathfrak{P}$ to $\mathfrak{A}$ plays an important role in extending the results valid in a separable Hilbert space to the non-separable case.[3] The process set up by Moore reveals itself in this setting as a natural generalization of the reduction of a denumerable system of vectors to an equivalent orthonormal system.

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[2] See Riesz, *Les Systèmes d'Équations Linéaires à une Infinité d'Inconnus*, p. 66.

[3] See Y. Y. Tseng, *The Characteristic Value Problem of Hermitian Functional Operators in a Non-Hilbertian Space*, University of Chicago thesis, 1933. Compare with F. Riesz, Szeged Acta, vol. 7 (1934–1935), pp. 34–38.

The balance of the chapter is taken up with convergence of modular vectors. We have a weak convergence (called mode 1) which involves convergence of μ_l for each p of $\mathfrak{P}$, and the ultimate boundedness of $M\mu_l$, and the usual strong convergence (called mode 2) equivalent to the convergence to zero of $M(\mu_l - \mu)$. In a later chapter it is proved that the former convergence is equivalent to the convergence of $L(\mu_l)$ to $L(\mu)$ for every linear continuous functional operation L on $\mathfrak{M}$ to $\mathfrak{A}$, that is, the current definition of weak convergence. There is also introduced the notion of a sum of a set of vectors, involving a very strong type of convergence: $\sum_l M_l$.

Chapter VI is devoted to modular matrices. The definition of modular or limited matrix is, as might very well be expected from an analogy with vectors, as follows: $\kappa^{12} = \kappa(p^1 p^2)$ on $\mathfrak{P}^1\mathfrak{P}^2$ to $\mathfrak{A}$ is modular relative to the positive hermitian matrices ϵ^1 on $\mathfrak{P}^1\mathfrak{P}^1$ to $\mathfrak{A}$ and ϵ^2 on $\mathfrak{P}^2\mathfrak{P}^2$ to $\mathfrak{A}$ if $S_{\sigma^1} S_{\sigma^2} \bar{\alpha}^1 \kappa^{12} \alpha^2$ is bounded for all σ^1 and σ^2 and all α^1 and α^2 such that $S_{\sigma^1} S_{\sigma^1} \bar{\alpha}^1 \epsilon^1 \alpha^1 \leqq 1$ and $S_{\sigma^2} S_{\sigma^2} \bar{\alpha}^2 \epsilon^2 \alpha^2 \leqq 1$. There follows a thoroughgoing study of conditions of modularity, and questions of various kinds of convergence of modular matrices. Relative uniform convergence which plays such a prominent role in Moore's first general analysis enters naturally in the equivalence of strong convergence, that is, $\lim_l M(\kappa_l^{12} - \kappa^{12}) = 0$ with that of $J^1 J^2 \bar{\mu}^1 \kappa_l^{12} \mu^2$ to $J^1 J^2 \bar{\mu}^1 \kappa^{12} \mu^2$ uniform relative to $M\mu^1 M\mu^2$. For hermitian matrices it develops that a weak modularity, namely, the boundedness of $S_\sigma S_\sigma \bar{\alpha} \kappa \alpha$ for all σ and α such that $S_\sigma S_\sigma \bar{\alpha} \epsilon \alpha \leqq 1$ is necessary and sufficient for modularity; further, if the number system $\mathfrak{A}$ is not the real number system, this is also true for any matrix κ on $\mathfrak{P}\mathfrak{P}$ to $\mathfrak{A}$ modular relative to ϵ, ϵ.

Chapter VII treats of linear functional operations, bilinear functional operations and linear functional transformations, especially those which are continuous or limited. In this setting every continuous linear functional operation is of the form $J\bar{\xi}\mu$, ξ modular, every limited or continuous bilinear functional operation determines and is determined by a modular matrix κ^{12}, namely, $\kappa^{12} = B(\epsilon^1, \epsilon^2)$ and $B(\bar{\mu}^1, \mu^2) = J^1 J^2 \bar{\mu}^1 \kappa^{12} \mu^2$, and every continuous linear transformation T on the modular class $\mathfrak{M}^2$ to $\mathfrak{M}^1$ is also determined by a limited matrix, namely, $\kappa^{12} = T\epsilon^2$ with $T\mu^2 = J^2 \kappa^{12} \mu^2$. Every completely continuous transformation, being defined as one in which weak convergence in $\mathfrak{M}^2$ implies strong convergence in $\mathfrak{M}^1$, is the strong limit, that is, in the sense of the modulus of the transformations, of a transformation of a finite number of dimensions each of which is completely continuous, and the form of this approximation is the simple expression $J_{\sigma^2} \kappa^{12} \mu^2 = S_{\sigma^2} \kappa^{12} \gamma_{\sigma^2}{}^2 \mu^2$, where γ_{σ^2} is the general reciprocal of $\epsilon_{\sigma^2}{}^2$. Various kinds of convergence of sets of linear, bilinear operations, and transformations are discussed. In many respects the last two chapters might be thought of as carrying over into the general setting the results of Hilbert and his followers on limited matrices and functional operations on ordinary Hilbert space.

The method of exposition in this second part follows closely the lines of the first part. Each chapter has an introduction giving an excellent survey of the material to be covered in the chapter, most theorems are stated not only in words, but also in the adaptation of the Peano symbolism introduced by Moore. To any one reading any considerable portion of this work, and consequently acquiring easily a familiarity with the symbolism used, this constant presentation of the same ideas in two forms unfortunately gets to be a little bit tiresome. The exposition is throughout very clear, very easily followed, and might even in some instances have assumed greater intelligence on the part of the reader. The reviewer was conscious of the paucity of references to the supporting literature, especially that current at the time when these developments of Moore were under way. While an isolationist point of view may have been justified at the time of development, the work would be enhanced historically and in comprehensibility if more frequent contacts with the literature were made available, and this would be in line with the spirit of E. H. Moore as the reviewer knew him thirty years ago.

To make an estimate of the value of this publication at this time is a little difficult. Much of it seems only historically worth while in the light of more recent developments in linear functional theory. The general limit has already shown its value in recent work. In the same way, the reviewer feels strongly that the notion of modularity is important, as well as the constructive procedure for hermitian operations on which Hilbert spaces are based. These two notions alone make this part worth while. Many of the results presented are basic to the parts of this publication to appear later, and so complete judgment must be deferred until these further developments are presented.

T. H. HILDEBRANDT

Lezioni di Analisi Matematica. Part 1. By Francisco Tricomi. 4th edition. Padova, Cedam, 1939. 8+328 pp.

It is to be understood that this is the first of two volumes on analysis and hence the author's aim is only to cover some of the traditional fundamentals of algebra and calculus.

The author introduces the subject with a discussion of determinants, and then applies them to the solution of linear equations. Next, real numbers, rational and irrational, are defined and some fundamental operations discussed, including the partition of rational numbers. The idea of "limit point," or point of condensation, is stressed.

Functions of one variable are defined and discussed, including the limit of a function, continuous function, and some properties resulting from continuity. In this chapter (V), the author is particularly generous with illustrative examples and figures.

Sixty-eight pages are devoted to the derivative. Some of the applications include: Rolle's theorem, theorem of mean value, theorem of L'Hospital, maximum and minimum, indeterminate forms, tangents, differential of arc, asymptotes, concavity, order of contact, curvature, osculating circle, and evolutes.

The indefinite integral and some elementary differential equations are more briefly treated. The definite integral is introduced, but the concept of a definite integral being the limit of a sum of products is omitted.

Although the chapter on series follows a discussion of the derivative and the integral, the treatment is entirely algebraic. It is confined to definition and properties of an infinite series, and rules for convergence and divergence of a series or sums and products of series.

After defining a complex number as a number pair, and showing some fundamental operations including De Moivre's formula, the author briefly shows that the preceding concepts of limit, series, function, derivative and integral as applied to real numbers could be developed along similar lines for complex numbers.

In a discussion of the rational integral equation of one unknown the reviewer was prepared to meet remainders, resultants, discriminants, methods of finding rational and irrational roots, symmetric functions, solution by formula of quadratic, cubic and quartic equations; but it was a pleasant surprise to find not just a reference to but a proof of D'Alembert's fundamental theorem of algebra.

The author closes with a discussion of linear transformations and their application to quadratic forms.

The book is well written, explanation is clear; whenever possible, theory is illustrated by both example and figure. No supplementary exercises are included, although the author points out that they are available in both printed and lithograph form.

AMOS BLACK

Irrationalzahlen. By Oskar Perron. 2d revised edition. (Göschens Lehrbücherei, vol. 1.) Berlin, de Gruyter, 1939. 8+199 pp.

As the author says in his interesting preface to this second edition, there are no great changes except in the fifth chapter, which is longer by twelve pages and includes Estermann's beautiful proof of Kronecker's approximation theorem and a section on the *Gleichverteilung.* The bibliography has been increased by about fifty per cent to correspond to the changes; but it is unfortunate that, of the three important works on the subject omitted from the bibliography of the first edition (and clearly indicated in the review in this Bulletin by A. J. Kempner (vol. 29 (1923), pp. 34–36)) only one has been added in this newer edition. Also, the references are still to whole tomes and no hint is given as to chapter and verse.

The preface of this new edition seemed in some ways the most interesting part, on account of the reason that the author gives for presenting Dedekind's theory of irrational numbers rather than the theory of Cantor and Méray. In 1921, he did not seem to think it necessary to give any reason for basing his treatment on Dedekind's work; but in 1939, he devotes most of the preface to justify his giving Dedekind's rather than Cantor's theory. He refers to an article by Bieberbach and the famous one by Hardy in "Nature" on the J-type and S-type of mathematicians. One could easily wonder just what lies behind these careful justifications. However that may be, we wish him well, for Perron has done yeoman service in writing textbooks for universities and *technische Hochschulen.*

OLIVE C. HAZLETT

Logic of Algebra. By Paul Dienes. (Actualités Scientifiques et Industrielles, no. 614.) Paris, Hermann, 1938. 76 pp.

In the preface Dienes says that it is his aim to clear the ground for a realistic discussion of the "crisis," and not to give a systematic description of the logical structure of algebra. To this end he has many discussions of the thought processes underlying our use of arithmetic and algebra. So long as he is discussing the more mathematical of these (counting, and the like), his treatment is sound, though antiquated. With some of the less mathematical thought processes, he is not so successful. In particular, his discussion of inference is extremely unsatisfactory.

BARKLEY ROSSER

La Mécanique Ondulatoire des Systèmes de Corpuscles. By Louis de Broglie. Paris, Gauthier-Villars, 1939. 6+223 pp.

Following the historical development of wave mechanics, the opening pages are devoted to an exposition of the analogies between the principles of classical mechanics and those of classical optics. After the general principles of quantum mechanics are formulated, the theory of first integrals is given a particularly extensive treatment. A short sketch of perturbation theory is given, followed by a discussion of the symmetric and anti-symmetric wave functions. "Spin" is then introduced and treated from a non-relativistic point of view. Pauli's exclusion principle is formulated and its experimental justification is discussed. In the last chapter applications are made to the theories of valences, of band spectra, and of the para- and ortho-spectra of hydrogen molecules.

There are a few trivial inaccuracies. The reviewer was greatly perplexed on page 133, until it occurred to him that c and d on line 12 were probably misprints for c^2 and d^2.

Also the reviewer personally does not believe in the cogency of the arguments based on the analogies between optics and classical mechanics. In fact, even as the author himself points out (p. 32), these analogies can not be followed too closely in arriving at the Schroedinger wave equation, which is of the first order with respect to the time rather than of the second order.

These adverse criticisms are, however, of a very minor nature. The purpose of the book, as stated in the author's preface, is to give the reader a bird's eye view of the vast structure of modern non-relativistic wave mechanics. There is no pretension whatever of any new mathematical results or methods; and even for the physicist the work must probably be regarded as purely expository. As such it appears to the reviewer to be outstandingly successful.

D. C. LEWIS

Les Nouvelles Méthodes du Calcul des Probabilités. By L. Bachelier. Paris, Gauthier-Villars, 1939. 8+71 pp.

The reviewer finds it rather difficult to justify the title. In a book dealing with *new* methods in probability and published in 1939 the reader expects to find something about the recent developments (to mention but the new treatment of the law of large numbers), even if the book contains 69 pages only. As it is, it is devoted to results due to the author himself, many dating as far back as 1906. These results are merely stated, which is hard on the reader, unless he decides to go through the original papers cited. (The author acknowledges this difficulty at the close of the book.)

The main features may be summarized as follows. (a) Reducing problems in probability to game-problems, which insures unity of treatment. (b) Introducing the time-element, so that if the number of trials (games) is very large, it may be treated as a continuous parameter, and each game is an element of a continuum.

The book is divided into eight chapters. Chapter I gives the classical formulae of Moivre, Poisson, Laplace. Chapter II is fundamental. It gives, first, a classification of probabilities. The conditions of the game may remain identical at each party (*uniformity*), or they may vary from party to party according to a certain given law; the latter, in turn, may depend for each party on its rank alone (*independence*), or on the eventual results of previous parties (*connectedness*). We further speak of probabilities of *first, second, third kind* according as the variables under consideration take all values from $-\infty$ to $+\infty$, or one of these variables is limited from above or from below, or from above and from below. The author next gives the probability that the loss of a player who plays t parties be x (assuming independence, but non-uniformity). The "hyperasymptotic formula" for this probability of the first kind is

$$\exp\left\{-\left[\int_0^t \psi'(t)dt + x\right]^2 \Big/ \int_0^t \phi'(t)dt\right\} \cdot \left[\pi\int_0^t \phi'(t)dt\right]^{-1/2}.$$

This replaces Laplace's formula. Here $\psi'(t)$, $\phi'(t)$ (>0) are arbitrary functions determined in each specific problem by its data. Thus, the "*elementary probability of first kind*" is

$$\exp\left\{-[\psi(t) + x]^2/\phi(t)\right\}dx[\pi\phi(t)]^{-1/2},$$

and the total mathematical expectation is

$$E = -\int_{-\infty}^{\infty} \frac{x\exp\left\{-[\psi(t) + x]^2/\phi(t)\right\}dx}{[\pi\phi(t)]^{1/2}} = \psi(t).$$

$\phi(t)$ is called "instability function," so that $\phi'(t)dt$, $\psi'(t)dt$ are respectively the "elementary instability function" and "elementary mathematical expectation" (relative to the time-element dt). In the "symmetric" case $\psi(t)=0$, and the formulas involve $\phi(t)$ only. In case of uniformity $\psi(t)=t\psi_1$, $\phi(t)=t\phi_1$, and the constants ψ_1, ϕ_1 are furnished by the data of the problem. We also find here expressions for the average, quadratic and probable deviations.

The following chapters extend the preceding notions and formulas to more complicated probabilities. Chapter III treats the loss in a game with connected probabilities. Use is made of the Maclaurin expansion for $\psi(x)$ (the existence of such expansion is stated, without further consideration). Chapters IV and V treat probabilities of second and third kinds respectively, under various conditions (unlimited number of plays, non-uniformity, periodicity). Chapter VI deals with several variables, that is, with several players and their respective losses $x_1, x_2, \cdots, x_{n-1}$. Here we find "ellipsoids of probability," on which "lie" $x_1, x_2, \cdots, x_{n-1}$ which at time t have the same probability. The last two chapters deal with mechanics. Chapter VII deals with "cinematic probabilities." A particle is moving (along a straight line or in plane or in space) at random—the motion being characterized by a given instability function; we ask what is the probability of its occupying a preassigned position at time t? The same problem is studied if, in addition to the above random displacement, the particle is activated by a force of attraction toward a fixed center (proportional to the distance). The last chapter (Chapter VIII) treats "dynamical probabilities," that is, the motion of a particle or of a system of particles (probability of preassigned position or of preassigned velocity at a given t) when the acting forces are random forces, totally or partially, for example, regarding their directions.

J. SHOHAT

NOTES

The Emergency Executive Committee of the International Congress of Mathematicians has decided definitely to postpone until some more favorable date the Congress which was to have been held in Cambridge in September, 1940. Notice to that effect is being sent to the invited speakers and to other parties interested.

The editors of the Bulletin wish to make grateful public acknowledgment of the services rendered by the following persons who have refereed papers: C. R. Adams, R. P. Agnew, Leon Alaoglu, R. G. Archibald, W. L. Ayres, Reinhold Baer, R. P. Bailey, R. W. Barnard, M. A. Basoco, Harry Bateman, Garrett Birkhoff, Henry Blumberg, Salomon Bochner, Richard Brauer, E. T. Browne, Herbert Busemann, C. C. Camp, E. W. Chittenden, Alonzo Church, J. A. Clarkson, A. H. Clifford, A. B. Coble, George Comenetz, A. H. Copeland, A. T. Craig, H. B. Curry, J. H. Curtiss, J. L. Doob, Jesse Douglas, R. D. Douglass, W. L. Duren, Ben Dushnik, Arnold Emch, H. J. Ettlinger, J. M. Feld, Aaron Fialkow, L. R. Ford, W. B. Ford, Orrin Frink, T. C. Fry, H. L. Garabedian, H. H. Goldstine, W. C. Graustein, O. H. Hamilton, M. R. Hestenes, Harold Hotelling, Ralph Hull, Witold Hurewicz, W. A. Hurwitz, M. H. Ingraham, Dunham Jackson, Nathan Jacobson, R. L. Jeffery, B. W. Jones, F. B. Jones, E. R. van Kampen, A. J. Kempner, S. C. Kleene, M. S. Knebelman, R. E. Langer, C. H. Langford, C. G. Latimer, D. H. Lehmer, J. H. Levin, Harry Levy, A. N. Lowan, R. G. Lubben, N. H. McCoy, E. J. McShane, Saunders MacLane, H. M. MacNeille, W. A. Manning, Morris Marden, W. T. Martin, Walther Mayer, Karl Menger, G. M. Merriman, H. H. Mitchell, Deane Montgomery, C. N. Moore, F. J. Murray, S. B. Myers, C. V. Newsom, Oystein Ore, W. F. Osgood, Gordon Pall, Hillel Poritsky, H. A. Rademacher, Tibor Radö, G. Y. Rainich, W. T. Reid, P. R. Rider, J. F. Ritt, J. H. Roberts, M. S. Robertson, I. J. Schoenberg, Wladimir Seidel, W. E. Sewell, I. M. Sheffer, J. A. Shohat, L. L. Silverman, Abraham Sinkov, L. L. Smail, I. S. Sokolnikoff, N. E. Steenrod, M. H. Stone, E. B. Stouffer, Otto Szász, Gabor Szegö, J. H. Taylor, E. W. Titt, W. J. Trjitzinsky, J. W. Tukey, S. M. Ulam, J. V. Uspensky, H. S. Vandiver, R. J. Walker, H. S. Wall, J. L. Walsh, L. E. Ward, G. C. Webber, J. H. M. Wedderburn, Louis Weisner, Hermann Weyl, A. P. Wheeler, G. T. Whyburn, W. M. Whyburn, D. V. Widder, Norbert Wiener, L. R. Wilcox, R. L. Wilder, A. R. Williams, Oscar Zariski, and Leo Zippin.

The editors of the Transactions wish to acknowledge the services of the following persons, not members of the Editorial Board, who have been consulted regarding papers offered for publication in volumes 45 and 46: Harry Bateman, Garrett Birkhoff, J. L. Doob, L. M. Graves, G. A. Hedlund, N. H. McCoy, Saunders MacLane, Deane Montgomery, H. A. Rademacher, H. W. Raudenbush, J. F. Ritt, O. F. G. Schilling, Wladimir Seidel, J. A. Shohat, I. S. Sokolnikoff, D. J. Struik, J. D. Tamarkin, J. M. Thomas, S. M. Ulam, H. S. Wall, J. L. Walsh, S. S. Wilks, and Aurel Wintner.

The report of the Joint Commission, appointed in 1937 by the Mathematical Association of American and the National Council of Teachers of Mathematics to consider the place of mathematics in secondary education, will be published early in 1940 as the yearbook of the National Council of Teachers of Mathematics.

The School of Mathematics of the Institute for Advanced Study each year allocates a small number of stipends to gifted young mathematicians and mathematical physicists to enable them to study and to do research work at Princeton. Candidates must have given evidence of ability in research comparable at least with that expected for the degree of Doctor of Philosophy. Blanks for application may be obtained from the School of Mathematics of the Institute, Fuld Hall, Princeton, New Jersey, and are returnable by February 15, 1940.

Professor W. D. Cairns of Oberlin College has been given the title of professor emeritus.

Professor C. E. Comstock of Bradley Polytechnic Institute has retired with the title of professor emeritus.

Professor K. D. Swartzel of the University of Pittsburgh has retired with the title of professor emeritus.

Mr. A. C. Washburne, actuary of the Berkshire Life Insurance Company since 1911, has retired with the title of actuary emeritus.

Professor A. H. Wilson of Haverford College has been given the title of professor emeritus.

Mr. Achille Bassi has been appointed to a professorship at the University of Bologna.

Dr. V. G. Iyer and Mr. V. Seetharaman have been appointed lecturers in mathematics at Annamalai University.

Dr. B. de Loor of the University of Pretoria has been promoted to a professorship.

Professor W. Ludwig of the Technical School of Dresden has retired.

Dr. B. Ramamurti of Annamalai University has been appointed professor and head of the department of mathematics at Government College, Ajmere.

Professor F. Rehbock has been appointed substitute professor at the Technical School of Braunschweig.

Dr. K. Strubecker has been appointed to an assistant professorship at the Technical School of Vienna.

Professor C. R. Adams of Brown University is spending the first semester of this year at the University of California.

President Frank Aydelotte of Swarthmore College has been elected director of the Institute for Advanced Study to succeed Dr. Abraham Flexner.

Assistant Professor Clifford Bell of the University of California at Los Angeles has been promoted to an associate professorship.

Assistant Professor H. A. Bender of the University of Akron has been promoted to an associate professorship.

Dr. J. W. Blincoe of the University of Tennessee has been promoted to an assistant professorship.

Assistant Professor Ann S. Buchanan of Southwestern Louisiana Institute has been promoted to an associate professorship.

Assistant Professor R. S. Burington of Case School of Applied Science has been promoted to an associate professorship.

Assistant Professor I. W. Burr of Antioch College has been promoted to an associate professorship.

Professor J. H. Butchart of Phillips University has been appointed to a professorship at William Woods College.

Assistant Professor J. W. Calkin of the University of New Hampshire has been appointed to an assistant professorship at Armour Institute of Technology.

Associate Professor J. A. Cooley of the University of Tennessee has been promoted to a professorship.

Mr. H. M. Cox, formerly assistant examiner in the University System of Georgia, has been appointed assistant professor and assistant to the Bureau of Instructional Research at the University of Nebraska.

Dr. A. E. Currier of the U. S. Naval Academy has been promoted to an assistant professorship.

Assistant Professor H. A. DoBell of the New York State College for Teachers has been promoted to a professorship.

Dr. J. E. Eaton has been promoted to an assistant professorship at Hofstra College.

Professor Arnold Emch of the University of Illinois has retired.

Dr. W. K. Feller of the University of Stockholm has been appointed to a lectureship at Brown University.

Mr. K. G. Fuller has been promoted to an assistant professorship at the Teachers College of Connecticut.

Dr. D. G. Fulton of Armour Institute of Technology has been appointed to an assistant professorship at Ohio Northern University.

Associate Professor J. J. Gergen of Duke University has been promoted to a professorship.

Professor W. C. Graustein of Harvard University has been appointed assistant dean of the faculty of arts and sciences.

Assistant Professor J. A. van Groos of Oregon State College has been promoted to an associate professorship.

Professor E. H. Hadlock of Chulalonghorn University, Siam, has been appointed to an assistant professorship at Boston University.

Dr. E. A. Hazelwood of State Teachers College, Mansfield, Pennsylvania, has been appointed to an assistant professorship at Texas Technological College.

Associate Professor E. E. Heimann has been promoted to a professorship at the State Teachers College of Ada, Oklahoma.

Assistant Professor E. R. Heinemann of Texas Technological College has been promoted to an associate professorship.

Professor T. H. Hildebrandt, on leave from the University of Michigan, is spending the first semester at Brown University.

Professor H. M. Hosford of the University of Arkansas has been appointed dean of the college of arts and sciences.

Dr. Witold Hurewicz has been appointed to an assistant professorship at the University of North Carolina.

Assistant Professor J. A. Hyden of Vanderbilt University has been promoted to an associate professorship.

Associate Professor C. M. Jensen of Kansas Wesleyan University has been promoted to a professorship.

Assistant Professor Fritz John of the University of Kentucky has been promoted to an associate professorship.

Assistant Professor Evan Johnson of Pennsylvania State College has been transferred from the Fayette Center to State College.

Assistant Professor W. J. Kirkham has returned to Oregon State College after two years' leave of absence as statistician with the Home Owners' Loan Corporation in Washington.

Dr. L. A. Knowler of Hunter College has been appointed to an assistant professorship at the University of Iowa.

Associate Professor R. C. Lamb of the U. S. Naval Academy has been promoted to a professorship.

Dr. V. V. Latshaw of Lehigh University has been promoted to an assistant professorship.

Associate Professor J. B. Linker of the University of North Carolina has been promoted to a professorship.

Assistant Professor Dora McFarland of the University of Oklahoma has been promoted to an associate professorship.

Associate Professor Roy MacKay of Eastern New Mexico Junior College has been promoted to a professorship.

Dr. M. S. Macphail of Acadia University has been promoted to an assistant professorship.

Assistant Professor Morris Marden of the University of Wisconsin Extension Division has been promoted to an associate professorship.

Assistant Professor S. L. Mason of the University of North Dakota has been promoted to an associate professorship.

Professor Karl Menger of the University of Notre Dame is conducting a fortnightly seminar at Armour Institute of Technology.

Assistant Professor J. L. Nickerson of Mt. Allison University has been appointed to an assistant professorship in the College of Physicians and Surgeons of Columbia University.

Dr. A. L. O'Toole has been appointed to a professorship at Mundelein College, Chicago.

Mr. J. C. Oxtoby has been appointed to an assistant professorship at Bryn Mawr College.

Dr. H. V. Park of North Carolina State College has been promoted to an assistant professorship.

Associate Professor B. C. Patterson of Hamilton College has been promoted to a professorship.

Professor H. R. Phalen, formerly of Armour Institute and Bard College, is at Brown University for the year 1939–40.

Dr. H. R. Pyle of Earlham College has been appointed dean.

Assistant Professor H. A. Rademacher of the University of Pennsylvania has been promoted to a professorship.

Dr. H. W. Raudenbush of Queens College has been promoted to an assistant professorship.

Assistant Professor C. B. Read has been made associate professor and acting head of the department of mathematics at the University of Wichita.

Assistant Professor Francis Regan of St. Louis University has been promoted to an associate professorship.

Mr. N. S. Risley of Fenn College has been promoted to an assistant professorship.

Lord Bertrand Russell will lecture at the University of California at Los Angeles during this academic year.

Dr. Samuel Saslaw has been appointed to an assistant professorship at the University of Miami.

Dr. Nathan Schwid of Montana State College has been appointed to an assistant professorship at the College of Mines and Metallurgy at El Paso, Texas.

Associate Professor J. A. Shohat of the University of Pennsylvania will be on leave of absence the second semester of 1939–40.

Associate Professor A. R. Sloan of Carson-Newman College has been promoted to a professorship.

Assistant Professor H. W. Smith of Oklahoma Agricultural and Mechanical College has been promoted to an associate professorship.

Assistant Professor W. F. Smith of New River State College, West Virginia, has been promoted to a professorship.

Dr. D. E. South of the University of Kentucky has been promoted to an associate professorship.

Associate Professor R. C. Staley of the University of North Dakota has been promoted to a professorship.

Associate Professor E. L. Thompson of Texas Technological College has been promoted to a professorship.

Mr. A. R. Turquette of Florida Southern College has been promoted to an associate professorship.

Dr. S. M. Ulam has been appointed to a lectureship at Harvard University.

Dr. S. E. Warschawski of Brown University has been appointed visiting assistant professor at Washington University.

Dr. D. L. Webb of Georgia School of Technology has been promoted to an assistant professorship.

Mr. C. W. Williams of Washington and Lee University has been appointed to a professorship at Armstrong Junior College, Savannah, Georgia.

Assistant Professor G. A. Williams of Oregon State College has been promoted to an associate professorship.

Associate Professor R. C. Yates of the University of Maryland has been appointed to an associate professorship at the Louisiana State University.

The following appointments to instructorships are announced: University of Alabama: Mr. E. S. Kennedy, Dr. C. L. Seebeck; University of Arkansas: Mr. J. R. F. Kent; Armour Institute of Technology: Dr. G. E. Hay, Dr. A. T. Lonseth; Berkeley Institute, Brooklyn, New York: Miss Melita A. Holly; Blue Ridge College, New Windsor, Maryland: Mr. M. E. Terry; Brooklyn College: Dr. Leon Gropper, Dr. Hyman Serbin; Brown University: Mr. O. H. Schmidt; University of California: Dr. A. P. Morse; University of California at Los Angeles: Dr. W. T. Puckett; Chicago Teachers College: Mr. Ralph Mansfield; College of the City of New York (Day Session): Dr. Max Shiffman; Compton Junior College, Compton, California: Mr. R. W. Rector; Duke University: Mr. A. V. Martin; Elmhurst College: Miss Mary M. Handel; George Washington University: Dr. J. W. Wrench; George School of Technology: Mr. C. L. Carroll, Dr. L. J. Green; Herzl Junior College: Dr. B. H. Gere; Hunter College: Mr. L. A. Aroian; Hunter College (Evening Session): Mr. F. C. Hall; Los Angeles City College: Dr. C. P. Brady; University of Michigan: Dr. H. H. Goldstine; University of Minnesota: Dr. Isaac Opatowski; University of Nebraska: Dr. D. H. Rock; University of Nevada: Mr. W. G. Palm; University of Notre Dame: Dr. A. O. Lindstrom; Oregon State College: Dr. C. B. Smith, Dr. Andrew Sobczyk; Pennsylvania State College at Altoona: Mr. E. H. Umberger; Pennsylvania State College in Pottsville: Mr. R. B. Kleinschmidt; Pennsylvania State College in Uniontown, Dr. C. E. Sealander; University of Rochester: Mr. H. P. Atkins; Swarthmore College: Dr. Herbert Busemann, University of Tennessee: Mr. D. M. Seward; Agricultural and Mechanical College of Texas: Dr. G. L. Gross; Tulane University: Dr. M. L. Kales; University of Utah: Mr. W. H. Myers; College of William and Mary: Dr. Wilfred Kaplan; Wright Junior College: Dr. W. H. Erskine; Yale University: Dr. S. A. Jennings, Mr. J. C. Montgomery, Mr. R. R. Stoll, Mr. W. G. Swope.

The death of Professor H. Wiener of the Technical School of Darmstadt is reported.

The death of Dr. Charles Hopkins of Tulane University on September 15, 1939, is reported. He had been a member of the Society since 1922.

Associate Professor W. H. Lyons of Kansas State College died on October 21, 1939.

The death of Dr. W. C. Mitchell of George Washington University has been reported. He had been a member of the Society since 1928.

Dean F. K. Richtmyer of Cornell University died November 7, 1939, at the age of fifty-eight.

Associate Professor A. A. Shaw of the University of Arizona died on October 11, 1939. He had been a member of the Society since 1928.

ABSTRACTS OF PAPERS

SUBMITTED FOR PRESENTATION TO THE SOCIETY

The following papers have been submitted to the Secretary and the Associate Secretaries of the Society for presentation at meetings of the Society. They are numered serially throughout this volume. Cross references to them in the reports of the meetings will give the number of this volume, the number of this issue, and the serial number of the abstract.

1. C. R. Adams and A. P. Morse: *Continuous additive functionals on the space (BV) and certain subspaces.*

In a recent paper (Transactions of this Society, vol. 40 (1936), pp. 421–438) Adams introduced for (BV), the space of functions $x(t)$ of bounded variation on $0 \leqq t \leqq 1$, the metric $(x, y) = \int_0^1 |x(t) - y(t)| dt + |T_0^1(x) - T_0^1(y)|$, where in general $T_0^1(z)$ stands for the total variation of $z(t)$ on $0 \leqq t \leqq 1$. Thus metrised, (BV) is not a Banach space; but it is complete, separable, and boundedly compact. On it there exist additive homogeneous functionals which are continuous at each point of a dense set and discontinuous at each point of its dense complement; and there exist additive functionals which are continuous without being uniformly so. The general form of the additive and uniformly continuous functional was found to be $\int_0^1 x(t)\alpha(t)dt$, where $\alpha(t)$ is essentially bounded; conversely, every such integral is such a functional. It is now shown that the general form of the additive and continuous functional on (BV) is (1) $\int_0^1 x(t)dg(t)$ with $g(t)$ continuous; that the general form of the additive and continuous functional on the subspace (CBV) of continuous functions of bounded variation is (1) with $g(t)$ bounded and continuous except at a countable set; and that the general form of the additive and continuous functional on the subspace (AC) of absolutely continuous functions is (1) with $g(t)$ bounded and continuous save at a set of measure zero. In each case, conversely, every such integral is a functional of the kind specified. (Received November 3, 1939.)

2. R. P. Agnew: *On rearrangements of series.*

Let E be the space of permutations $x = \{x_1, x_2, \cdots\}$ of the positive integers in which the metric is that of Fréchet. Let $\sum c(n)$ be a convergent series of real terms for which $\sum |c(n)| = \infty$. It is shown that, for each $x \in E$ except those of a set of the first category, the inferior and superior limits of the partial sums of the series $\sum c(x_n)$ are respectively $-\infty$ and $+\infty$. In particular, the set of $x \in E$ for which $\sum c(x_n)$ converges is of the first category. (Received October 23, 1939.)

3. A. A. Albert: *On ordered algebras.*

The note consists of a proof that every ordered algebra is a field. It will appear very soon in this Bulletin. (Received October 16, 1939.)

4. J. V. Atanasoff: *Generalized Taylor expansions.* Preliminary report.

It is easy to see that by employing the idea of functionals one may view a Taylor expansion as a typical biorthogonal development. In this paper a generalization of

Taylor's expansion is exhibited which retains several of the properties of Taylor's expansion including a simple formula for the remainder. It is then shown that all biorthogonal developments may be regarded as special cases of these generalized Taylor expansions. (Received November 4, 1939.)

5. A. A. Aucoin: *Diophantine equations of degree n.*

This paper generalizes the results of an earlier paper by W. V. Parker and the author (National Mathematics Magazine, vol. 13 (1938), pp. 115–117). Solutions are obtained for the equation $f(x_1, \cdots, x_p) = g(y_1, \cdots, y_q)$ where f and g are homogeneous polynomials of degrees n and m respectively, with integral coefficients, and f is such that for a set of integers $a_1, \cdots, a_p$ (not all zero) all the partial derivatives of f of all orders less than $n-1$ vanish for $x_i = a_i$. Two examples of a function f which satisfy the conditions of the theorem are given. One is the determinant $D(x) = |a_{ij}x_{ij}|$ and the other is the product of n linear factors in n unknowns. In the case of the latter function, the equation is also solved by an entirely different method. (Received October 24, 1939.)

6. W. L. Ayres: *A note on the definition of arc-sets.*

In order that a subset A of a Peano space P be an arc-set or A-set either of the following two conditions is necessary and sufficient: (1) Every set separating two points (in the weak *coupure* sense) of A in A separates them in P: (2) A is connected, $\bar{A} \cdot \bar{C}$ is a single point for each component C of $P-A$, and this point y belongs to A if $\bar{A}$ contains two continua S and T such that $S \cdot T = y$. (Received November 3, 1939.)

7. E. T. Bell: *Transformed multiplicative diophantine equations.*

The equations discussed are polynomials of any degree in any number of indeterminates, with integer coefficients, equated to zero, transformable rationally (but not in general birationally) into pure multiplicative equations. *All* integer solutions are found, and a general method of solution, with numerous examples, is given. An extension to systems of diophantine equations is sketched; a simple example is the system $xy+yz+zx = uw+wy+yu$, $x^2+z^2 = u^2+w^2$ in six indeterminates. (Received November 6, 1939.)

8. P. O. Bell: *Projective analogues of the congruence of normals.*

Both the normal and the projective normal at a general point of a non-ruled surface in ordinary space may be defined as cusp axes of certain integral invariants. Moreover, the projective normal, like the normal, is intrinsically connected with the surface, and the curves which correspond to the developables of the projective normal congruence resemble the lines of curvature by forming also a conjugate net. The author shows that these properties, which have been regarded as peculiarly characteristic of a unique substitute for the normal, are shared likewise by every member of an infinite class of such congruences. A method is outlined for the geometric determination of a general congruence of this class and the extremals of the associated integral invariant are interpreted geometrically. Finally certain significant special cases are considered and their associated canonical forms are introduced. (Received October 28, 1939.)

9. Garrett Birkhoff: *An ergodic theorem for arbitrary semi-groups.*

First, convergence with respect to any transitive ordering relation is defined. This specializes to Moore-Smith convergence, and allows one to speak of convergence for the means of the transforms of any element, under any group or semi-group of transformations. This new definition thus enables one to formulate ergodic theorems for arbitrary semi-groups of linear operators, a problem suggested to the author by L. Alaoglu. The main result of the present paper is a universal ergodic theorem for, semi-groups of linear isometries and contractions. Surprisingly, the theorem does not apply if the hypothesis of modulus unity is weakened to modulus two. (Received October 21, 1939.)

10. Salomon Bochner: *Finitely additive integral.*

Generalizing a construction recently given for point functions (Annals of Mathematics, (2), vol. 40 (1939), pp. 769–799) we show that a finitely additive positive functional on a partially ordered vector space without any limit requirement can be interpreted as an integral over a suitable Boolean algebra. (Received November 7, 1939.)

11. H. F. Bohnenblust: *Axiomatic characterization of l_p spaces.*

The notion of l_p spaces can be extended to include similar spaces defined over any Boolean algebra with a measure function (cf. H. Freudenthal, Akademie van Wetenschappen, Amsterdam, vol. 39 (1936), p. 641). Similarly a variety of spaces can be defined corresponding to the value p equals infinity. The spaces thus obtained, which include the classical l_p and L_p spaces, are characterized axiomatically as partially ordered Banach spaces in which the norm satisfies the following condition: If a, b and c, d are two pairs of elements such that the inf $(|a|, |b|)$ and the inf $(|c|, |d|)$ are equal to zero and the norms of a and c and the norms of b and d are equal, then the norms of $a+b$ and of $c+d$ are equal. (Received November 4, 1939.)

12. D. G. Bourgin: *Closure of products of functions.*

Let E_s and E_t be sets in n and m dimensional euclidean spaces, respectively. The points of these sets are denoted by s and t. The symbol $T(E)$ denotes a certain class of function spaces in which the domain of the function is E. The sense of the results obtained is: If $[\phi_\mu(s)\psi_\nu(t)]$, $(\mu, \nu=0, 1, \cdots)$, is a sequence of functions in $T(E_s \times E_t)$, then a necessary and sufficient condition for closure is that the sequences $[\phi_\mu(s)]$, $[\psi_\nu(t)]$ have the closure property in $T(E_s)$ and $T(E_t)$ respectively. The complex Hilbert space, the L_p space $(p \geqq 1)$ and the C space are included in $T(E)$. (Received November 3, 1939.)

13. D. G. Bourgin and R. J. Duffin: *The Dirichlet problem for the vibrating string.*

The string equation $y_{xx}-y_{tt}=0$ is investigated subject to Dirichlet conditions on the boundary of a rectangle formed by parallels to the coordinate axes. Considerations of existence and uniqueness of solutions pivot on the character of the parameter γ, where $\gamma=\tau/vS$. Here S is the string length, v is the velocity of wave propagation and τ is the time interval. For instance it is sufficient for the existence of a solution of class C^2 that the data, besides satisfying certain corner conditions, be of class C^{4+k} where γ is an algebraic number of degree k. This condition is not necessary but some such restriction must be introduced for sufficiency assertions. The uniqueness proof is based on a novel use of Fourier transforms and the method has wide application. (Received November 3, 1939.)

14. A. L. Foster: *The structure of regularly ordered natural systems with 3 primes.*

The principal result of this paper is the determination of the structure of all regularly ordered natural systems with 3 primes. The method is an extension of that previously given for 2 primes (Proceedings of the National Academy of Sciences, vol. 24, no. 4). A natural system is a system with single composition $(N, \circ)$ which is closed, associative, commutative, denumerably infinite, and with unique prime (irreducible) factorization. $(N, \circ)$ is regularly ordered if its elements are arranged in a sequence such that $\sigma<\tau\rightarrow\rho_0\sigma<\rho_0\tau$ ($<$ means precedes). (Received November 4, 1939.)

15. O. H. Hamilton: *A fixed point theorem for a cell in a Hilbert space.*

It is shown in this paper that if M is a closed cell in a Hilbert space, then every continuous transformation of M into a subset of itself leaves some point of M invariant. The method employed in this proof can also be used to give a new and simple proof of the well known fixed point theorems for cells in n-dimensional space. (Received November 2, 1939.)

16. A. E. Heins: *The solution of the discrete wave equation.*

Solutions of the discrete wave equation $f(x+1, t)+f(x-1, t)-f(x, t+1)-f(x, t-1)=0$ are considered here for values of $t>0$ (x, t not integers). Symmetric and nonsymmetric, finite and infinite boundary conditions are treated. One need only prescribe one initial condition here, that is, $f(x, t)$ when $[t]=0$, since the derived solution with the aid of the given difference satisfies the second initial condition automatically, namely $f(x, t)$, when $[t]=1$. (Received November 9, 1939.)

17. Einar Hille: *Contributions to the theory of Hermitian series. II. The representation problem.*

A necessary and sufficient condition in order that the Fourier-Hermite series of an analytic function $f(z)$ shall exist and converge to the function in the strip $-\tau<y<\tau$ is that $f(z)$ be holomorphic in the strip and that to every β, $(0<\beta<\tau)$, there shall exist a finite $M(\beta)$ such that $|f(x+iy)|\leqq M(\beta)\exp\left[-(\beta^2-y^2)^{1/2}|x|\right]$ for $-\beta\leqq y\leqq\beta$. In particular, the Hermitian series of an entire function of exponential type can never converge outside of the real axis. (Received October 16, 1939.)

18. P. G. Hoel: *The errors involved in evaluating correlation determinants.*

In evaluating a correlation determinant by the method of Chio, successive reductions introduce errors of rounding which become increasingly important as the order of the determinant increases or the value of the determinant decreases. Through the use of operational methods an upper bound is obtained for the magnitude of these errors. By considering the first order terms in more detail, probability bounds are also obtained for these errors. (Received November 1, 1939.)

19. Mark Kac: *On a problem concerning probability and its connection with the theory of diffusion.*

The paper gives a probabilistic approach to the problem of diffusion. It is shown that the classical solution of the differential equation of diffusion is an asymptotic formula for the statistical problem under consideration. (Received October 23, 1939.)

20. E. P. Lane: *A theorem on surfaces.*

The purpose of this note is to prove the following theorem: If the asymptotic curves on an analytic non-ruled surface S in ordinary space belong to linear complexes, then the asymptotic curves of each family on S are projectively equivalent, not only to each other, but also to all the non-rectilinear asymptotic curves on all the asymptotic ruled surfaces of S which are composed of the tangents of the asymptotic curves of the other family on S, constructed at the points of the various asymptotic curves of the first family on S. (Received October 7, 1939.)

21. J. P. LaSalle: *Linear functions and functionals in linear topological spaces.* Preliminary report.

In this paper a number of Banach's theorems on linear operations are extended (a) to convex linear topological spaces by use of the pseudo metric of von Neumann, and (b) by use of the pseudo norm of Hyers to linear topological spaces which satisfy an additional postulate (the postulate may be weaker than that of convexity). Some theorems other than extensions of Banach's also appear. (Received November 4, 1939.)

22. Rhoda Manning: *On the derivatives of the sections of bounded power series.*

Let $f(z)$ be a power series with sections $s_n(z)$, such that $|f(z)| \leqq 1$ in $|z| < 1$, and $f(0) = 0$. I. Schur and G. Szegö have shown (Sitzungsberichte der preussischen Akademie, 1925, p. 560) that for odd n the radius r_n of the largest circle $|z| \leqq r_n$ in which $|s'_{n+1}(z)| \leqq 1$ is a root of the equation $1-2r-r^2+(-1)^n\{(2n+4)r^{n+1}+(2n+2)r^{n+2}\} = 0$. The author proves that this result holds for all n provided $n \geqq 11$. As a consequence, $r_n \downarrow \rho = 2^{1/2}-1$, n even, $n \geqq 12$. It is also shown that the r_n, $(n=2, 4, \cdots, 10)$, are algebraic numbers, connected by the following order relations (for simplicity n is written for r_n): $1<2<3<4<5<6<7<9<8<11<13< \cdots <\rho< \cdots <14<12<10$. Hence for even n, $n \geqq 10$, the sections $s'_{n+1}(z)$ remain bounded "longer" than $f'(z)$ itself. For large n, asymptotic expressions for r_n are given. (Received October 30, 1939.)

23. W. T. Martin: *Analytic functions and multiple Fourier integrals.*

The paper considers the class of entire functions $f(z_1, \cdots, z_n)$ satisfying relations of the form $\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} |f(x_1+iy_1, \cdots, x_n+iy_n)|^2 dx_1 \cdots dx_n < Ae^{a(|y_1|+ \cdots +|y_n|)}$ for all finite values of $y_1, \cdots, y_n$. It is proved that this class is identical with the class of functions having Fourier transforms which vanish outside a certain bounded region and that growth-function of f, namely $h(\lambda) = \lim_{\rho\to\infty}(1/2\rho) \log \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} |f(x_1+i\lambda_1\rho, \cdots, x_n+i\lambda_n\rho)|^2 dx_1 \cdots dx_n$, is equal to the supporting function $k(\lambda)$ of the convex body K which is the intersection of all convex bodies in whose exterior the Fourier transform of f vanishes. These results are related to those of Plancherel and Pólya (Commentarii Mathematici Helvitici, vol. 9 (1936–1937), pp. 224–248; vol. 10 (1937–1938), pp. 110–163). Analogous results are also obtained for functions in an octant $I_m[z_k] > 0$, $(k=1, \cdots, n)$. (Received November 2, 1939.)

24. A. B. Mewborn: *Abstract local geometry of paths.* Preliminary report.

The local geometry of a space of paths in a topological space with postulated Banach coordinates is considered from the standpoint of the representations of these paths in distinguished coordinate systems analogous to normal coordinates. It is shown that allowable transformations of coordinates can be defined in a space with this structure, and yield a linear connection form. It is proposed to study the affine, projective and descriptive properties of this space by means of this connection. (Received November 4, 1939.)

25. A. D. Michal: *Differentials in abstract abelian topological groups.*

Let G_1 and G_2 be abstract abelian groups (written additively) with continuous operations in a Hausdorff topology. A differential is defined for functions on a Hausdorff neighborhood in G_1 to G_2. The fundamental theorems on group sum and difference, and composition of functions are proved. A differential is additive and continuous in the increment and is a "first order approximation" in a prescribed manner to the group increment of the function. A differential is proved unique for all values of the increment in an existing abelian topological subgroup of G_1. The uniqueness of a differential for all values of the increment in G_1 is proved under an additional postulate on the topological groups. This additional postulate is automatically satisfied whenever G_1 and G_2 are linear topological spaces. (Received November 4, 1939.)

26. F. J. Murray: *Nullifying functions.*

A measurable function, defined on the unit interval, is termed nullifying, if it takes a set of measure one into a set of measure zero. In this note a function $f(x)$ is given with the property that $f(x)+\rho x$ is nullifying for every value of the parameter ρ. (Received November 4, 1939.)

27. W. H. Myers: *Finite linear groups with abelian subgroups of variety* 3.

H. F. Blichfeldt (*Finite Collineation Groups*, University of Chicago Press, 1917, p. 96) proves that no finite group of linear transformations which is primitive can contain a transformation (other than a similarity transformation) which in canonical form has its multipliers when located on the unit circle lying not more than 60° on either side of some one of them; and (page 101) no primitive linear group can contain an abelian subgroup of variety 3 and order $k\phi$ where $k \geqq 30$. In this paper, by extending the first-mentioned theorem somewhat, it is proved that if G is a finite group of linear transformations in n variables which contains an abelian subgroup K of variety 3 and order $k\phi$, where $k>16$ and $k\neq 20$ or 25, then G cannot be a primitive group. (Received November 2, 1939.)

28. G. B. Price: *Convex operators.*

The process of forming the convex extension of a given set may be thought of as a transformation which carries the set into a second set, its convex extension. This transformation is accomplished by means of a certain operator, whose construction and properties are known, and which may be called the ordinary convex operator. This paper describes, in the first place, the construction of other operators with properties similar to those of the convex operator. The ordinary convex operator can be applied to sets in a vector space. This paper introduces not only new convex operators for vector spaces, but also convex operators for a wide variety of more general spaces. In the second place, this study investigates the properties of the class of convex operators. Certain operations can be performed on members of the class, and there are numerous interesting subclasses. Finally, the paper examines the properties of sets which are obtained by transforming a given set by one of the convex operators. A later paper will develop the applications of these convex operators in the theory of integration. (Received October 31, 1939.)

29. E. J. Purcell: *A multiple null-correspondence and a space Cremona involution of order $2n-1$.*

Consider a curve C of order n having $(n-1)$ points on a straight line d, and a curve C' of order m having $(m-1)$ points on a second line d' (m, n any integers). A general point P determines a ray r intersecting C once and d once. P also determines a ray r' intersecting C' once and d' once. Rays r and r' determine a plane which is the null-plane of P in a $(1, mn, m+n)$ correspondence. This paper discusses this correspondence and also a Cremona involution of order $2n-1$ which arises when the curves are in special position. This paper will appear soon in this Bulletin. (Received October 31, 1939.)

30. Tibor Radó and P. V. Reichelderfer: *On cyclic transitivity.*

This paper is a continuation and extension of a previous paper by T. Radó (see abstract 45-5-223). Let $\mathfrak{R}$ be a binary relation defined in a set 1 of elements; write $a_1\mathfrak{R}a_2$ to express the fact that the elements a_1 and a_2 of 1 are in the $\mathfrak{R}$-relation. The following concept is fundamental. The binary relation $\mathfrak{R}$ is said to be *cyclicly transitive* if, for every finite cyclicly ordered set of distinct elements $a_1, a_2, \cdots, a_n$ of 1 satisfying $a_1\mathfrak{R}a_2\mathfrak{R}\cdots\mathfrak{R}a_n\mathfrak{R}a_1$, it is true that $a_i\mathfrak{R}a_j$ for every choice of the subscripts i and j. In the first part of the paper there is developed an abstract theory which parallels closely the cyclic element theory for Peano spaces originated by G. T. Whyburn. In the second part of the paper this abstract theory is applied to the special case of Peano spaces. (Received October 9, 1939.)

31. W. C. Randels: *On the absolute summability of Fourier series. III.*

It is proved that, if a function $f(x)$ is such that, in the neighborhood of every point x, it is identical with a function whose Fourier series is absolutely summable $|(C, 1)|$, then the Fourier series of $f(x)$ is absolutely summable $|(C, 1)|$. This is analogous to a theorem of Wiener for absolute convergence. (Received October 20, 1939.)

32. G. E. Raynor: *On Serret's integral formula.*

The paper is concerned chiefly with a method leading to the evaluation of various definite integrals arising in gamma function theory. The principal tool employed is a formula due to Alfred Serret which appears in his classic paper, *Note sur quelques formules de calcul intégral*, Journal de Mathématiques Pures et Appliquées, vol. 8 (1843), p. 7. (Received October 11, 1939.)

33. Fred Robertson: *The general differential operator.*

The object of this paper is the unification of theories of general integration and differentiation by the use of a second order partial differential equation. The discussion includes the derivation of the differential equation, the relation of its solutions to generalized integration and differentiation and the development of the concept of the logarithmic operator. The theory is then applied to the solution of an integral equation with a logarithmic kernel. (Received November 2, 1939.)

34. D. T. Sigley: *Sets of conjugate subgroups in direct product groups.*

The distribution of the subgroups of a group G which is the direct product of finite abstract groups is studied, with applications to the problem of the enumeration of the subgroups of G. The particular cases which are considered are those in which the constituents of the direct products are abelian, hamiltonian, or non-abelian groups whose orders are the product of two distinct prime numbers. The problem of the construction of an abstract group in which the number of sets of conjugate subgroups is equal to the number of units in the order of the group is initiated. (Received November 3, 1939.)

35. E. R. Smith: *Generating functions for certain confluent hypergeometric functions.*

Generating functions corresponding to certain rational values of the subscripts of the Whittaker functions $W_{k,m}$, where $k-m-1/2$ is not a negative integer, are found by elementary methods. The function is first expressed as a curvilinear integral taken over a closed path enclosing the positive real axis. The integral is evaluated for a number of important sets of functions. It is also shown that the generating functions for integral values of k and m satisfy a partial differential equation of the second order. (Received November 9, 1939.)

36. R. H. Tripp: *Stresses in an orthotropic elastic layer.*

In an elastic isotropic layer of finite depth h supported on a rigid base, the stresses and displacements may be determined from a function $\phi(x, y)$ satisfying $\nabla^4\phi(x, y)=0$ interior to the region and known conditions on the boundary. In an orthotropic finite layer, where the elastic constants are known in orthogonal directions, the stress function satisfies $\nabla^4\phi(x, ky)=0$ where k is a constant. The stresses and displacements throughout the layer are investigated by examining the properties of integrals of the form $\int_0^\infty f(\alpha, y)\cos \alpha x/h\, d\alpha$. (Received November 2, 1939.)

37. E. P. Vance: *On locally non-separating and locally non-alternating transformations.*

The transformation $T(A)=B$ is said to be locally non-separating at a point p provided for any point p and neighborhood U_p of p there exists a neighborhood V_p lying in U_p such that if u and v are any two points of $V_p-T^{-1}(T(p))$, then $u+v$ lies in a connected subset of $U_p-T^{-1}(T(p))$. T will be called locally non-separating on A if it is locally non-separating at all points of A. A similar definition is made for a locally non-alternating transformation. Necessary and sufficient conditions are proved for both transformations. It follows that if T is locally non-separating and for no point x of B does $T^{-1}(x)$ contain an open set, then T is completely non-separating (and therefore non-separating). Other necessary conditions are given for both transformations and the transformations are applied to special spaces. The main result is that if T is locally non-separating on A, then B must be Peanian, regardless of A. (Received October 13, 1939.)

38. Morgan Ward: *Residuated distributive lattices.*

A residuated lattice $\mathfrak{S}$ with ascending chain condition is called a P-lattice if $a \supset b$ in $\mathfrak{S}$ only if there exists an element c such that $ac=b$. $\mathfrak{S}$ is called an S-lattice if every irreducible is primary and if for any two elements a, b the residuals $a:b$ and $b:a$ are coprime. Both S-lattices and P-lattices are known to be distributive, but in a distributive residuated lattice with chain condition, the properties of being a P-lattice or S-lattice are completely independent of one another. In this paper, the decompositions of elements into products and cross-cuts of irreducibles and primaries are determined, and shown to be unique. Both types of lattice may be characterized by simple properties of their primes and primaries. (Received November 6, 1939.)

39. W. F. Whitmore: *Convergence theorems for functions of two complex variables.*

Lindelöf has stated a theorem to the following effect: Let $f(z)$ be an analytic function of one complex variable, defined and bounded in the upper half plane. If $f(z)$ converges to a definite limit for approach to infinity along the negative real axis, then it also converges to this limit inside the angular domain $0<\eta<\arg z<\pi$. A similar result is here established for functions of two complex variables, with the aid of the notions of distinguished surface and functions of extended class introduced by Bergmann (Mathematische Annalen, vol. 109 (1934), pp. 324–348, and Composito Mathematica, vol. 6 (1939), pp. 305–335). A theorem is first obtained for the case of the bicylinder $|z_1|<r$, $|z_2|<1$. Using this as a domain of comparison, the result is extended to a more general class of domains of the form $E[z_1=th(z_2, \lambda), |z_2|<1, 0\leqq t<1, 0\leqq\lambda\leqq 2\pi, h(z_2, 0)=h(z_2, 2\pi)]$, where the function $h(z_2, \lambda)$ satisfies certain conditions of regularity. (Received October 20, 1939.)

40. Max Wyman: *Curl differential equations in abstract spaces.*

Let E be a Banach space, and let $F(x, \xi_1, \xi_2)$, $G(\xi_1, \xi_2)$ be bilinear functions of ξ_1, ξ_2 on E^3 to E, and E^2 to E respectively. Differential equations of the form (1) $\phi(x, \xi_1; \xi_2)-\phi(x, \xi_2; \xi_1)=F(x, \xi_1, \xi_2)$, (2) $\phi(x, \xi_1; \xi_2)-\phi(x, \xi_2; \xi_1)=G(\phi(x, \xi_1), \phi(x, \xi_2))$ arise in abstract transformation group theory, and it is of interest to know under what conditions these equations possess solutions. It is possible to prove the following three theorems. Theorem 1: A necessary and sufficient condition for (1) to have a solution $\phi(x, \xi)$ linear in ξ, and such that $\phi(x, \xi_1; \xi_2; \xi_3)$ exists continuous in x for $\|x\|<K$, is that (a) $F(x, \xi_1, \xi_2)=-F(x, \xi_2, \xi_1)$ for $\|x\|<K$, (b) $F(x, \xi_1, \xi_2; \xi_3)$ exists continuous in x for $\|x\|<K$, (c) $F(x, \xi_1, \xi_2; \xi_3)+F(x, \xi_2, \xi_3; \xi_1)+F(x, \xi_3, \xi_1; \xi_2)=0$ for $\|x\|<K$. Theorem 2: If $G(\xi_1, \xi_2)=-G(\xi_2, \xi_1)$, then equation (2) always possesses a solution $\phi(x, \xi)$, which possesses a continuous second Fréchet differential for all x in E. Theorem 3: If $G(G(\mu, \nu), \lambda)+G(G(\nu, \lambda), \mu)+G(G(\lambda, \mu), \nu)=0$, and if it is required that $\phi(x, x)=g(x; x)$ where $g(x)$ is a given function, then there exists a unique solution of (2) which will meet this condition. (Received November 4, 1939.)

41. Max Wyman: *Projective non-holonomic tensor analysis.*

In considering a projective tensor analysis theory for a Hausdorff space H, with coordinates in a Banach space E, Professor A. D. Michal introduces the normed ring R of linear transformations on E to E (see A. D. Michal, *General projective differential geometry*, Proceedings of the National Academy of Sciences, vol. 23 (1937)). He postulates an inner product in E, and a contraction in R. In exactly the same way one can obtain a projective non-holonomic tensor analysis theory, if in addition to H and E another Banach space E_1 is considered. The defining equations of parallel displacement along the curve $x=x(t)$ are taken to be $dV/dt+K(x, V, dx/dt)=\alpha(t)V$, where $K(x, V, \xi)$ is a non-holonomic linear connection, and $\alpha(t)$ is a numerical scalar field. From this it can easily be shown that the most general change of linear connection which preserves parallelism is of the form $K'(x, V, \xi)=K(x, V, \xi)+\phi(x, \xi)V$ where

$\phi(x, \xi)$ is linear in ξ and scalar valued. It is possible to define what is meant by a projective non-holonomic flat space, and to give necessary and sufficient conditions for a space to be projectively flat. (Received November 4, 1939.)

42. Max Zorn: *Topological studies in the theory of analytic functions.* II. *Potential functions.* Preliminary report.

This paper points out simple topological properties of potential functions in the plane. In particular, it furnishes a characterization of conjugate pairs. (Received November 6, 1939.)

43. R. P. Agnew: *On Tauberian theorems for double series.*

Let s_{mn} and σ_{mn} denote respectively the sequence of partial sums and the C_1 transform of a double series $\sum u_{mn}$ in which m and n take values 1, 2, 3, $\cdots$. The question whether $\sigma_{mn} \rightarrow s$ and the Tauberian condition $mnu_{mn} < K$ imply $s_{mn} \rightarrow s$ was raised and left unanswered by Knopp in his recent paper *Limitierungs-Umkehrsätze für Doppelfolgen*, Mathematische Zeitschrift, vol. 45 (1939), pp. 573–589; p. 581. It is shown that neither the condition $mnu_{mn} < K$ nor any one of several stronger conditions will serve. The situation is the same for many other methods of summability, including the Cesàro methods of all positive orders and the Abel power series method. Questions involving iterated limits of s_{mn} and σ_{mn} are also answered. (Received December 1, 1939.)

44. A. A. Albert: *On p-adic fields and cyclic algebras.*

Cyclic fields over a p-adic field are considered and it is shown that there are only a finite number of such fields of a given degree. The number is explicitly determined in the case where the ramification order is prime to p. It is also shown that in this case every abelian non-cyclic field is the direct product of two cyclic fields, one of which is completely ramified. The results are used to prove that there are only a finite number of algebraic extensions of a given degree over a p-adic field. The paper concludes with a discussion of the question of the existence of a cyclic field of degree n over the rational field such that its composite with the algebraic number centrum K of the division algebra D of degree n over K splits D. (Received November 13, 1939.)

45. H. W. Alexander: *On the edge of regression of a general surface.*

In this paper the analogue for a general surface of the edge of regression of a developable surface is considered, and its properties, both local and in the large, are developed. The edge of regression is a curve E along which the mean curvature is infinite. Any plane normal to E through a point P of E cuts the surface in a curve having an algebraic cusp at P. The curve E is also the envelope of both sets of asymptotic lines. An index j_R is defined for a region R of an orientable surface, and is used to define the index of an umbilic. Finally, the Euler characteristic of a region entirely bounded by the edge of regression is expressed in terms of the indices of the umbilics in the interior. (Received November 24, 1939.)

46. Stefan Bergman: *On the approximation of functions satisfying a linear partial differential equation.*

In this paper a method is developed for the calculation of a sequence of approximations V_n of a function U given by its values $\tilde{U}$ on the boundary $\mathfrak{C}$ of a convex domain $\mathfrak{B}$ and satisfying in $\mathfrak{B}$ the equation $L(U) \equiv U_{xy} + a_1 U_x + a_2 U_y + a_3 U = 0$,

$z=x+iy$, $\bar{z}=x-iy$, $U_z=\partial U/\partial x-i\partial U/\partial y$, $U_{\bar{z}}=\partial U/\partial x+i\partial U/\partial y$, a_k being an analytic function of z, $\bar{z}$ for $k=1, 2, 3$. A method is presented for the calculation of a sequence of particular solutions W_k, $(k=1, 2, \cdots)$, of $L(W_k)=0$, which are complete for certain classes of functions $F(z, \bar{z}): L(F)=0$, $z \in \mathfrak{G}$, $\mathfrak{G}$ being an arbitrary convex domain whose boundary curve satisfies certain conditions. In the case of the first boundary problem $V_n=\sum_{k=1}^{n} A_k^{(n)} W_k$ is calculated with the assumption that $A_k^{(n)}$ are defined by the condition $\int_{\mathfrak{C}}(\tilde{U}-V_n)^2 ds=\min$. Analogous processes can be employed for the solution of certain characteristic value problems and in the theory of equations of hyperbolic type. (Received November 17, 1939.)

47. R. P. Boas: *A completeness theorem.*

Let $G(x) \in L^2(-\pi, \pi)$. It is shown that the set of functions $\{e^{2nix}, G(x)e^{(2n+1)ix}\}$ is complete in $L^2(-\pi, \pi)$ if and only if $G(x+\pi)+G(x) \neq 0$, $(-\pi<x<0)$, except perhaps on a set of measure zero. By the use of a well known theorem of Paley and Wiener (*Fourier Transforms in the Complex Domain*, American Mathematical Society Colloquium Publications, vol. 19, 1934, p. 13), this result can be transformed into a uniqueness theorem for entire functions of exponential type π which belong to L^2 on the real axis. An interesting special case is that if $f(z)$ is such a function, q is a positive integer, and $f(2n)=f^{(q)}(2n)=0$, $(n=0, \pm 1, \pm 2, \cdots)$, then $f(z)\equiv 0$. (Received November 13, 1939.)

48. R. P. Boas: *Some uniqueness theorems for entire functions.*

Let $\phi(z)$ be regular in the rectangle $|x|<L$, $|y|<\pi$, so that $\phi(z)=\sum_{\nu=0}^{\infty} a_\nu z^\nu$, $|z|<\min(L, \pi)$. If $f(z)$ is an entire function of exponential type, the series $\sum_{\nu=0}^{\infty} a_\nu f^{(\nu)}(z)$ is either convergent or summable by the method of Mittag-Leffler, and may be denoted by $\phi(D)f(z)$, where D stands for d/dz. Consider the class of functions $f(z)$, of exponential type, satisfying $f(iy)=O(e^{k|y|})$, $|y|\to\infty$, $k<\pi$; and $f(x)=O(e^{l|x|})$, $|x|\to\infty$, $l<L$. A necessary and sufficient condition that any $f(z)$ of this class should vanish identically if $f(2n)=\phi(D)f(2n)=0$, $(n=0, \pm 1, \pm 2, \cdots)$, is that $\phi(z+i\pi/2)-\phi(z-i\pi/2)\neq 0$, $|x|<L$, $|y|<\pi$. An interesting special case is that if $f(z)$ belongs to the class, with $l<(\pi/2)\cot \pi(1-1/r)/2$, then $f(z)\equiv 0$ if r is an odd integer and $f(2n)=f^{(r)}(2n)=0$, $(n=0, \pm 1, \pm 2, \cdots)$ (compare the preceding abstract). (Received November 13, 1939.)

49. D. G. Bourgin: *The Dirichlet problem for the damped wave equation.*

This paper treats the Dirichlet data problem for $y_{xx}-y_{tt}+Ky^2=0$. The fundamental contour is a rectangle or parallelogram with sides equally inclined to the characteristic directions. The existence and uniqueness properties of solutions are closely interrelated with results in elementary number theory. The theory is richer than that of the undamped equation because of the presence of two basic parameters, $\alpha=T/m$ and $\beta=(Km)^2$, where $T\pi$, $m\pi$ and K are the two side lengths and the damping coefficient respectively. In particular, there are now various degrees of non-uniqueness of solutions. Generalizations are indicated to other contours and equations. (Received November 21, 1939.)

50. A. T. Brauer: *On the density of the sum of sets of positive integers. II.*

This paper is the continuation of the author's paper in the Annals of Mathematics, (2), vol. 39 (1938), pp. 322–340. Let $\alpha_1 \leqq \alpha_2 \leqq \cdots \leqq \alpha_n$ be the densities of the sets $A_1, A_2, \cdots, A_n$ of positive integers, and $\gamma < 1$ the density of the Schnirelmann sum $S = A_1 + A_2 + \cdots + A_n$. If the famous conjecture that $\gamma \geqq \alpha_1 + \alpha_2 + \cdots + \alpha_n$ were correct, then one would obtain in particular that $\alpha_1 + \alpha_2 + \cdots + \alpha_n < m/n$ for all n and m with $n \geqq m$. For $m = 1$, this is the theorem of Khintchine and for $n = m = 2$ the theorem of Schnirelmann. In this paper it is proved that the inequality is correct in all events, if n and m satisfy the condition $1/(n-1) + 1/(n-2) + \cdots + 1/(n-m+1) \leqq m/n$. Moreover the inequalities of I. Schur (Sitzungsberichte der preussischen Akademie der Wissenschaften, 1936, pp. 269–297) for $\alpha_1 + \alpha_2 + \cdots + \alpha_n$ and $\gamma/(\alpha_1 + \alpha_2 + \cdots + \alpha_m)$ are essentially improved. (Received November 20, 1939.)

51. Richard Brauer and H. S. M. Coxeter: *A generalization of theorems of Schönhardt and Mehmke on polytopes.*

Let G be an irreducible finite group of orthogonal transformations in an n-dimensional space, and let $V_1, V_2, \cdots, V_k$ be the distinct h-dimensional linear subspaces which can be obtained from one such subspace (for instance V_1) by the transformations of G. Then the arithmetic mean of the orthogonal projections of any vector x on $V_1, V_2, \cdots, V_k$ is equal to $(h/n)x$. I. Schur's formulas for the coefficients of such a group are used to prove this, as well as a more general theorem in which the projections are not restricted to be orthogonal. (Received November 18, 1939.)

52. J. W. Calkin: *Semi-bounded forms and self-adjoint boundary value problems.*

This paper develops further the author's theory of abstract boundary conditions (Transactions of this Society, vol. 45 (1939), pp. 369–442), with special reference to boundary value problems with which a semi-bounded bilinear symmetric form in Hilbert space is associated. (Received November 16, 1939.)

53. Leonard Carlitz: *A set of polynomials.*

For $m = a_0 + a_1 p^n + \cdots + a_k p^{nk}$, $(0 \leqq a_i < p^n)$, we define $G_m(t) = \psi_0^{a_0}(t)\psi_1^{a_1}(t) \cdots \psi_k^{a_k}(t)$, where $\psi_k(t)$ is the "linear" polynomial discussed by the author (Duke Mathematical Journal, vol. 1 (1935), p. 137). In this paper we use $G_m(t)$ to derive certain results concerning the arithmetic of polynomials with coefficients in the $GF(p^n)$. (Received November 21, 1939.)

54. Leonard Carlitz: *On certain sums involving polynomials in a Galois field.*

This note is concerned with certain sums extended over the set of polynomials of fixed degree. Some of the results were proved earlier, but all are derived here in a more direct manner. In particular the sums $\sum M^{p^{nk}-1}$ and $\sum M^{1-p^{nk}}$ are evaluated. (Received November 21, 1939.)

55. R. F. Clippinger: *On matrix products of positive powers of given matrices.* Preliminary report.

In the special case considered two second order matrices are given. The problem is equivalent to that of finding the region filled by solutions of $dY/dt = Y|\rho \log A + \mu \log B|$ through the unit matrix of the second order, where ρ and μ are arbitrary positive functions of t. It is shown that if the determinants of A and B are one, this region is that filled by $A^{\alpha}B^{\beta}A^{\gamma}$. This theorem serves as a lemma to show that the region in the special case considered is that filled by $A^{\alpha}B^{\beta}A^{\gamma}$. (Received November 21, 1939.)

56. L. W. Cohen: *On topological completeness.*

The complete space S^* constructed for the space S in the author's paper, *On imbedding a space in a complete space*, Duke Mathematical Journal, vol. 5 (1939), is shown to be complete in the sense of A. Weil. It is also shown that Weil's uniform space is a special case of the space S of that paper. (Received January 5, 1940.)

57. A. H. Copeland: *Transformations on probability sequences.*

In 1919 von Mises developed a theory of probability based on two axioms and four types of transformations. The first axiom concerned the definition of probabilities as limits of success ratios. Roughly the second stated that no selection made in advance could increase one's likelihood of winning a game of chance. This defined the range of applicability of one transformation—namely, that of selection. von Mises' system possesses a strong intuitional appeal but logical objections have been raised against the second axiom. To avoid these objections a number of authors have devised alternative systems in many of which there has been a tendency to include only transformations by selection. This tendency is unfortunate since all four transformations are necessary in solving probability problems. Moreover, Greville has recently indicated certain classical problems which cannot be handled by von Mises' system. Although Greville's transformations are capable of handling these problems, a restricted label space is employed and the selections must be separately treated. It is the purpose of this paper to generalize Greville's theory so that the label space is an abstract space and the selections as well as the other three transformations are special cases of the transformations used. (Received November 21, 1939.)

58. J. J. De Cicco: *The horn angle in the geometry of element-series.*

This is another extension of the horn angle problem which was developed by Kasner (*Conformal geometry*, Proceedings of the Fifth International Congress of Mathematicians, vol. 2 (1912), p. 81; and *Equilong invariants and convergence proofs*, this Bulletin, vol. 23 (1917), pp. 341–347). The deviation between two series S_1 and S_2 contained in a field F is the distance between the centers of their tangent turbines constructed at a common element E. If this is zero, the two series are said to form a horn angle (of first order). A transformation between two fields is said to be equideviate if it preserves the deviation between any two series. Under the group of equideviate transformations, a horn angle possesses the unique invariant $M_{12}=(\Gamma_2-\Gamma_1)^2/(d\Gamma_2/dU_2-d\Gamma_1/dU_1)$, where $d\Gamma_i/dU_i$ is the variation of the geodesic curvature Γ_i with respect to the angle U_i between any two lineal elements of the series S_i at the common element E of the horn angle. Two horn angles of two different fields may be mapped into one another by an equideviate transformation if and only if they have the same measure M_{12}. (Received November 13, 1939.)

59. J. J. De Cicco: *The magnilong near-Laguerre transformations.*

In a previous paper, *The analogue of the Moebius group of circular transformations in the Kasner plane*, this Bulletin, vol. 45 (1939), pp. 936–943, the author discussed the point transformations with respect to the maximum number of vertical parabolas preserved. This discussion is isomorphic to that of the line transformations with respect to the maximum number of circles preserved. A nonmagnilong transformation carries at most 2 ∞^2 circles into circles. On the other hand, a magnilong transformation (not a Laguerre transformation) carries at most ∞^1 circles into circles. In this paper, the author obtains the set of all magnilong near-Laguerre transformations (those, which convert exactly ∞^1 circles into circles). These are of the form L_2TL_1, where L_1 and L_2 are Laguerre transformations and T is any one of the three transformations e^z, $\log z$, z^n, $(z=x+jy, j^2=0)$.The family preserved is a linear pencil of circles in the sense of higher circle geometry. (Received November 13, 1939.)

60. A. H. Diamond: *Postulates for the theory of strict implication.*

In 1932 Lewis gave a set of postulates for the theory of strict implication. Among the primitive symbols listed by Lewis is the symbol $=$. Lewis, however, gives an alleged definition of this symbol as follows. Definition: $p=q. = .p \prec q \cdot q \prec p$. Huntington in his American Academy paper of 1937 gives a set of postulates for Lewis's theory of strict implication in which he takes as primitive symbols all those of Lewis's list except $=$, and defines $=$ legitimately in terms of the others. In giving these postulates, however, Huntington departs from Lewis's method of stating the postulates as primitive sentences of a language S_1 and states them instead in the so-called word-language. It is the purpose of this paper to give a set of postulates for strict implication following Lewis's method of stating the postulates as the primitive sentences of a language S_1 and using as primitive symbols all those listed by Lewis including the symbol $=$. A postulate system is thus obtained which is free of Lewis's circularity in the use of the symbol $=$ and, moreover, a system which affords greater freedom of interpretation than does Huntington's system. (Received November 21, 1939.)

61. R. P. Dilworth: *Ideals in Birkhoff lattices.*

Let $\mathfrak{S}$ be an arbitrary lattice. A sublattice $\mathfrak{a}$ is an ideal if $x \supset a$, $x \in \mathfrak{S}$, $a \in \mathfrak{a}$ implies $x \in \mathfrak{a}$. The ideals of $\mathfrak{S}$ form a lattice $\mathfrak{L}$ containing $\mathfrak{S}$ as a sublattice of principal ideals. $\mathfrak{S}$ is said to be a Birkhoff lattice if $\mathfrak{a}$ covers $\mathfrak{a} \cap \mathfrak{b}$ implies that $\mathfrak{a} \cup \mathfrak{b}$ covers $\mathfrak{b}$ in $\mathfrak{L}$. This clearly reduces to the usual definition if $\mathfrak{S}$ is archimedean. By means of the ideals much of the arithmetical theory of archimedean Birkhoff lattices is extended to general Birkhoff lattices. In particular, the Kurosch-Ore decomposition theorem for modular lattices in its most general form is shown to follow from a simple exchange property of independent bases. (Received November 22, 1939.)

62. J. L. Doob: *On a certain type of family of chance variables.*

Let t vary in any simply ordered set, and let $\{x_t\}$ be a family of chance variables. The chance variables are said to have the property $\mathcal{E}$ if, whenever $t_1 < t_2 < \cdots < t_n < t_{n+1}$, the conditional expectation of $x_{t_{n+1}}$ for given values of $x_{t_1}, \cdots, x_{t_n}$ is equal to x_{t_n}: $E[x_{t_1}, \cdots, x_{t_n}; x_{t_{n+1}}] = x_{t_n}$, with probability 1. Let $\cdots, x_{-1}, x_0, x_1, \cdots$ be a sequence of chance variables with the property $\mathcal{E}$. Then (i) the x_j for $j \leqq n$ are uniformly integrable (any n); (ii) $E|x_j|$ (the expectation of $|x_j|$) is monotone non-decreasing; (iii) $\lim_{j\to-\infty} x_j$ exists and is finite, with probability 1; (iv) if $\lim_{j\to+\infty} E|x_j| < \infty$, $\lim_{j\to+\infty} x_j$ exists and is finite with probability 1. These theorems and other related ones can be applied to a series of independent chance variables $\sum_1^\infty y_j$ with $Ey_1 = Ey_2 = \cdots = 0$, since the partial sums of such a series have the property $\mathcal{E}$. Let $\{x_t\}$, $(-\infty < t < \infty)$, be a family of chance variables with the property $\mathcal{E}$. It is shown that the random function x_t can be considered (in a sense made precise in the paper) as a function of t which is continuous except perhaps on a denumerable set, and which for all t has the property that $\lim_{h\downarrow 0} x(t+h)$ exists if $E|x_t \log |x_t|| < \infty$. (Received November 20, 1939.)

63. R. J. Duffin and J. J. Eachus: *Sets invariant under a polynomial operator.*

An investigation is made of certain properties of curves and domains which are transformed into themselves under a polynomial operator. For example, the boundary and interior of the unit circle are invariant under the polynomial z^n. In general, the closed curves that arise may be quite complicated, non-analytic to say the least. The use of nonlinear q-difference equations is found to facilitate the study of analytic invariant curves. (Received November 21, 1939.)

64. Samuel Eilenberg: *An invariance theorem for subsets of S^n.*

Let A and B be two homeomorphic subsets of the n-sphere S^n. If the number of components of S^n-A is finite, then it is equal to the number of components of S^n-B. The proof uses Vietoris cycles and duality theorems. (Received November 20, 1939.)

65. Samuel Eilenberg: *Fixed points for periodic transformations.*

Let R be a commutative ring with a unit element and p a prime number such that $px \neq 1$ for all $x \in R$. Given a separable, metric, finite dimensional space X in which every cycle with coefficients in R bounds, every periodic transformation of X with the period p has a fixed point. In particular R can be taken to be the ring of integers reduced modulo q where q is a multiple of p. The theorem does not hold if R is the ring of integers reduced modulo q not a multiple of p. The theorem generalizes a fixed point theorem of P. A. Smith (Annals of Mathematics, (2), vol. 35 (1934), pp. 572–578). (Received November 20, 1939.)

66. Samuel Eilenberg: *On the homotopy type of S^n.*

A necessary and sufficient condition is given in order that an infinite polyhedron have the homotopy type of the n-sphere S^n. The condition is expressed in terms of homotopy groups and infinite cohomology groups. As an application the author proves that if S^{r-n-1}, $(n \neq 1)$, is topologically imbedded into S^r then S^r-S^{r-n-1} has the homotopy type of S^n if and only if the fundamental group of S^r-S^{r-n-1} vanishes. A condition is also given for the case $n=1$. (Received November 20, 1939.)

67. Samuel Eilenberg and E. W. Miller: *On 0-dimensional upper-semi-continuous collections.*

A collection Φ of closed subsets of a compact metric space X is said to be upper-semi-continuous if $F \cdot \lim\inf F_i \neq 0$ implies $\lim\inf F_i \subset F$ for every sequence $F, F_1, F_2, \cdots$ of sets of Φ. The class Φ can be considered as a separable metric space. The following theorem is proved: Let X be a unicoherent Peano continuum and let Φ be 0-dimensional. If none of the sets $F \in \Phi$ cuts X between two points x_1 and x_2, then the set $\sum F$, for $F \in \Phi$, does not cut X between x_1 and x_2. (Received November 20, 1939.)

68. Benjamin Epstein: *Growth properties of analytic functions of two complex variables.* II. Preliminary report.

Using methods developed previously (abstract 45-9-311), the author is able to deduce a number of other growth theorems for functions regular in domains $\overline{\mathfrak{M}}^4$: $E[\mathfrak{R}e z_k \geq 0, k=1, 2]$. Roughly stated the results show that a function $f(z_1, z_2)$ regular in $\overline{\mathfrak{M}}^4$ cannot sink too low in value on certain non-analytic (three-dimensional) hypersurfaces without compensating with high growth properties in other hypersurfaces. As an example of the results the following theorem is stated: Given $f(z_1, z_2)$ regular in $\overline{\mathfrak{M}}^4$ satisfying the conditions: (a) $\log |f(\pm iy_1, \pm iy_2)| \leqq -\mathfrak{H}_1|y_1|^{\alpha_1}-\mathfrak{H}_2|y_2|^{\alpha_2}$; (b) $\log |f(r_1e^{i\phi_1}, \pm iy_2)| \leqq -\mathfrak{H}_3r_1^{\beta_1}-\mathfrak{H}_4|y_2|^{\beta_2}$; (c) $\log |f(\pm iy_1, r_2e^{i\phi_2})| \leqq -\mathfrak{H}_5r_2^{\gamma_1} -\mathfrak{H}_6|y_1|^{\gamma_2}$; (d) $\log |f(r_1e^{i\phi_1}, r_2e^{i\phi_2})| \leqq M_1r_1^{\delta_1}+M_2r_2^{\delta_2}$, where $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, \delta_1, \delta_2$ may be determined, with $\mathfrak{H}_i$, $(i=1, \cdots, 6)$, M_1, M_2 as fixed positive constants, and where the inequalities are satisfied on certain hypersurfaces, then the function is identically zero in $\mathfrak{M}^4$. $\overline{\mathfrak{M}}^4$ is swept out by a class of hypersurfaces and the constants $\alpha_i, \beta_i, \gamma_i, \delta_i$, $(i=1, 2)$, are related in a specific fashion to the class used. The results are also extended to considerations of integrals taken over zero-lines. (Received November 20, 1939.)

69. H. L. Garabedian: *A new formula for the Bernoulli numbers.*

A unique method is given for deriving what appears to be a new formula for the Bernoulli numbers: $B_{k+1}=((-1)^{k+1}(k+1)/(2^{k+1}-1))\sum_{i=0}^{k}(\Delta^i a_0/2^{i+1})$, $a_n=(1+n)^k$, $(k=0, 1, 2, \cdots)$. This formula gives all of the Bernoulli numbers, including the zero ones, together with the algebraic signs which are sometimes attributed to them. The derivation involves a theorem established recently by the author (this Bulletin, vol. 45 (1939), pp. 592–596). (Received November 18, 1939.)

70. G. N. Garrison: *Quasi-groups.*

A subset of the elements of a finite quasi-group S (a multiplicative system in which $ax=b$ and $ya=b$ are uniquely solvable for x and y) is called an invariant complex if for every pair of elements a, b in S there exists a corresponding element c in S such that $(Ha)(Hb)=Hc$. It follows that the elements of S fall into disjoint classes of the form Ha and that these form a quotient quasi-group which is a homomorphism of S. The principal result is then that every homomorphism of S is given by the quotient quasi-group of a suitably chosen invariant complex. Invariant complexes with special properties $a \subset Ha$ or $(Ha)(Hb)=H(ab)$ are also discussed. (Received November 18, 1939.)

71. Abe Gelbart: *On the growth of a function of two complex variables given by its power series expansion on certain hypersurfaces.*

In this paper the author deals with the growth of a function $f(z_1, z_2)$ given by its power series expansion $\sum_{mn=0}^{\infty}a_{mn}z_1^m z_2^n$ on a sum-manifold of surfaces which are topologically different from the type previously considered (abstract 45-11-418). With the use of the integral formula as developed by Bergman (Matematicheskii Sbornik, vol.1 (43) (1936)) a lower bound for max $[M(G^2, f)]$ is obtained, where $M(G^2, f)$ is the maximum-modulus of f on G^2, G^2 being a part of the intersections of the hypersurfaces $i_1^3=E[z_2=Ar^s e^{i\lambda_1},\ 0\leqq\lambda_1\leqq 2\pi]$, $i_{k+1}^3=E[z_1=Ar^s e^{i\lambda_{k+1}}+c_k z_2,\ 0\leqq\lambda_{k+1}\leqq 2\pi,\ k=1,\ 2]$, which form the boundary of a domain $\mathfrak{M}^4$; f is analytic in $\overline{\mathfrak{M}}^4$. Further, a surface H^2 is considered that lies in the hypersurface $S_{\lambda_1=0}^{2\pi}I_1^2(\lambda_1, r)$, which is that part of the boundary of $\mathfrak{M}^4$ that belongs to i_1^3. Bergman has associated with such surfaces two numbers α_1, α_2 (Mathematische Annalen, vol. 109 (1936), p. 347). A lower limit for $M(H^2, f)$ is obtained depending only on the coefficients a_{mn} and α_1, α_2. Analogous results can be obtained for more general types of hypersurfaces. (Received November 27, 1939.)

72. P. W. Gilbert: *Two-to-one transformations on linear graphs.* Preliminary report.

Given a linear graph G, denote by $E(G)$ the negative of the Euler characteristic of G, and by T an exactly two-to-one continuous transformation defined over G. A necessary condition for the existence of T is that $E(G)$ be even. This condition is not sufficient, as is shown by an example, for the case $E(G)=2$, where T does not exist. However, for the case where the maximal cyclic elements of G are simple closed curves, the above necessary condition is also sufficient. In the general case, a method is described for reducing G to a subset H, such that T can be defined over G if and only if it can be defined over H. The cases $E(G)=0$ and 2 are completely treated. (Received November 22, 1939.)

73. André Gleyzal: *A general theorem on the structure of linear orders.*

The paper defines a number of operations on simply ordered sets and, more generally, on order types. With a given order type α, there are associated certain order types denoted by α^E, α^I, α^T, and so on. These order types may be characterized by means of closure and minimal properties. It turns out that if A is any order, and α any order type, A is either of type α^T or may be formed by substituting orders of type α^T for elements of an order no proper segment of which has property α. As a particular application of this theorem, it is shown that every order of cardinal $\aleph_\lambda$ contains either the normal order ω_λ, or its reverse, or contains an order every proper segment of which has $\aleph_\lambda$ elements. (Received November 20, 1939.)

74. Harriet M. Griffin: *The abelian quasi-group.*

This paper is concerned with the study of the abelian quasi-group when certain conditions with respect to coset expansions are imposed by stated associative laws. Detailed study is made of the minimal quasi-group of units, and definite laws for the order of elements are developed. Certain problems about structure and quotient quasi-groups for this case are answered. (Received November 21, 1939.)

75. O. G. Harrold (National Research Fellow): *Continua of finite degree and certain product sets.*

In this paper the class of continua of finite degree, which has been studied by G. T. Whyburn (American Journal of Mathematics, vol. 55 (1933), pp. 11–16) and the author (abstract 45-9-321), is identified with each of the classes of continua defined by the following properties: (1) M is a locally connected continuum such that to each other pair of closed, disjoint sets A, B in M there exists a finite collection of disjoint, perfect sets $N^1, N^2, \cdots, N^k$ such that any continuum in M intersecting both A and B contains some N^i; (2) if K, K^i, $(i=1, 2, \cdots)$, are nondegenerate continua in the continuum M with $\lim K^i=K$, there exists an integer n for which $\prod_n^\infty K^i$ is uncountable. (Received November 16, 1939.)

76. O. G. Harrold (National Research Fellow): *Exactly $(k, 1)$ transformations on linear graphs.*

This paper initiates a study of exactly $(k, 1)$ continuous transformations defined on a linear graph A. The principal results are: for $k=2$, $B=f(A)$ is a graph, and f is locally interior on A at all but possibly a finite number of points; for $k>2$, $B=f(A)$ need not be a graph but is a stably regular curve, and f is locally interior on A except possibly for a closed set of dimension zero. For $k=2$ a formula is given which limits decidedly the types of graphs on which an exactly $(2, 1)$ transformation can be defined. It is shown that an exactly $(2, 1)$ mapping on a simple closed curve gives a simple closed curve and that the transformation f is topologically equivalent to either (a) $w=z^2$ on $|z|=1$, or, (b) $w=z^2$ on $|z|=1$ for $I(z)\geqq 0$ and $\overline{w}=z^2$ on $|z|=1$ for $I(z)\leqq 0$. Examples are given which show there is no such simple behavior for $k>2$. (Received November 16, 1939.)

77. O. G. Harrold (National Research Fellow): *The dimensionality of continuous transforms of an n-cell.*

A classical result in dimension theory due to W. Hurewicz states that the dimensionality of a compact metric space is raised at most $k-1$ units by a continuous $(k, 1)$ transformation. Since it is possible to map a 1-cell continuously onto a 2-cell by a $(3, 1)$ mapping, the question arises as to whether or not this can be accomplished by a continuous $(2, 1)$ transformation. In answer to this we show that the dimensionality of an n-cell is raised at most $k-2$ units by a continuous $(k, 1)$ mapping, $k>1$. This result is the best obtainable for an n-cell. (Received November 16, 1939.)

78. M. H. Heins: *On the conformal mapping of a multiply-connected region into itself.*

The work of Ritt and Aumann-Carathéodory on the conformal mapping of a multiply-connected region into itself suggests the following problem: what is the behavior of the iterates of a function $w(z)$ analytic and single-valued in a multiply-connected region G_z with at least three boundary points such that $w(G_z)\subset G_z$? Excluding the case where there exists a point $z_0\subset G_z$ such that $z_0=w(z_0)$, and positing that G_z be a region of finite connectivity $p>1$ bounded by p Jordan curves, one may show that either the iterates $w_n(z)$ of $w(z)$ converge continuously to a single point of the boundary of G_z, or else $w(z)$ is necessarily a one-to-one conformal map of G_z onto itself. The proof is based on Julia's lemma and on the behavior in the neighborhood of $|x|=1$ of a function which maps $|x|<1$ onto G_z^∞, the universal covering surface of G_z. (Received November 21, 1939.)

79. M. S. Hendrickson: *On certain properties of arbitrary functions.*

The present paper is related to the article of Henry Blumberg (Acta Mathematica, vol. 65, pp. 263–282), whose principal theorem characterizes, in a certain significant sense, the internal structure of the general real function in terms of associated measurable functions of restricted character. The present paper presents various companion theorems to this theorem of Blumberg, which are based on generalized types of continuity, related to, but somewhat different from, those treated in the literature. The theory of this generalized continuity is developed in certain directions. (Received November 20, 1939.)

80. A. D. Hestenes: *On the solutions of a certain functional equation* $\phi_c(x)-c\log\phi_c(x)=x-c\log x+1$. Preliminary report.

In a paper shortly to appear in the Annales de l'Institute Henri Poincaré, G. D. Birkhoff has discussed a class of linear functional equations (termed by him ϕ-difference equations) which seem to be of comparable importance with linear and q-difference equations. Roughly speaking these equations of Birkhoff are the linear equations in which the operations $x'=x+1$ and $x'=qx$ in these classical cases are replaced by $x'=\phi_c(x)$ where $\phi_c(x)$ is defined by the relation $\phi_c(x)-c\log\phi_c(x)=x-c\log x+1$. In his paper Birkhoff has treated only the homogeneous ϕ-difference equation of the first order. The author proposes, in this paper, to discuss the values of the underlying function of Birkhoff type $\phi_0(x)=x+1+c/x+c_2/x^2+\cdots$ in the complex plane. (Received November 22, 1939.)

81. A. D. Hestenes: *A solution of the nonhomogeneous ϕ-difference equation of the first order.* Preliminary report.

This paper presents solutions of the nonhomogeneous ϕ-difference equation $y(\phi_0(x))-a(x)y(x)=b(x)$ in which $\phi_0(x)$ is a function of Birkhoff type. These solutions are shown to be analytic in certain regions of the complex plane and are asymptotically represented there by a formal series solution $S(x)$. (Received November 22, 1939.)

82. A. D. Hestenes: *On certain matrix ϕ-difference equations.* Preliminary report.

The author extends the methods used in solving the Birkhoff homogeneous and nonhomogeneous ϕ-difference equations to a special nth order case in which the roots of a certain characteristic equation are distinct and different from zero in the homogeneous case $Y(\phi_0(x))=A(x)Y(x)$ and different from zero and one in the nonhomogeneous case $Y(\phi_0(x))-A(x)Y(x)=B(x)$. The results are parallel to those of the classical linear difference equation. In a subsequent paper the author hopes to extend these methods to the most general linear ϕ-difference equations. (Received November 22, 1939.)

83. M. R. Hestenes: *A theorem on quadratic forms and its application in the calculus of variations.*

In the present paper it is shown that if $Q(\xi, \eta)$ is a quadratic form in $2n$ variables $\xi_1, \cdots, \xi_n, \eta_1, \cdots, \eta_n$ such that $Q(\xi, \eta)>0$ whenever the $n\times 2$ dimensional matrix (ξ_i, η_i) has rank one, then there exists an n-rowed skew-symmetric matrix (S_{ik}) such that the form $Q(\xi, \eta)+S_{ik}\xi_i\eta_k$ is positive definite. The method used is an extension of the method used by McShane (abstract 45-5-209) for the case $n=3$. With the help of this result it is shown that if $z_i(x, y)$ $((x, y)$ on A; $i=1, \cdots, n)$ is an admissible surface S for the calculus of variations integral $\iint f(x, y, z, p, q)dx\,dy$, where $p_i=\partial z_i/\partial x$, $q_i=\partial z_i/\partial y$, such that the Legendre form $Q(\xi, \eta)$ for the integral f is positive on S whenever the matrix (ξ_i, η_i) has rank one, then there is an invariant integral $\iint g(x, y, z, p, q)dx\,dy$ such that the Legendre quadratic form for $f+g$ is positive definite on S. The results described here will be submitted for publication with those of McShane (abstract 45-5-209). (Received November 24, 1939.)

84. Dunham Jackson: *Note on certain orthogonal polynomials.*

A well known property of the kernel associated with an ordinary system of orthogonal polynomials, as set forth for example in Szegö's Colloquium volume (vol. 23, p. 39), is extended to orthogonal trigonometric sums, and more generally to other sets of orthogonal polynomials on plane curves. (Received November 18, 1939.)

85. R. D. James: *A problem analogous to the problem of prime pairs.*

It has long been conjectured that there is an infinite number of primes p such that $p-2$ is also a prime. At present no proof is known although numerical evidence indicates that the statement is true. In this paper it is shown that there is an infinite number of primes p such that $p-k$ is the sum of two primes. Here k is any odd integer, positive or negative. By taking $k=p'+2$, where p' is any odd prime, it follows that there is an infinite number of primes p such that $p-2$ is the sum of three primes, one of which is arbitrary. The method of proof is to establish the asymptotic formula $2A(N) = N^2(\log N)^{-3}S(k)+O(N^2 \log \log N (\log N)^{-4})$, where $A(N)$ is the number of solutions in primes p_1, p_2, p_3 not exceeding N of the equation $p_1-k=p_2+p_3$, and where $S(k) \geqq c > 0$. The proof is a modification of that given by Vinogradov for Goldbach's problem (Comptes Rendus de l'Académie des Sciences de l'URSS, vol. 15 (1937), pp. 169–172). (Received November 20, 1939.)

86. Walter Jennings: *Some implications of the continuum hypothesis.*

A theorem is derived which subsumes various results due to Sierpiński and Lusin and has as corollaries a number of apparently new implications of the continuum hypothesis. (Received November 18, 1939.)

87. B. W. Jones: *Related genera of quadratic forms.* Preliminary report.

This paper shows that with every genus g of quadratic forms and every prime number p there is associated, by a set of transformations, a genus or set of genera G, having the property that the numbers represented by the forms of G are $1/p$ times the multiples of p represented by the forms of g. At least for positive forms, Siegel's expression for the number of representations of any number by the forms of g is a simple function of the corresponding expressions for G. One use of this process is to be able to eliminate, by successive steps of this kind, all the factors common to the number represented and the determinant of the form. Another use is the establishment of relationships between the number of classes of g and the number of classes of G. (Received November 20, 1939.)

88. F. B. Jones: *Almost cyclic elements and simple links of a continuous curve.*

The principal object of this paper is to define two classes of subsets of a non-locally compact continuous curve: (1) the *cyclic nucleae*, which are closely related to basic sets, and (2) the *almost cyclic elements*, which are closely related to cyclic elements and which *are* cyclic elements if the continuous curve is locally compact. A few of the properties of these sets are investigated to the extent that the reader can supply others by analogy to known propositions concerning cyclic elements. It is also pointed out that if H is a nondegenerate almost cyclic element of a locally connected complete Moore space in which the Jordan curve theorem holds true, then considered as a space, H is a locally connected complete Moore space in which the Jordan curve theorem holds true but which contains no cut point. (Received November 20, 1939.)

89. F. B. Jones: *Certain consequences of the Jordan curve theorem.*

If the Jordan curve theorem holds true in a locally connected Moore space M, it is well known that M need be neither a subset of a plane nor even metric. However, M has many properties of a locally connected inner-limiting subset of the plane. Besides the properties of the space itself, the properties of its subcontinua are investigated. Particular emphasis has been placed on separation theorems. (Received November 20, 1939.)

90. Mark Kac: *On the partial sums of the exponential series.*

The formula $(1/\pi^{1/2})\int_{-\infty}^{\omega} e^{-u^2}du = \lim_{x\to\infty} e^{-x}\sum x^k/k!$, where the summation extends over all $k < x+\omega(2x)^{1/2}$, is valid for every real ω. The case $\omega = 0$ was proved by

Ramanujan. A connection with a number theoretical problem is indicated. (Received November 16, 1939.)

91. E. R. van Kampen: *On uniformly almost periodic multiplicative and additive functions.*

An additive (multiplicative) function $f(n)$, $(n=1, 2, \cdots)$, is a function which may be represented in the form $\sum_p$ $(\prod_p)$ f_p, where p runs through all primes and $f_p(n)=f(p^k)$ if $p^k | n$ and $p^{k+1} \nmid n$. The following theorem is proved: An additive (real-valued multiplicative) function f is uniformly almost periodic if and only if the above representation of f as an infinite sum (product) is uniformly convergent and each term (factor) is u.a.p. The sufficiency of this condition is of course evident. In case of a general multiplicative function the question whether or not the above conditions are necessary depends on whether or not they are necessary in the case of a u.a.p. function f which is additive (mod 2π). (Received November 22, 1939.)

92. Edward Kasner and J. J. De Cicco: *The conformal near-Moebius transformations.*

In a previous paper, *Characterization of the Moebius group of circular transformations*, Proceedings of the National Academy of Sciences, vol. 25 (1939), pp. 209–213, the point transformations of the plane with reference to the maximum number of circles preserved were discussed. A nonconformal point transformation converts at most $2\infty^2$ circles into circles. A conformal transformation not of the Moebius type carries at most $2\infty^1$ circles into circles (excluding the minimal lines). In this paper, the authors determine the set of all conformal near-Moebius transformations (those which preserve exactly $2\infty^1$ circles). These are of the form M_2TM_1, where M_1 and M_2 are Moebius transformations and T is any one of the three transformations e^z, $\log z$, z^n. The two families preserved are two orthogonal linear pencils of circles. The conformal near-collineation problem, solved by Kasner in the paper *The problem of partial geodesic representation*, Transactions of this Society, vol. 7 (1906), pp. 200–206, is a special case of this. Any conformal near-collineation is of the form S_1TS_2, where S_1 and S_2 are similitudes. The family preserved is a pencil of straight lines. (Received November 13, 1939.)

93. Edward Kasner and J. J. De Cicco: *Transformation theory of integrable double-series of lineal elements.*

In this paper, the authors completely solve a problem in the theory of differential elements of three-space. A collection of ∞^2 lineal elements of space is said to form a double-series. If ∞^1 unions can be found whose lineal elements coincide exactly with those of a given double-series S, then S is called an integrable double-series. Thus an integrable double-series S is the configuration obtained by constructing at each point of a surface a single tangent direction. The fundamental result is that the entire group of lineal element transformations which carry every integrable double-series into an integrable double-series is exactly the group of extended point transformations. This paper will be published in full in this Bulletin. (Received November 20, 1939.)

94. J. L. Kelley: *On the hyperspaces of a given space.* Preliminary report.

A study is made of the sets $S(A)$ and $C(A)$ consisting respectively of all closed and all closed and connected subsets of a metric separable space A. Particular attention is given the classification of properties of A in terms of the hyperspaces. For example, if for $x \varepsilon A$ we designate by C_x the collection of elements in $C(A)$ which contain x, then if x is a local separating point, C_x is found to have interior points. If A is Peanean, this is a necessary and sufficient condition that x be a local separating point. Theorems on local separating points are obtained from corresponding theorems on separating points. (Received November 21, 1939.)

•95. C. G. Latimer: *On a certain equation.*

Let a, b, c, d be integers such that $abcd \neq 0$, not all have the same sign, each is without a square factor greater than 1 and a, b, c have no common prime factor. The integers a, b, c determine an integer D such that *if* $\theta = (abcd)^{1/2}$ *is rational* $ax^2+by^2+cz^2+dw^2=0$ *has a solution in integers, not all zero, if and only if* $D = -1$. *If* θ *is irrational, such a solution exists if and only if no prime factor of* D *is the product of two distinct prime ideals in the field* $F(\theta)$. The equation is reduced to the form $\zeta^2 = abcdw^2$, where ζ is an element in a quaternion algebra $\mathfrak{A}$ with the fundamental number D. A theorem on the quadratic subfields of $\mathfrak{A}$ and the theorem that $\mathfrak{A}$ is a non-division algebra if and only if $D = -1$ is then applied. The result seems simpler than former results, all of which require considerable separation of cases. (See Mordell, this Bulletin, vol. 38 (1932), p. 277; Archibald, this Bulletin, vol. 37 (1931), p. 607, where numerous other references are given.) (Received November 20, 1939.)

96. Norman Levinson: *A proof of Hardy's theorem on the zeros of the zeta function.*

Let $Z(t) = \Xi(t)e^{\pi t/4}/(t^2+1/4)$ where $\Xi(t)$ is the entire function having as its zeros the non-trivial zero of $\zeta(s)$. Then a certain Mellin transform of $e^{-\pi t/4}Z(t)$ is $\phi(x) = 2\sum e^{-n^2x} - \pi^{1/2}/x^{1/2}$. The fact that $\phi(x) \to \infty$ as $x \to 2\pi i$ demonstrates at once that the integral of $|Z(t)|$ over $(0, \infty)$ diverges. The fact that $\phi(x)$ is bounded as $x \to \pi i$ shows that the integral of $e^{-\delta t}Z(t)$ over $(0, \infty)$ is bounded as $\delta \to 0$. But these two facts imply that $Z(t)$ must have an infinity of zeros. (Received November 20, 1939.)

97. F. A. Lewis: *Generators of permutation groups simply isomorphic with $LF(2, p)$.*

From the results of this paper the generators of any group of the class may be readily obtained. (Received November 20, 1939.)

98. A. N. Lowan: *Note on upper bounds of the derivatives of certain transcendental functions.*

Starting with certain integral representations of the functions $J_0(z)$ and e^{-x^2}, upper bounds of the absolute values of the kth derivatives of these functions are obtained. Specifically, $|(d^k/dz^k)J_0(z)| < I_0(r \sin \theta)$ and $|(d^k/dx^k)e^{-x^2}| < 2^k\{k/2\}!$, where $\{k/2\}$ designates the integer which is either equal to $k/2$ if k is even or the integer immediately smaller than $k/2$ if k is odd. Similarly, starting with the integral representation of the first derivative of arc tan x, an upper bound of the modulus of the kth derivative of the latter function is found to be $x^{-k}(k-1)!$. The knowledge of these upper bounds is of paramount importance in determining the numbers of terms required in the Taylor expansions of the functions under consideration. The computation of tables of these functions by the Project for the Computation of Mathematical Tables (Work Projects Administration, New York City) is now in progress. (Received November 10, 1939.)

99. R. G. Lubben: *Separabilities of higher orders and related properties.*

A point set M is strongly α-separable provided that there exists N such that $\bar{N} \supset M \supset N$ and either N is countable or its power is less than α; if this is true for α the power of M, M is semi-separable. M is almost perfectly α-compact in itself provided that if $M \supset N$, α is not greater than the power of N, and δ is less than the power of N, then M contains a limit point of N of an order relative to N at least as great as δ. If $\alpha > \aleph_0$ is a regular cardinal, the following are equivalent properties of point sets in a space satisfying Hausdorff's Axioms A and B: (1) each point set of power α contains a limit point of itself; (2) each point set either is strongly α-separable, or it contains a limit point of itself of an order not less than α; (3) each point set is almost perfectly α-compact in itself; (4) if M has a regular power which is not less than α, then either M is semi-separable or it is similar to the set of all its complete limit points which belong to it. (Received November 13, 1939.)

100. H. F. MacNeish: *A set of postulates for a finite geometry represented by the Pappus configuration.*

Nine postulates are stated both geometrically and abstractly which define a geometry with three points on a line and three lines through a point. Postulates 6 and 8 are dual, and Postulates 7 and 9 are dual. The duals of Postulates 1, 2, 3, 4, 5 are proved as theorems. Therefore, in view of the fact that the definitions of the set have duality, the entire finite geometry has duality. A euclidean parallel postulate holds. Duality leads to the idea of parallel points. The Pappus theorem is proved as a proposition. The Pappus configuration of nine points and nine lines is proved to satisfy the nine postulates and includes all elements of the finite geometry. The complete independence proof for the postulates is given. Triangles are classified according as lines joining the vertices, and intersections of the sides exist in the finite geometry. A geometry of triangles is given leading to two special Desargues' theorems. The geometry of quadrilaterals is also considered in this paper. (Received November 16, 1939.)

101. H. F. MacNeish: *A sufficient condition for integrability.*

The paper contains a generalization of the condition that $F^n(x)$ be integrable to the case that $\prod_1^r F_i^{n_i}(x)$ be integrable. The condition is that one of the functions say $F_j^{n_j}$ is such that $F_j^{n_j} = k(\prod F_i)(\sum (n_i+1)F_i'/F_i)$, $(i=1, 2, \cdots, r; i \neq j)$. It follows that $\int \prod_1^r F_i^{n_i}(x)dx = k\prod F_i^{n_i+1}(x)+c$, $(i=1, 2, \cdots, r; i \neq j)$. By this process many functions may be integrated easily, which are not included in the well known categories of functions integrable by standard methods, and which might be considered as not integrable in a finite number of terms. (Received November 21, 1939.)

102. J. D. Mancill: *On the Carathéodory condition for unilateral variations.*

The two formulations and proofs of the Carathéodory condition in the calculus of variations given by Graves (American Journal of Mathematics, vol. 52 (1930), pp. 13–19) do not necessarily apply to the case when the minimizing curve may have arcs in common with the boundary of the region of admissible variations. The purpose of this note is to show how his first formulation and proof can be modified so as to be applicable to unilateral (one-sided) variations in the plane. (Received November 20, 1939.)

103. J. D. Mancill: *On the Jacobi condition for unilateral variations.*

The proof of the Jacobi condition in the calculus of variations by means of the Kneser envelope theorem does not necessarily apply to all extremal arcs of the minimizing curve in the case when the minimizing curve may have arcs in common with the boundary of the region of admissible curves. So far as the author knows no use has yet been made of the second variation in this connection. In the present paper additional properties of the envelope of a one-parameter family of extremals in the plane are derived, by means of which, with the aid of the Kneser theorem, a Jacobi condition is proved for the case of unilateral (one-sided) variations in the plane. Incidentally, the results of this paper are of historical interest since they eliminate the restriction, in the case of the geometric formulation of the Jacobi condition, that the envelope have a regressive branch at its point of contact with the minimizing curve. (Received November 20, 1939.)

104. Karl Menger: *On Cauchy's theorem in the real plane.*

If $p(x, y)=q(x, y)=f(x+y)$ where f is a continuous nowhere differentiable function on one real variable, then $\int pdx+qdy$ is independent of the path though neither p nor q has any partial derivative. It is sufficient for the independence that $\lim (\Delta q/\Delta x-\Delta p/\Delta y)=0$ holds for certain rectangular nets of points. This condition is satisfied not only in the classical case that $\partial p/\partial y$ and $\partial q/\partial x$ exist and are equal and continuous everywhere, but also if existence, equality and certain continuity properties of $\partial p/\partial y$ and $\partial q/\partial x$ are guaranteed on a denumerable set only; it is satisfied even for some nowhere differentiable functions. By introducing subdivided nets we get a condition that is both sufficient and necessary in order that the integral be independent of the path. (Received November 13, 1939.)

105. A. N. Milgram: *Iterations of mappings of a set on its square.*

Let M be an abstract set, f and g two mappings of M on itself such that if $x \in M$, $y \in M$ there exists $t \in M$ for which $x=f(t)$, $y=g(t)$. Then for $x_1, \cdots, x_k$ any elements of M, and $n_1 \leqq n_2 \leqq \cdots \leqq n_k$, $m_1 < m_2 < \cdots < m_k$ two sets of positive integers, the equations $x_1=f^{n_1}g^{m_1}(t), \cdots, x_k=f^{n_k}g^{m_k}(t)$ will always have a solution, where f^n means f iterated n times. If M is the interval (0, 1) and f and g are continuous, the above gives a continuous mapping of (0, 1) on the k-dimensional unit cube. If M is compact space, and f and g are continuous, then even each denumerable set of such equations has a solution. Thus if M is (0, 1), taking $x_k=f^{n_k}g^{m_k}(t)/k$, one obtains a continuous mapping of (0, 1) on the fundamental cube in Hilbert space. (Received November 17, 1939.)

106. Harlan C. Miller: *A theorem concerning closed and compact point sets which lie in connected domains.*

In this paper it is proved that in a connected space satisfying Axioms 0, 1, and 2 of R. L. Moore's *Foundations of Point Set Theory* every closed and compact point set is a subset of a compact continuum. (Received November 28, 1939.)

107. Deane Montgomery and Leo Zippin: *A Hilbert axiom for topological transformation groups of space.*

The paper shows that Hilbert's Axiom III in his *Grundlagen der Geometrie* (Mathematische Annalen, vol. 56 (1902)) can be replaced by its more intuitive and simpler "two-point" form. This materially weakened axiom suffices for Hilbert's development of plane geometries and also for the extension of this program to three-space, recently given by the authors (abstract 45-11-402). (Received November 27, 1939.)

108. R. L. Moore: *Concerning accessibility.*

In this paper it is shown that in a locally peripherally compact space (F. B. Jones, Transactions of this Society, vol. 42 (1937), p. 53) satisfying Axioms 0, 1 and 2 of the author's *Foundations of Point Set Theory* (American Mathematical Society Colloquium Publications, vol. 13, 1932), every continuum with a compact boundary is compactly connected; and if the boundary of the domain D is a compact point set β, then (1) if no component of D is closed, the components of D form a semi-contracting collection, (2) if D is connected, $\overline{D}$ contains a compact continuum containing β. (Received November 18, 1939.)

109. R. L. Moore: *Concerning domains whose boundaries are compact.*

In this paper it is shown that, in a space satisfying Axioms 0–5 of the author's *Foundations of Point Set Theory,* (1) no domain with a compact boundary has uncountably many components, (2) if N is a continuum whose boundary is a subset of a compact continuous curve with no continuum of condensation, then N is itself a continuous curve. (Received November 22, 1939.)

110. I. M. Niven: *Integers of quadratic fields as sums of squares.*

L. J. Mordell (Mathematische Zeitschrift, vol. 35 (1932)) gives necessary and sufficient conditions that a positive integral binary quadratic form be expressible as a sum of squares of integral linear forms. Since every integer of a quadratic algebraic field $R(m^{1/2})$ is expressible in an infinitude of ways as a quadratic form in 1, $m^{1/2}$, it is possible to give necessary and sufficient conditions that such an integer be a sum of squares of integers of the field. It is shown that every integer of the form $a+2b\theta$, where $\theta^2=-m$, m being a positive integer, is expressible as a sum of three squares of integers of the field $R(\theta)$; in case m is congruent to three modulo four, every integer of the field is so expressible. Necessary and sufficient conditions that a Gaussian integer be expressible as a sum of two squares of Gaussian integers are established. Similar results are also obtained for real quadratic integers. (Received November 20, 1939.)

111. Rufus Oldenburger: *Symbolic elements in dynamics.*

In the present paper relations are obtained between the properties of periodicity, minimalness, and recurrence of a symbolic trajectory T and these properties for the rays based on T. Since symbolic elements are essentially rays, similar considerations hold for symbolic trajectory and elements based on this trajectory. It is on the use of these elements that much of the theory of trajectories in dynamics is based. It is proved, in particular, that if all elements based on a symbolic trajectory T are recurrent, T is recurrent, while conversely, if $T=T^{-1}$ and T is recurrent, all elements based on T are recurrent. It is further proved that the property of minimalness of a symbolic trajectory T as defined in terms of elements based on T is equivalent to minimalness as defined in terms of rays based on T and that periodicity of all elements or all rays based on T is equivalent to the periodicity of T. If a trajectory T is recurrent and the recurrency functions of the elements based on T are bounded, these properties are possessed by each limit trajectory of T. (Received November 20, 1939.)

112. W. A. Patterson: *Inverse problems of the calculus of variations for multiple integrals.*

For a partial differential equation of the form $F\equiv A_{\alpha\beta}(x_1, \cdots, x_n, z, p_1, \cdots, p_n)p_{\alpha\beta} + B(x_1, \cdots, x_n, z, p_1, \cdots, p_n)=0$, $p_i=\partial z/\partial x_i$, $p_{ij}=\partial^2 z/\partial x_i\partial x_j$, $(i, j, \alpha, \beta=1, 2, \cdots, n)$, where α and β are umbrals and the functions $A_{ij}=A_{ji}$ and B are analytic but otherwise arbitrary, a function $M(x_1, \cdots, x_n, z, p_1, \cdots, p_n)\neq 0$ and such that the equation $M\cdot F=0$ has a self-adjoint equation of variation is called a multiplier. When $F=0$ admits a multiplier, the integral hypersurfaces of $F=0$ are the extremals of an associated variation problem of the form $\int\int f(x_1, \cdots, x_n, z, p_1, \cdots, p_n)dx_1dx_2 \cdots dx_n=\min$. In the present paper the generality of the multiplier M and the form of the integrand function f in the most general associated variation problem is determined for certain important classes of partial differential equations of the form $F=0$. (Received November 21, 1939.)

113. R. S. Phillips: *On linear transformations.*

The purpose of the paper is to give an integral characterization of linear limited and completely continuous transformations both on the common Banach spaces to an arbitrary Banach space and vice versa. The author first studies the abstractly valued function spaces and integrals needed in these characterizations. As an application of this work, he shows that completely continuous transformations on the spaces L^p, l^p, c, C, $(1\leqq p\leqq\infty)$, to an arbitrary Banach space are approximable in the norm by degenerate transformations. In this, use is made of the theorem characterizing compact sets in L^p, $(1\leqq p\leqq\infty)$, which is also proved. A final section is devoted to the extension of linear limited transformations. (Received November 16, 1939.)

114. G. B. Price: *On the theory of integration.*

This paper treats the integrals of functions defined on an abstract space with values in a Banach space. Furthermore, the measure function itself is abstract, that is, a transformation defined over the Banach space rather than a number. The convex operators recently introduced by the author are used to obtain a generalization of the upper and lower integrals. Sufficient conditions that a function be integrable are given. The methods yield a unified treatment of a large class of the more common integrals, a class which includes the Riemann integral, the Lebesgue integral, various Stieltjes integrals, various line integrals, and the integral of a function of a complex variable along a curve in the complex plane. (Received November 20, 1939.)

115. Eric Reissner and H. A. Wood: *On boundary value problems of bi-potential theory for an infinite sector.*

The solution of the equation $\nabla^2\nabla^2 w=0$ is derived for the case that the values of the function and its normal derivative are prescribed along the boundaries of a sector (in polar coordinates (r, θ) along $\theta=\alpha$ and $\theta=-\alpha$). The method consists in superimposing an infinite number of elementary solutions such as $r^n \cos n\theta$, $r^n \sin n\theta$, $r^{n+2}\cos n\theta$, $r^{n+2}\sin n\theta$ which leads to a solution in the form of a Mellin integral. The properties of this solution are discussed. (Received November 16, 1939.)

116. R. M. Robinson: *On the mean values of an analytic function.*

The main result obtained is that if a function $f(z)$ is regular for $|z|<1$, and if the mean of $|f(z)|$ on $|z|=r$ is less than or equal to 1 for each $r<1$, then (for any integer $p>1$) the mean of $|f(z)|^p$ on $|z|=r$ is less than or equal to 1 for $r\leqq p^{-1/2}$, but not in general for larger values of r. (Received November 28, 1939.)

117. Barkley Rosser: *On the computation of logarithms to a large number of decimal places.*

To find log N to a large number of decimal places, it is desirable to find an M very near N such that log M is known to the requisite number of places. Then, if $N=M(1+\epsilon)$, ϵ will be very near zero, and the series for log $(1+\epsilon)$ will converge rapidly. In this paper are tables which, for any N, enable one to find an M of the form $2^a3^b5^c7^d$ (where a, b, c, and d are integers) with $|\epsilon| \leqq .000007$. Then log M is known to 250 decimal places. Table 1 enables one to find an M_1 of the desired form such that $N=M_1(1+\epsilon_1)$ and $|\epsilon_1| \leqq .0024$. Table 2 gives (in order of magnitude) certain values of $2^a3^b5^c7^d$ which are near unity. By looking in Table 2 for a value near $1+\epsilon_1$, one can find an M_2 such that $1+\epsilon_1=M_2(1+\epsilon_2)$ and $\epsilon_2 \leqq .000094$. Table 3 is similar to Table 2 except that values nearer unity are given. By looking in Table 3, one can find an M_3 such that $1+\epsilon_2=M_3(1+\epsilon_3)$ and $|\epsilon_3| \leqq .000007$. (Received November 22, 1939.)

118. A. C. Schaeffer and Gabor Szegö: *Polynomials whose real part is bounded on a Jordan curve in the complex plane.*

Let C be a closed Jordan curve in the complex z-plane consisting of a finite number of analytic arcs forming angles not equal to 0 or 2π with each other. Let $f(z)$ belong to the class of polynomials of degree n or less whose real part is bounded by 1 on C. It is shown that at any point z_0 on C the order of magnitude of max $|f'(z)|$ as n becomes infinite is n^α where $\alpha\pi$ is the exterior angle of C at z_0. For the same class, restricted by the additional condition that $\mathfrak{I}f(z)$ vanishes at a fixed interior point of C, the maximum of $|\mathfrak{I}f(z)|$ over C is also discussed; the rate of growth of this maximum as n becomes infinite is log n. The method used is based on conformal mapping, and especially on the theorems of Osgood-Taylor concerning the behavior of the map function near the boundary. The results of the present paper are well known in the special case in which C is the unit circle. (Received November 20, 1939.)

119. I. J. Schoenberg: *On metric arcs of vanishing Menger curvature.*

A topological proof is given of the so-called n-lattice theorem for metric arcs (Menger, Mathematische Annalen, vol. 103 (1930), p. 487). In addition, Menger's conditions (loc. cit., pp. 488–492) on a metric arc, under which identically vanishing curvature implies the congruence of the arc with the euclidean segment, are replaced by weaker conditions in terms of Ptolemy's classical inequality in elementary geometry. The paper will appear in the Annals of Mathematics. (Received November 21, 1939.)

120. G. E. Schweigert: *A local property arising from certain interior transformations.*

Let $T(A)=B$ be an interior transformation where A is compact and metric and B is a Peano continuum. If U is an open set in A such that $T(U)$ is connected and if x is any point of U, there exists a point y of B and a connected set K in U such that y is distinct from $T(x)$ and K intersects both $T^{-1}T(x)$ and the inverse of y. A certain form of this result will combine with other necessary conditions to characterize particular non-alternating interior transformations. (Received November 19, 1939.)

121. W. T. Scott: *Interpolation by continued fractions.*

In this paper it is shown that interpolation by continued fractions is always possible, even when the well known Thiele interpolation formula fails. The method given here leads to the general corresponding type continued fraction (Leighton and Scott, this Bulletin, vol. 45 (1939), pp. 596–605). A generalization of the Thiele remainder formula for continued fractions is obtained. (Received November 20, 1939.)

122. C. E. Sealander: *Some third order irregular boundary value problems.*

In a previous paper (abstract 44-1-4), the author discussed expansions in series of characteristic functions of the differential system $u'''(x)+p(x)u'(x)+[\rho^3+q(x)]u(x)=0$, $u(0)=u'(0)=u(\pi)=0$, in which the functions $p(x)$ and $q(x)$ were restricted to have special forms. The results of that paper have been extended to the case where these functions are only required to have derivatives of all orders on an interval I of which $x=0$ is an interior point. It is shown that a function $f(x)$ will have an expansion in a series of characteristic functions of this system which converges uniformly to $f(x)$ in every interval $0\leqq x\leqq a<\pi$ interior to I, provided that $f(x)$ has a continuous second derivative in $(0, \pi)$ and that the series $f(x)-w_1(x)/\rho^3+w_2(x)/\rho^6-\cdots$, where the $w_n(x)$ are defined by recurrence relations involving $f(x)$, $p(x)$, and $q(x)$ converges uniformly in I and its second derived series converges at $x=0$. Regions of convergence of the series for $f(x)$ when x is complex are determined. The analogous problem for the system $u'''(x)+\lambda x^\nu u(x)=0$, $u(-\alpha)=u'(-\alpha)=u(\beta)=0$, $(\alpha, \beta>0)$, λ being a complex parameter and ν a real positive integer, in which the coefficient of the parameter may change sign, is then shown to reduce to the above problem by a series of transformations of variable. (Received November 27, 1939.)

123. Seymour Sherman: *Some new properties of transfinite ordinals.*

It is well known that if A is an ordinal, then A is uniquely representable in the form $\sum_{i=0}^{n-1}\omega^{\alpha_i}a_i$, where $A>\alpha_0>\cdots>\alpha_{n-1}$ and $\omega>a_0\cdots a_n>0$. This sum is the Cantor normal form of A. A necessary and sufficient condition that x be a left factor of A is that either $x\leqq\omega^{\alpha_{n-1}}$ or $x=\omega^{\alpha_j}r_j+\sum_{i=j+1}^{n-1}\omega^{\alpha_i}a_i$, $r_jp_j=a_j$, $0\leqq j\leqq n-1$. A necessary and sufficient condition that x be a right factor of A is that either $x=\sum_{i=0}^{j-1}\omega^{-\alpha_j+\alpha_i}+\omega^{\alpha_j}r_j$, $r_jp_j=a_j$, $0\leqq j\leqq n-1$, or $x=\sum_{i=0}^{n-1}\omega^{-\beta+\alpha_i}a_i$, $\beta\leqq\alpha_{n-1}$. These theorems are used in the further characterization of the sets of right and left factors of ordinals. In addition it is proved that $(\alpha+\beta)\gamma\leqq\alpha\gamma+\beta\gamma$ for α, β, and γ ordinals. Necessary and sufficient conditions that the equality hold are given. (Received November 20, 1939.)

124. M. F. Smiley: *A note on the Jacobi condition for extremaloids.*

Recently McShane (Duke Mathematical Journal, vol. 5 (1939), pp. 184–206) gave a treatment of the Jacobi condition for extremals which permitted a simultaneous discussion of this condition for the parametric and nonparametric problems. It is the purpose of this note to formulate the Jacobi condition for extremaloids in a similar fashion. In order to obtain flexibility at the corners we find it necessary to alter McShane's (Bliss') definition of normality of solutions of the Jacobi equations. No attempt is made to discuss questions of tensor invariance. (Received November 21, 1939.)

125. H. W. Smith: *The oscillation of solutions of the differential boundary value problems of the fourth order.*

The general ordinary linear homogeneous differential equation of the fourth order, whose coefficients are continuous functions of x and λ and are of class C' with respect to x, is reduced to the system: $y_1'=p(x, \lambda)y_2$, $y_2'=q(x, \lambda)y_3$, $y_3'=r(x, \lambda)y_4$, $y_4'=s(x, \lambda)y_1$. The p, q, r are essentially positive and it is assumed that s does not change sign on

$a \leqq x \leqq b$. Then if p, q, r, s are monotonic functions of λ on $\Lambda_1 < \lambda < \Lambda_2$, having finite upper bounds while $|p\ q\ r\ s| \to 0$ as $\lambda \to \Lambda_1$ and having positive lower bounds while $|p\ q\ r\ s| \to \infty$ as $\lambda \to \Lambda_2$, it is shown that if $s < 0$ there exists an infinite sequence of values of λ for each of which the system has a solution satisfying the conditions $y_1(a) = y_1'(a) = y_1(b) = 0$ and one other condition (not implying $y_1'(b) = 0$) which may be imposed at either a or b. The solution corresponding to λ_k has k zeros on $a < x < b$. If $s > 0$, there exists a similar sequence for which the system has solutions satisfying the conditions $y_1(a) = y_1'(a) = y_1(b) = y_1'(b) = 0$, the solution corresponding to λ_k has k zeros on $a < x < b$. (Received November 16, 1939.)

126. D. C. Spencer: *On mean one-valent functions.*

Suppose $f(z)$ is regular for $|z| < 1$, and that $f(z)$ maps the unit circle on a Riemann domain W. Let $W(R)$ denote the area (regions covered multiply being counted multiply) of that portion of W which lies inside the circle $|w| \leqq R$. Then, if $W(R) \leqq p\pi R^2$ for all $R > 0$, where p is a positive number (not necessarily integral), $f(z)$ is said to be mean p-valent. In this paper the family of mean one-valent functions (which includes schlicht functions) is studied. If $f(z) = z + a_2 z^2 + \cdots$ is mean one-valent and d is the distance of the boundary of its W from $w = 0$, then in particular the following three results are true: (i) $|a_2| \leqq 2$; (ii) $d \geqq 1/4$; and (iii) $|z|/(1+|z|)^2 \leqq |f(z)| \leqq |z|/(1-|z|)^2$. Equality can occur in any one of the inequalities only if f is the Koebe function. The results are classical for schlicht functions. (Received November 16, 1939.)

127. Otto Szász: *On strong summability of Fourier series.*

A series $\sum_{n=0}^{\infty} u_n$ is said to be strongly summable $(C, 1)$ with index k and sum s if $(n+1)^{-1}\sum_{j=0}^{n} |s_j - s|^k \to 0$ as $n \to \infty$, where $k > 0$ and $s_n = \sum_{j=0}^{n} u_j$. In 1913 Hardy and Littlewood obtained a sufficient condition for strong summability with index 2 of the Fourier series of an integrable function $f(t)$ at $t = x$. Recently Fejér gave two new proofs of this result for the case in which $F(t)$ is continuous at $t = x$. In the present paper several new proofs are given for the general Hardy-Littlewood theorem and for the special case treated by Fejér. Moreover, necessary and sufficient conditions for the strong summability of Fourier series are obtained. The methods are elementary. A typical result is this: Let $A_n = \int\int \psi(t)\psi(u) \cot t/2 \cot u/2\ Q_n(u, t)\,dt\,du$, $(0 \leqq u \leqq t,\ u + t < r)$, where $f(t) \in L$, $\psi(t) = [f(x+t) + f(x-t)]/2$, $Q_n(u, t)$ denotes $\sum_{j=0}^{n} (n+1-j)^2 \sin jt \sin ju$. Then $A_n = o(n^5)$ is necessary and sufficient for strong summability of index 2 of the Fourier series of $f(t)$ at $t = x$. (Received November 21, 1939.)

128. A. W. Tucker: *The algebraic structure of complexes.* II. Preliminary report.

Let X, Y, Z be complexes in a general algebraic sense (see abstract 45-11-437 and a forthcoming note in the Proceedings of the National Academy of Sciences). Then a suitably chosen "tensor," contravariant in X and Y and covariant in Z, determines: (1) a product cycle on XYZ^*, (2) a chain-mapping of Z in XY or a cochain-mapping of XY in Z, (3) a chain-mapping of Y^*Z in X or a cochain-mapping of X in Y^*Z, (4) a product cocycle on X^*Y^*Z. Items (2) and (3) may also be given interpretation as "cup" or "cap" products (Whitney, Annals of Mathematics, (2), vol. 39 (1938), pp. 397–432). In like fashion a general "tensor" determines a variety of mappings, graphs, and products. If two "tensors" of the same type give results which are equivalent, or homologous, in any one respect, they agree also in all other respects. This is the essence of the invariance properties of the "tensors." (Received November 24, 1939.)

129. L. I. Wade: *Certain quantities transcendental over the field $\Phi(x)$, where $\Phi = GF(p^n)$.*

Let $\psi(t) = \sum_{j=0}^{\infty}(-1)^j t^{p^{nj}}/F_j$ so that $\psi(\xi E) = 0$ for all polynomials E in x with coefficients in the finite field Φ and ξ is fixed (L. Carlitz, Duke Mathematical Journal, vol. 1 (1935), p. 137). It is proved that ξ is transcendental with respect to the field $\Phi(x)$. More generally $\psi(\alpha)$ is transcendental, where $\alpha \neq 0$ is algebraic over $\Phi(x)$. From this it follows that $\lambda(\alpha)$ is also transcendental, where $\psi(\lambda(t)) = t$. (Received November 21, 1939.)

130. H. S. Wall: *Continued fractions and totally monotone sequences.*

The object of this paper is to characterize completely totally monotone sequences in terms of continued fractions. It is shown that the sequence $c_0, c_1, c_2, \cdots$ has the property $\Delta^m c_n \geqq 0$, $(m, n = 0, 1, 2, \cdots)$, if and only if the power series $c_0 - c_1 x + c_2 x^2 - \cdots$ has a corresponding continued fraction of the form $c_0/1 + g_1 x/1 + g_2(1-g_1)x/1 + g_3(1-g_2)x/1 + \cdots$, which may or may not terminate, where $0 < g_n < 1$, $(n \geqq 1)$, in case the continued fraction is non-terminating, while the last partial quotient may be irregular in the case of the terminating continued fraction. (Received November 21, 1939.)

131. A. D. Wallace: *Relatively non-alternating transformations.*

Let $T(A) = B$ be a continuous transformation defined on the metric continuum A and let G be a collection of closed subsets of B; the transformation T is said to be non-alternating relative to G (or G n-a) if for no pair (b, Y), with b a point of B and Y an element of G, does there exist a separation $A - T^{-1}(Y) = A_1 + A_2$ such that $A_1 \cdot T^{-1}(b) \neq 0 \neq A_2 \cdot T^{-1}(b)$. This generalizes a definition due to G. T. Whyburn (American Journal of Mathematics, vol. 56 (1934), pp. 294–302) since a non-alternating transformation may be defined as non-alternating relative to the set of all points of B. In this paper the author carries over many known theorems concerning non-alternating transformations to G n-a transformations. It is interesting to note that T is monotone if and only if it is non-alternating relative to the collection of all closed subsets of B. (Received November 20, 1939.)

132. M. E. Wescott: *Sets of Newton polynomials analogous to Laguerre's polynomials.*

Let $x_0 < x_1 < x_2 < \cdots$ be an infinite set of discrete variates with interval $h = x_{i+1} - x_i$, $(i \geqq 0, x_0 = 0)$. Set $a = (1+h^2)^{1/h^2}$. The expansion of $h^{-n} a^{x+nh} \Delta_h^n [x^{(n)} a^{-x}]$, where $x^{(n)} = x(x-h) \cdots [x-(n-1)h]$, yields, for $n = 0, 1, 2, \cdots$, a set of Newton polynomials $\{\mathcal{L}_n(x)\}$ analogous to Laguerre's polynomials. The set is orthogonal on the discrete range $[0, \infty]$ with weight function a^{-x}, normalizes to $[(n!)^2 a^{(n+1)h}]/K$, $K = (a^h - 1)/h$, possesses a recurrence relation, and satisfies a second order difference equation. As $h \to 0$, the set reduces to the continuous Laguerre polynomials $\{L_n(x)\}$, and all properties of the discrete set become their counterparts in the continuous set. For $h = 1$, the set reduces to a set $\{l_n(x)\}$, which is orthogonal with weight function 2^{-x} on the range of all positive integers and zero. This set normalizes to $(n!)^2 2^{n+1}$. The coefficients of the Newton polynomial $l_n(x)$ are precisely those of the Laguerre polynomial $L_n(x)$. The polynomials $\{l_n(x)\}$ may be used to approximate an empirical function $\{y_x\}$, under a least square criterion, through the relation $y = \sum_{k=0}^{n} c_k l_k(x)$. Here $c_k = \sum_{r=0}^{k} \{(-1)^r [C_{k,r}]^2 (k-r)! M_r\}/(k!)^2 2^{k+1}$, where $M_r = \sum_{x=0}^{\infty} 2^{-x} x^{(r)} y_x$ is the rth observed moment. M_r converges for empirical data because an index p always exists such that $y_x = 0$ for all $x \geqq p$. (Received November 28, 1939.)

133. Norbert Wiener: *A canonical series for symmetric functions in statistical mechanics.*

A symmetric function of a set of particles can under certain conditions be expanded in a series of which the nth term is an n-tuple sum over the particles. This method is used to express symmetric functions of the positions of a set of molecules of a monatomic gas at time t in terms of the positions at time 0. The distribution function of the particles is deduced. (Received November 22, 1939.)

134. R. L. Wilder: *Local connectedness and generalized manifolds.*

Although existing definitions of generalized n-manifolds (g. n-m.) usually imply that the manifold is locally connected in all dimensions, this local connectedness is of the "ϵ-δ" type and not the type which provides for each point arbitrarily small neighborhoods U such that $p^i(U)=0$ for all i. (In the case of local 0-connectedness the two types are well known to be equivalent when possessed by a space at all points, but this is not the case for local i-connectedness in either the homology or homotopy sense when $i>0$.) One result of this is that a g. 3-m. can be constructed with points which cannot be ϵ-separated by 2-spheres nor, indeed, by 2-manifolds. The following stronger type of definition will eliminate this feature: An M_0 is a pair of points. Assuming an M_{n-1} defined an M_n is a compact metric continuum whose "small" r-cycles ($r<n$) bound on M_n, and such that if $p \varepsilon M_n$, $\epsilon>0$, there exists a "sphere-like" M_{n-1} such that $M_n - M_{n-1} = A+B$ separate, $p \varepsilon A \subset S(p, \epsilon)$, and the basis cycle of M_{n-1} bounds on $\bar{A}$ and $\bar{B}$. (Received November 21, 1939.)

135. Alexander Wundheiler: *Are complex numbers vectors?*

Against the widespread belief, the complex numbers are not isomorphic with vectors, as the complex multiplication does not correspond to any invariant operation on vectors. But if a distinguished direction α is introduced in the (metric) vector-plane, the complex product u of the numbers $x=x^1+ix^2$, $y=y^1+iy^2$ becomes equivalent to an invariantive operation performed on the three vectors $u^h=\eta^h_{\cdot klm}x^k y^l a^m$, where $u^h=\delta^h_{\cdot k}g_{lm}+g\cdot g\cdot g^{hj}\epsilon_{jk}\epsilon_{lm}$, $(h, j, k, l, m=1, 2)$. (g_{lm} is the metric tensor, $g=|g_{lm}|$, $\delta^k_{\cdot b}$ is the Kronecker delta, $\epsilon_{jk}=+1, 0, -1$ as $j<k$, $j=k$, $j>k$.) $a=1$ gives the familiar product. Thus, a complex number is a vector in a metric plane with a distinguished direction. Analogous results apply to quaternions. (Received November 20, 1939.)

136. J. W. T. Youngs: *A remark on cyclic transitivity.*

Given a set 1 of elements x, a class Γ of subsets E of 1 generates a binary relation $a\ R(\Gamma)\ b$ defined as follows: for every $x\neq a, b$ there is a set $E \varepsilon \Gamma$ such that $a+b \subset E \subset 1-x$. If the class Γ satisfies certain requirements, the relation $R(\Gamma)$ is reflexive, symmetric and cyclicly transitive, and an abstract theory of cyclic elements can be developed in terms of $R(\Gamma)$. (See Radó, abstract 45-5-223; Radó and Reichelderfer, abstract 46-1-30.) The object of this note is to investigate the results of requiring that Γ have two properties: P_1: 0, 1, $x \varepsilon \Gamma$; P_2: if E_1, $E_2 \varepsilon \Gamma$, $E_1 \cdot E_2 \neq 0$ and $a+b \subset E_1+E_2$, then there exists a set $E \varepsilon \Gamma$ such that $a+b \subset E \subset E_1+E_2$. It is to be noticed that in a Peano space arcs have P_2. (Received November 21, 1939.)

137. Max Zorn: *Alternative rings with nilpotent elements.*

This paper contains a proof for the analogue of Engel's theorem in alternative systems, which satisfy the ascending chain condition for subrings. It discusses the relations to the theory of the radical and completes the theory of linear alternative algebras in this respect. (Received November 20, 1939.)

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FOURIER EXPANSIONS OF MODULAR FORMS AND PROBLEMS OF PARTITIONS

HANS RADEMACHER

The subject which I am going to discuss in this lecture excels in the richness of its ramifications and in the diversity of its relations to other mathematical topics. I think therefore that it will serve our present purpose better not to attempt a systematic treatment, beginning with definitions and proceeding to lemmas, theorems, and proofs, but rather to look around and to envisage some outstanding marks scattered in various directions. I hope that the intrinsic relationships connecting the problems and theorems which I shall mention will nevertheless remain quite visible.

A good deal of the investigations about which I shall report can be subsumed under the heading of analytic number theory, and, more specifically, analytic additive number theory. It would, however, be a misplacement of emphasis if we were to look upon analysis, which here means function theory, only as a tool applied to the investigation of number theory. It is more the inner harmony of a system which I wish to depict, properties of functions revealing the nature of certain arithmetical facts, and properties of numbers having a bearing on the character of analytic functions.

Whereas the multiplicative number theory, which deals with questions of factorization, divisibility, prime numbers, and so on, goes back more than 2000 years to Euclid, the history of additive number theory, in any noteworthy sense, begins with Euler less than 200 years ago. In his famous treatise, *Introductio in Analysin Infinitorum* (1748), Euler devotes the sixteenth chapter, "De partitione numerorum," to problems of additive number theory. A "partition" is, after Euler, a decomposition of a natural number into summands which are natural numbers, for example, $6=1+1+4$. We can impose various restrictions on the summands; they may belong to a specified class of numbers, let us say odd numbers, or squares, or cubes, or primes; it may be required that they be all different; or their number may be preassigned. I wish to speak here only about unrestricted partitions. By the way, only the parts are essential, not their arrangement, so that we do not count two decompositions as different if they differ only in the order of the summands; we can therefore take the summands ordered according to their size.

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[1] An address delivered before the Williamsburg meeting of the Society, December 29, 1938, by invitation of the Program Committee.

The first step in any characterization will be to count the partitions. Thus we have the following 11 partitions of 6:

$$6,\quad 1+5,\quad 2+4,\ 3+3,\quad 1+1+4,\quad 1+2+3,\quad 2+2+2,$$
$$1+1+1+3,\ 1+1+2+2,\ 1+1+1+1+2,\ 1+1+1+1+1+1.$$

We write $p(6)=11$, and in general denote the number of partitions of n by $p(n)$.

As far as the method is concerned, Euler made the simple remark that, since we have $x^m \cdot x^n = x^{m+n}$, exponents of powers can easily be combined in an additive manner, and therefore products of power series can be used as "generating functions." In our case, for the unrestricted partitions, he found by simple reasoning the generating function

$$f(x) = 1 + \sum_{n=1}^{\infty} p(n)x^n = \frac{1}{(1-x)(1-x^2)(1-x^3)\cdots}. \tag{1}$$

Indeed, in a partition of n we can first collect the equal summands and thereby express n as a sum of multiples of 1, 2, 3, $\cdots$,

$$n = m_1 \cdot 1 + m_2 \cdot 2 + m_3 \cdot 3 + \cdots, \qquad m_j \geqq 0. \tag{2}$$

On the other side

$$\frac{1}{(1-x)(1-x^2)(1-x^3)\cdots} = \sum_{m_1=0}^{\infty} x^{m_1} \cdot \sum_{m_2=0}^{\infty} x^{m_2 \cdot 2} \cdot \sum_{m_3=0}^{\infty} x^{m_3 \cdot 3} \cdots,$$

and the power x^n therefore occurs as often in the product as n can be written in the form (2); but the number of solutions of the diophantine equation (2) is precisely $p(n)$.

Euler later investigated the infinite product

$$P(x) = (1-x)(1-x^2)(1-x^3)\cdots \tag{3a}$$

appearing in the right member of (1). He found, first empirically[2] and later with conclusive proof,[3] that

$$P(x) = 1 - x - x^2 + x^5 + x^7 - - + + = \sum_{\lambda=-\infty}^{\infty} (-1)^{\lambda} x^{\lambda(3\lambda-1)/2}. \tag{3b}$$

The discovery of the equality expressed by (3a) and (3b) marks a highly important event in the history of our science. It is the first time that a ϑ-function, in a special case, appears in the literature; moreover it appears here immediately in its two aspects: as a power series with exponents formed by a *quadratic* expression of the index of summation, and secondly as an infinite product.[4] This equation leads to the identity

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[2] *Découverte d'une loi tout extraordinaire des nombres par rapport à la somme de leurs diviseurs, Opera Omnia,* (1), vol. 2, pp. 241–253.

[3] *Demonstratio theorematis circa ordinem in summis divisorum observatum* (1754–1755), *Opera Omnia,* (1), vol. 2, pp. 390–398).

$$\left(1 + \sum_{n=1}^{\infty} p(n)x^n\right) \sum_{\lambda=-\infty}^{\infty} (-1)^{\lambda} x^{\lambda(3\lambda-1)/2} = 1$$

which furnishes a formula of recurrence for $p(n)$. Formulas of recurrence have been used, indeed, for the construction of tables of $p(n)$ for values of n up to 600 (MacMahon,[5] Gupta[6]). The function $p(n)$ is very rapidly increasing with n. Some specimens are

$$p(10) = 42, \quad p(100) = 1905\ 69292, \quad p(200) = 397\ 29990\ 29388,$$
$$p(600) = 4580\ 04788\ 00814\ 43085\ 53622.$$

Outside the range from 1 to 600 a few isolated values of $p(n)$ have been computed by D. H. Lehmer.[7] That, of course, could not be done by a formula of recurrence, but has been made possible by certain independent representations of $p(n)$, which belong to the main part of this report. Let me mention only that the largest known value of $p(n)$ is $p(14031)$, which turns out to be a figure of 127 digits.

In spite of the profound discoveries in the field of ϑ-functions and related functions due to Jacobi, Riemann, Klein, and Poincaré, the situation of our problem remained unchanged for more than one and a half centuries after Euler's investigations. It was not until 1917 that

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[4] In a letter to Fuss, the first editor of Euler's unpublished works, Jacobi writes (1848), "Ich möchte mir bei dieser Gelegenheit noch erlauben, Ihnen zu sagen, warum ich mich so sehr für diese EULERsche Entdeckung interessiere. Sie ist nämlich der erste Fall gewesen, in welchem Reihen aufgetreten sind, deren Exponenten eine arithmetische Reihe *zweiter* Ordnung bilden, und auf diese Reihen ist durch mich die Theorie der elliptischen Transcendenten gegründet worden. Die EULERsche Formel ist ein spezieller Fall einer Formel, welche wohl das wichtigste und fruchtbarste ist, was ich in reiner Mathematik erfunden habe . . ." (quoted from Euler, *Opera Omnia*, (1), vol. 2, p. 192, footnote).

[5] This table was published by Hardy and Ramanujan in the paper referred to in footnote 8.

[6] *A table of partitions*, Proceedings of the London Mathematical Society, (2), vol. 39 (1935), pp. 142–149; *A table of partitions* II, Proceedings of the London Mathematical Society, (2), vol. 42 (1937), pp. 546–549.

[7] *On a conjecture of Ramanujan*, Journal of the London Mathematical Society, vol. 11 (1936), pp. 114–118. *An application of Schläfli's modular equation to a conjecture of Ramanujan*, this Bulletin, vol. 44 (1938), pp. 84–90.

G. H. Hardy and S. Ramanujan published their fundamental paper, *Asymptotic formulae in combinatory analysis.*[8] As I have already mentioned, the denominator in (1) is closely related to ϑ-functions. In a commentary on Riemann's *Collected Works*, and in another paper,[9] Dedekind had made detailed studies of the function

$$\eta(\tau) = e^{\pi i\tau/12}\prod_{m=1}^{\infty}(1 - e^{2\pi i m\tau}), \qquad \Im(\tau) > 0, \tag{4}$$

which, under the substitution $x = e^{2\pi i\tau}$, is essentially the denominator in question.

Hardy and Ramanujan started with Cauchy's formula applied to the equation (1):

$$p(n) = \frac{1}{2\pi i}\int_C \frac{f(x)}{x^{n+1}}\,dx. \tag{5}$$

Since $f(x)$ is regular inside the unit circle and has the unit circle as its natural boundary, C has to be a closed curve inside $|x| = 1$, surrounding the point $x = 0$.

Now the usual approach to a complex integral is to utilize the freedom in the path of integration, whether we use the calculus of residues or the method of steepest descents or similar devices. In any case we trace the path of integration in such a way that it passes through that region in which the function, by the overwhelming amount of its value, gives the heaviest contribution on a relatively short piece of the path. The function $f(x) = P(x)^{-1}$ tends rapidly to infinity if we approach radially the point $x = 1$, since each term in $P(x)$ tends to zero. Thus the neighborhood of $x = 1$ will yield the most essential contribution. The next heaviest singularity is located at $x = -1$ in whose vicinity every other factor of $P(x)$ comes close to zero. In this way the roots of unity enter, according to their denominators, but decreasing in weight with increasing denominators. The path C is taken as a circle around 0 rather close to the unit circle, and we cut it into parts, each part corresponding to a neighborhood of just one root of unity. The assembly of all proper fractions h/k with $k \leq N$ is called the Farey series of order N, to the use of which Hardy and Ramanujan were quite naturally led by these considerations.

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[8] *Asymptotic formulae in combinatory analysis*, Proceedings of the London Mathematical Society, (2), vol. 17 (1918), pp. 75–115.

[9] *Schreiben an Herrn Borchardt über die Theorie der elliptischen Modulfunktionen*, Journal für die reine und angewandte Mathematik, vol. 83 (1877), pp. 265–292; also *Gesammelte Werke*, vol. 1, pp. 175–201.

This is, of course, only a sketch of the procedure of integration. Its details have to be furnished by a study of the function $f(x)$ in the neighborhood of a root of unity $e^{2\pi ih/k}$. This is given by the formula

$$(6) \qquad f(e^{2\pi ih/k-2\pi z/k}) = \omega_{h,k} z^{1/2} \exp\left(\frac{\pi}{12kz} - \frac{\pi z}{12k}\right) f(e^{2\pi ih'/k-2\pi/kz}), \qquad \Re(z) > 0.$$

Here $\omega_{h,k}$ is a certain $24k$th root of unity. This formula is of importance in our problem since in the neighborhood of the root of unity $e^{2\pi ih/k}$ we have z small, $1/z$ large, and both z and $1/z$ with positive real parts. Such a large $1/z$ involves a very small value of $e^{2\pi ih'/k-2\pi/kz}$ so that, because of (1), $f(e^{2\pi ih'/k-2\pi/kz})$ can with good approximation (which has, of course, to be appraised) be replaced by 1. This gives an elementary approximation function for $f(x)$ in the neighborhood of $e^{2\pi ih/k}$. By means of this treatment of (5), and appropriate estimations of the errors which are made when the function $f(x)$ is replaced by certain approximation functions, Hardy and Ramanujan arrived at the asymptotic formula

$$(7) \quad p(n) = \frac{1}{2\pi 2^{1/2}} \sum_{k \leq \alpha n^{1/2}} A_k(n) k^{1/2} \frac{d}{dn}\left(\frac{\exp\left\{\frac{\pi}{k}\left[\frac{2}{3}(n - 1/24)\right]^{1/2}\right\}}{(n - 1/24)^{1/2}}\right) + O(n^{-1/4}),$$

with

$$(8) \qquad A_k(n) = \sum_{h \bmod k,\, (h,k)=1} \omega_{h,k} e^{-2\pi inh/k}.$$

This formula is remarkable in analytic number theory because of its error term which tends to zero as n increases. The constant involved in the error term was not determined; actual computation of $p(100)$ and $p(200)$, however, showed that a relatively small number of terms suffice to give a value which differs from the true value by only a few thousandths of a unit. Hardy and Ramanujan raised the question whether the series (7) extended indefinitely converges or not. D. H. Lehmer[10] has shown recently, by a study of the $A_k(n)$, that the infinite series (7) is divergent.

The method applied by Hardy and Ramanujan was further developed by Hardy and Littlewood and applied to other problems, in particular to Waring's problem, where it leads also to asymptotic results. In the present case, however, Hardy and Ramanujan were not aware of the full strength of their method. When it is applied with more refinement of the estimates, and when in particular the coupling of N (the order of the Farey dissection) with n (the given integer) is abolished, and N is made to go to infinity with a fixed n, then we obtain the following exact formula[11] for $p(n)$:

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[10] *On the Hardy-Ramanujan series for the partition function*, Journal of the London Mathematical Society, vol. 12 (1937), pp. 171–176.

$$(9)\qquad p(n) = \frac{1}{\pi 2^{1/2}} \sum_{k=1}^{\infty} A_k(n) k^{1/2} \frac{d}{dn}\left(\frac{\sinh \dfrac{\pi}{k}\left[\dfrac{2}{3}(n - 1/24)\right]^{1/2}}{(n - 1/24)^{1/2}} \right),$$

where the infinite series is absolutely convergent. The formula (7) not only appears as a consequence of formula (9), but also the error term in (7) can be definitely estimated. This, by the way, made it possible to compute the function $p(n)$ for the great values of n mentioned above.

The formula (9) has now been the starting point for further results. We saw that the formula (6) forms the basis for the evaluation of the integral (1). The formula (6) in turn is derived from the theory of the function $\eta(\tau)$. We have

$$(10)\qquad \eta(\tau') = \epsilon(a, b, c, d)\left[- i(c\tau + d)\right]^{1/2}\eta(\tau), \qquad c > 0,$$

$$(10\text{a})\qquad \tau' = \frac{a\tau + b}{c\tau + d}, \qquad \begin{vmatrix} a & b \\ c & d \end{vmatrix} = 1,$$

where a, b, c, d are integers and ϵ is a certain 24th root of unity. This formula is connected with (6) through the substitution

$$\tau = \frac{iz + h}{k}, \qquad \tau' = \frac{i/z + h'}{k}, \qquad hh' \equiv -1 \pmod{k}.$$

The substitutions (10a), together with the case $c=0$:

$$(10\text{b})\qquad \tau' = \tau + b,$$

for which we have, directly from (4),

$$(10\text{c})\qquad \eta(\tau') = e^{\pi i b/12}\eta(\tau),$$

are called modular substitutions. They form an infinite discontinuous group. All these considerations can be applied to modular forms in general.[12] Let us give, in brief, a definition of this concept.

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[11] Rademacher, *A convergent series for the partition function* $p(n)$, Proceedings of the National Academy of Sciences, vol. 23 (1937), pp. 78–84; *On the partition function* $p(n)$, Proceedings of the London Mathematical Society, (2), vol. 43 (1937), pp. 241–254.

A modular form is a homogeneous analytic function of two variables ω_1, ω_2 defined for $\mathfrak{J}(\omega_2/\omega_1) > 0$:

$$H(\lambda\omega_1, \lambda\omega_2) = \lambda^r H(\omega_1, \omega_2); \tag{11}$$

the parameter r, which we take as real, is the dimension of the form. A modular form is invariant under modular transformations:

$$\begin{aligned} \omega_2' &= a\omega_2 + b\omega_1, \\ \omega_1' &= c\omega_2 + d\omega_1, \end{aligned} \qquad \begin{vmatrix} a & b \\ c & d \end{vmatrix} = 1,$$

$$H(\omega_1', \omega_2') = H(\omega_1, \omega_2).$$

Examples are the invariants

$$g_2 = 60 \sum{}' \frac{1}{(2m_1\omega_1 + 2m_2\omega_2)^4}, \qquad g_3 = 140 \sum{}' \frac{1}{(2m_1\omega_1 + 2m_2\omega_2)^6}.$$

A nonhomogeneous notation is often preferable: By means of (11) we have

$$\omega_1'^{\,r} H(1, \omega_2'/\omega_1') = \omega_1^r H(1, \omega_2/\omega_1)$$

and, with $\omega_2/\omega_1 = \tau$, $\omega_2'/\omega_1' = \tau'$, $\tau' = (a\tau + b)/(c\tau + d)$,

$$H(1, \tau') = (c\tau + d)^{-r} H(1, \tau). \tag{12}$$

If r is not an integer, we have to determine the branch of $(c\tau + d)^{-r}$. In order to avoid this difficulty we admit a slight generalization. We consider functions $F(\tau)$ for $\mathfrak{J}(\tau) > 0$ with the property

$$F(\tau') = \epsilon(a, b, c, d) \cdot (-i(c\tau + d))^{-r} F(\tau), \tag{13a}$$

where $c > 0$ and $|\epsilon| = 1$ and $(-i(c\tau + d))^{-r}$ stands for the principal branch. The case $c = 0$ has to be mentioned separately:

$$F(\tau + 1) = \epsilon_1 F(\tau) = e^{2\pi i\alpha} F(\tau), \qquad 0 \leqq \alpha < 1. \tag{13b}$$

A function $F(\tau)$ having the properties (13a) and (13b) may now be called a "modular form," in spite of its nonhomogeneous notation; in particular (13a) shows that it is of dimension r.

From (13b) we derive

$$e^{-2\pi i\alpha(\tau+1)} F(\tau + 1) = e^{-2\pi i\alpha\tau} F(\tau),$$

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[12] Rademacher and Zuckerman, *On the Fourier coefficients of certain modular forms of positive dimension*, Annals of Mathematics, (2), vol. 39 (1938), pp. 433–462.

and note the periodicity in τ modulo 1. Therefore we have a Fourier development

$$F(\tau) = e^{2\pi i\alpha\tau} \sum_{n=-\infty}^{\infty} a_n e^{2\pi i n\tau}, \tag{14}$$

which will converge in the whole upper half-plane if $F(\tau)$ is assumed to be regular there. We then call $F(\tau)$ an "entire" modular form. One further restriction is important: we assume that (14) contains only a finite number of terms with negative exponents, or, as we can say, $F(\tau)$ has a pole at the parabolic point $\tau = i\infty$ measured in the uniformizing variable $e^{2\pi i\tau}$.

In our previous case we had

$$\eta(\tau)^{-1} = e^{2\pi i(23/24)\tau} \sum_{n=-1}^{\infty} p(n+1) e^{2\pi i n\tau}.$$

The partition function appears here therefore as the Fourier coefficients of a modular function of dimension $+1/2$ (since $\eta(\tau)$ in (10) has the dimension $-1/2$). Our method now enables us to determine the coefficients a_n for $n \geqq 0$ from the principal part of the pole at the parabolic point $\tau = i\infty$, that is, from the coefficients a_n with $n < 0$.

I cannot go into details of the application of the Hardy-Ramanujan-Littlewood method. Only one essential point need be mentioned: it is important that r be positive. Indeed, this r is responsible for a term z^r corresponding to $z^{1/2}$ in (6), and, with z approaching zero, helps in a decisive way to ensure convergence.

The result is this:

If $F(\tau)$ is an entire modular form of positive dimension r,

$$F\left(\frac{a\tau + b}{c\tau + d}\right) = \epsilon(a, b, c, d)(-i(c\tau + d))^{-r} F(\tau), \qquad c > 0,$$

$$F(\tau + 1) = e^{2\pi i\alpha} F(\tau), \qquad 0 \leqq \alpha < 1,$$

$$F(\tau) = e^{2\pi i\alpha\tau} \sum_{m=-\mu}^{\infty} a_m e^{2\pi i m\tau},$$

then

$$a_m = \sum_{\nu=1}^{\mu} a_{-\nu} \sum_{k=1}^{\infty} \frac{1}{k} A_{k,\nu}(m) \left(\frac{\nu - \alpha}{m + \alpha}\right)^{(r+1)/2} \cdot I_{r+1}\left((4\pi/k)\left[(\nu - \alpha)(m + \alpha)\right]^{1/2}\right), \qquad m \geqq 0, \tag{15}$$

with

$$A_{k,\nu}(m) = \sum_{0 \leqq h < k,\, (h,k)=1} \epsilon(h', -(hh'+1)/k, k, -h)^{-1} \cdot \exp\left\{-(2\pi i/k)((\nu-\alpha)h' + (m+\alpha)h)\right\}.$$

Before I come to consequences of these equations for coefficients, I wish to mention two important extensions. We have taken into consideration only modular forms invariant with respect to the full modular group

$$\tau' = \frac{a\tau + b}{c\tau + d}, \qquad \begin{vmatrix} a & b \\ c & d \end{vmatrix} = 1.$$

But we can just as well consider a subgroup of the full modular group. It is only necessary that its fundamental region have a finite number of parabolic points. The most important and best known subgroups of this sort are the congruence subgroups, that is, those in which a, b, c, d are subjected to certain congruence restrictions. The principal congruence subgroup modulo p is that which has

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{p}.$$

Zuckerman[13] has carried out this generalization.

Secondly we can overcome to a certain extent the restriction that the dimension must be positive. For this purpose we have to be more careful with our estimates. The sums $A_{k,\nu}(m)$ can immediately be estimated as $|A_{k,\nu}(m)| \leqq \sum 1 = \phi(k)$. The problem of a better estimate has not been fully solved. But in certain simple cases we can get

$$A_{k,\nu}(m) = O(k^{2/3+\epsilon}). \tag{16}$$

These improved estimates were first begun by Kloosterman[14] after whom these sums are now named, and later continued by Esterman,[15] Salié,[16] Davenport,[17] Lehmer,[18] and others. In this way we can easily include the case with dimension zero in our reasonings. A very important function belongs to that class, the modular invariant $J(\tau)$:

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[13] *On the coefficients of certain modular forms belonging to subgroups of the modular group*, Transactions of this Society, vol. 45 (1939), pp. 298–321.

[14] *Asymptotische Formeln für die Fourierkoeffizienten ganzer Modulformen*, Abhandlungen aus dem mathematischen Seminar der Hamburgischen Universität, vol. 5 (1927), pp. 337–352.

[15] *Vereinfachter Beweis eines Satzes von Kloosterman*, Abhandlungen aus dem mathematischen Seminar der Hamburgischen Universität, vol. 7 (1939), pp. 82–98.

[16] *Zur Abschätzung der Fourierkoeffizienten ganzer Modulformen*, Mathematische Zeitschrift, vol. 36 (1933), pp. 263–278.

[17] *On certain exponential sums*, Journal für die reine und angewandte Mathematik, vol. 169 (1933), pp. 158–176.

[18] *On the series for the partition function*, Transactions of this Society, vol. 43 (1938), pp. 271–295.

$$(17)\qquad \begin{gathered} J(\tau) = \frac{g_2^3}{\Delta} = \frac{g_2^3}{g_2^3 - 27g_3^2}, \qquad J\left(\frac{a\tau + b}{c\tau + d}\right) = J(\tau), \\ 12^3 J(\tau) = e^{-2\pi i\tau} + c_0 + \sum_{n=1}^{\infty} c_n e^{2\pi i n\tau}. \end{gathered}$$

The coefficients c_n of this fundamental function can be found by our method[19] as

$$(18)\qquad \begin{gathered} c_n = \frac{2\pi}{n^{1/2}} \sum_{k=1}^{\infty} \frac{1}{k} A_k(n) I_1((4\pi/k)n^{1/2}), \qquad n \geqq 1, \\ A_k(n) = \sum_{h \bmod k,\,(h,k)=1} e^{-(2\pi i/k)(nh+h')}. \end{gathered}$$

It is of interest that a few years ago Petersson[20] found these coefficients (which are integers) by a completely different method, which operates with modular functions of negative dimensions. Indeed, $J'(\tau)$ is of dimension -2 as is readily seen,

$$J'\left(\frac{a\tau + b}{c\tau + d}\right)\frac{1}{(c\tau + d)^2} = J'(\tau),$$

from the differentiation of (17). Modular forms of negative dimension have, however, been studied as far back as Eisenstein, who found the partial fraction series for g_2 and g_3, and later by Poincaré among others. These investigations lie outside the field of our present discussion.

If I have in this way outlined definitions, methods, and direct results, I wish now to survey briefly a few consequences of our theory and some remaining problems. Let me begin with function theoretical consequences. Looking at formula (15), we see that the a_m $(m \geqq 0)$ depend linearly on $a_{-\mu}, \cdots, a_{-1}$. If all these should happen to be equal to zero, our analysis would go through just as well and would lead to $a_m = 0$ for $m \geqq 0$. Hence we have that an entire modular form of positive dimension which is regular also at the parabolic point (or points) of the fundamental region vanishes identically. This is remarkable in so far as it is not true for modular forms of negative dimension, as for example $g_2(1, \tau)$, $g_3(1, \tau)$, $\vartheta_3(0|\tau)$, and $\eta(\tau)$ show.

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[19] Rademacher, *The Fourier coefficients of the modular invariant* $J(\tau)$, American Journal of Mathematics, vol. 60 (1938), pp. 501–512.

[20] *Ueber die Entwicklungskoeffizienten der automorphen Formen*, Acta Mathematica, vol. 58 (1932), pp. 169–215.

These facts have a bearing on the generalized Kloosterman sums, which occur in the estimates. For the case of $\eta(\tau)^{-1}$ we know

$$A_k(n) = O(k^{1/2+\epsilon}), \tag{18a}$$

and for $J(\tau)$ (which leads to the original Kloosterman sums)

$$A_k(n) = O(k^{2/3+\epsilon}n^{1/3}),$$

after Salié and Davenport. It is highly probable that in the latter case a sharper estimate like (18a) will be true. However, our reasonings show that such estimates cannot be expected in all cases. Indeed, if there existed a positive constant α such that for all Kloosterman sums belonging to all modular forms we should have

$$A_{k,\nu}(n) = O(k^{1-\alpha}), \tag{18b}$$

it would be possible to repeat our investigations of the coefficients, for example, for $\eta(\tau)^{\alpha/2}$. Since this function (of dimension $-\alpha/4$) is regular at infinity, the estimate (18b) would imply the absurd consequence that all coefficients of $\eta(\tau)^{\alpha/2}$ vanish identically. An estimate better than the trivial $O(k)$ for the generalized Kloosterman sums can therefore be obtained only under certain special conditions and not uniformly for all real dimensions.

We can use the exact coefficients which we have found again in our Fourier series. We have

$$12^3J(\tau) = e^{-2\pi i\tau} + c_0 + \sum_{n=1}^{\infty} \frac{2\pi}{n^{1/2}} e^{2\pi in\tau} \sum_{k=1}^{\infty} \frac{1}{k} A_k(n) I_1(4\pi n^{1/2}/k). \tag{19}$$

This series, on the other hand, determines the function directly. It is clear that it satisfies the relation $J(\tau+1) = J(\tau)$. But it must also be invariant with respect to all other modular substitutions. Since all of them are generated by $\tau' = \tau+1$, $\tau' = -1/\tau$, it is of interest to show directly $J(-1/\tau) = J(\tau)$ by means of the series, or, in other words, to show that the series defines a modular function. This, indeed, can be done, as I have shown in a recent paper.[21] The proof consists of a rearrangement of terms of a conditionally convergent double series.

I hope that this proof has a prospect of further development for the problem of existence. Up to the present we have only discussed modular forms of positive dimension which are given by certain other definitions (infinite products and so on). But the problem is to construct new ones with given principal parts at the parabolic points. For negative dimensions we have a powerful principle of generation in the Eisenstein-Poincaré series, which have no analogue here for reasons of convergence.

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[21] *The Fourier series and the functional equation of the absolute modular invariant* $J(\tau)$, American Journal of Mathematics, vol. 61 (1939), pp. 237–248.

The exact formulas for the coefficients lead to a sort of analytic continuation which I wish to mention at least briefly. I exemplify this idea with the treatment of $f(x) = 1 + \sum_{n=1}^{\infty} p(n)x^n$. Introducing our value for $p(n)$ we obtain

$$f(x) = 1 + \frac{1}{\pi 2^{1/2}} \sum_{n=1}^{\infty} x^n \sum_{k=1}^{\infty} \sum_{h \bmod k,\,(h,k)=1} \omega_{h,k} e^{-2\pi i h n/k} \cdot k^{1/2} \frac{d}{dn}\left(\frac{\sinh \frac{\pi}{k}\left[\frac{2}{3}(n - 1/24)\right]^{1/2}}{(n - 1/24)^{1/2}} \right)$$

$$(20) \qquad = 1 + \frac{1}{\pi 2^{1/2}} \sum_{k=1}^{\infty} \sum_{h \bmod k,\,(h,k)=1} \omega_{h,k} k^{1/2} \sum_{n=1}^{\infty} (x e^{-2\pi i h/k})^n \cdot \frac{d}{dn}\left(\frac{\sinh \frac{\pi}{k}\left[\frac{2}{3}(n - 1/24)\right]^{1/2}}{(n - 1/24)^{1/2}} \right)$$

$$= 1 + \frac{1}{\pi 2^{1/2}} \sum_{h \bmod k,\,(h,k)=1} \omega_{h,k} k^{1/2} \Phi_k(x e^{-2\pi i h/k}),$$

with

$$(20\text{a}) \quad \Phi_k(z) = \sum_{n=1}^{\infty} \frac{d}{dn}\left(\frac{\sinh \frac{\pi}{k}\left[\frac{2}{3}(n - 1/24)\right]^{1/2}}{(n - 1/24)^{1/2}} \right) z^n = \sum_{n=1}^{\infty} g_k(n) z^n.$$

Here

$$g_k(n) = \frac{d}{dn}\left(\frac{\sinh \frac{\pi}{k}\left[\frac{2}{3}(n - 1/24)\right]^{1/2}}{(n - 1/24)^{1/2}} \right)$$

$$(20\text{b}) \qquad = \frac{d}{dn} \sum_{\nu=0}^{\infty} \frac{[(\pi/k)(2/3)^{1/2}]^{2\nu+1}}{(2\nu + 1)!} (n - 1/24)^{\nu}$$

$$= \sum_{\nu=1}^{\infty} [(\pi/k)(2/3)^{1/2}]^{2\nu+1} \frac{\nu}{(2\nu + 1)!} (n - 1/24)^{\nu-1} .$$

is a transcendental function of order 1/2 in the variable n. Hence by a theorem of Wigert $\Phi_k(z)$, which originally is defined only in $|z|<1$, can be extended over the whole z-plane and has only an isolated essential singularity at $z=1$. The formula (20) therefore represents the splitting up of $f(x)$ into functions each having just one essential singularity. All singularities together form the natural boundary $|x|=1$.

Moreover, the series (20) has now a meaning for $|x|>1$. It turns out that it converges there also. Simple considerations show that, for $|z|>1$, we get

$$\Phi_k(z) = \sum_{m=0}^{\infty} \frac{d}{dm}\left(\frac{\sin \frac{\pi}{k}\left[\frac{2}{3}(m+1/24)\right]^{1/2}}{(m+1/24)^{1/2}}\right) z^{-m}$$

and so, for $|x|>1$, if we introduce $f^*(x)$ instead of $f(x)$,

$$\begin{aligned} f^*(x) &= 1 + \frac{1}{\pi 2^{1/2}} \sum_{h \bmod k,\,(h,k)=1} \omega_{h,k} k^{1/2} \\ &\quad \cdot \sum_{m=0}^{\infty} \frac{d}{dm}\left(\frac{\sin \frac{\pi}{k}\left[\frac{2}{3}(m+1/24)\right]^{1/2}}{(m+1/24)^{1/2}}\right)\cdot\left(xe^{-2\pi i h/k}\right)^{-m} \\ &= 1 + \sum_{m=0}^{\infty} p^*(m) x^{-m}, \end{aligned}$$

let us say. Here

$$\begin{aligned} p^*(m) &= \frac{1}{\pi 2^{1/2}} \sum_{k=1}^{\infty} A_k(-m) k^{1/2} \frac{d}{dm}\left(\frac{\sin \frac{\pi}{k}\left[\frac{2}{3}(m+1/24)\right]^{1/2}.}{(m+1/24)^{1/2}}\right) \\ &= -p(-m), \end{aligned}$$

formally. Now $f^*(x)=0$ identically. Indeed, it turns out that we get $p(0)=1$, as is to be expected, and $p(-m)=0$ for $m \geqq 1$. This was first proved by Petersson[22] in a recent publication, making use again of modular forms of negative dimension. It can, however, also be proved by means of the Hardy-Ramanujan method.

Similar "continuations" beyond the natural boundary can be effectuated for all those power series arising from modular forms of nonnegative dimension. For $J(\tau)$ we obtain, for example,

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[22] *Die linearen Relationen zwischen den ganzen Poincaréschen Reihen von reeller Dimension zur Modulgruppe*, Abhandlungen aus dem mathematischen Seminar der Hamburgischen Universität, vol. 12 (1938), pp. 415–472.

$$J(\tau) = F(e^{2\pi i\tau}) = F(x), \qquad |x| < 1,$$

with $F^*(x) = 5/12$, $(|x| > 1)$.

I come finally to some arithmetical consequences of the method connected mainly with partitions. If we admit only odd integers as parts, then the generating function for the number $q(n)$ of partitions into odd summands is

$$g(x) = 1 + \sum_{n=1}^{\infty} q(n)x^n = \frac{1}{(1-x)(1-x^3)(1-x^5)\cdots}$$
$$= \frac{(1-x^2)(1-x^4)\cdots}{(1-x)(1-x^2)\cdots} = \frac{f(x)}{f(x^2)}.$$

This can be treated by our method, but with the difficulty that we have to deal with modular forms of dimension zero and belonging to a congruence subgroup modulo 2. This has been done by Hua in an unpublished paper. The modular form here would be $\eta(2\tau)/\eta(\tau)$. As a result Hua obtains again a convergent series for $q(n)$, of which Hardy and Ramanujan had given only the first few terms.

There are a number of problems of this sort concerning restricted partitions, which can be attacked by our method. For example Niven[23] has recently determined the number of partitions of a number into summands of the form $6n \pm 1$.

A further consequence which seems more interesting to me is the following: Ramanujan discovered, first empirically, the properties

(a) $p(5m+4) \equiv 0 \pmod{5}$,

(b) $p(7m+5) \equiv 0 \pmod{7}$,

(c) $p(11m+6) \equiv 0 \pmod{11}$.

He was able to prove (a) and (b) easily but remarked that they would be obvious consequences of the identities

$$\sum_{m=0}^{\infty} p(5m+4)x^m = 5\,\frac{\prod(1-x^{5\nu})^5}{\prod(1-x^{\nu})^6},$$
$$\sum_{m=0}^{\infty} p(7m+5)x^m = 7\,\frac{\prod(1-x^{7\nu})^3}{\prod(1-x^{\nu})^4} + 49x\,\frac{\prod(1-x^{7\nu})^7}{\prod(1-x^{\nu})^8}.$$

He did not give a proof of these identities. Proofs were later given by Darling[24] and Mordell[25] and recently again by Watson[26] in a comprehensive paper which covers much more. Now our method can be directly applied to these identities.[27] We know the coefficients of the left member and can express the right member in the same way. All we need to do is then to compare coefficients, which turns out to be not completely trivial since the expressions need a slight transformation.

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[23] *On a certain partition function*, to be published.

Zuckerman[28] has found a new identity of this sort by our method. If we write $\prod(1-x^\nu)=\phi(x)$, then

$$\begin{aligned}\sum_{m=0}^{\infty} p(13m+6)x^m = 11\,\frac{\phi(x^{13})}{\phi(x)^2} &+ 36\cdot 13x\,\frac{\phi(x^{13})^3}{\phi(x)^4} + 38\cdot 13^2x^2\,\frac{\phi(x^{13})^5}{\phi(x)^6}\\ &+ 20\cdot 13^3x^3\,\frac{\phi(x^{13})^7}{\phi(x)^8} + 6\cdot 13^4x^4\,\frac{\phi(x^{13})^9}{\phi(x)^{10}} + 13^5x^5\,\frac{\phi(x^{13})^{11}}{\phi(x)^{12}}\\ &+ 13^5x^6\,\frac{\phi(x^{13})^{13}}{\phi(x)^{14}}\cdot\end{aligned}$$

It is regrettable that this identity does not lead to arithmetical properties of the Ramanujan sort since the factor 13 does not appear as a factor of every term of the right member.

The connection between modular functions and partitions seems to be accidental. Analogues of these concepts may be found in algebraic fields. However they are not connected as in the rational case by formulas of the type (1). Our method could probably be carried over to the discussion of these modular functions, but they would not yield any information concerning these partition functions.

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[24] *Proofs of certain identities and congruences enunciated by S. Ramanujan*, Proceedings of the London Mathematical Society, (2), vol. 19 (1921), pp. 350–372.

[25] *Note on certain modular relations considered by Messrs. Ramanujan, Darling, and Rogers*, Proceedings of the London Mathematical Society, (2), vol. 20 (1922), pp. 408–416.

[26] *Ramanujans Vermutung über Zerfällungsanzahlen*, Journal für die reine und angewandte Mathematik, vol. 179 (1938), pp. 97–128.

[27] Rademacher and Zuckerman, *A new proof of two of Ramanujan's identities*, Annals of Mathematics, (2), vol. 40 (1939), pp. 473–489.

[28] *Identities analogous to Ramanujan's identities involving the partition function*, Duke Mathematical Journal, vol. 5 (1939), pp. 88–110.

NOTE ON COMPLEMENTED MODULAR LATTICES

R. P. DILWORTH

1. **Introduction.** In this note we study those elements of a complemented modular lattice whose complements are unique. We show that these elements are simply the neutral[1] elements of the lattice. It is also shown that an element with unique complement decomposes the lattice into a direct product of sublattices.[2] Hence if the lattice is indecomposable, each element not the null or unit element must have at least two complements. In case the lattice is of finite dimensions these results give a new proof of the Birkhoff-Menger[3] theorem that a complemented modular lattice of finite dimensions is a direct product of projective geometries and a Boolean algebra.

Although the existence of points and divisor-free elements is postulated, no chain conditions are assumed and the proofs are purely combinatorial.

2. **Notation and definitions.** Let $\mathfrak{S}$ denote a closed, complemented, modular lattice with null element z and unit element i. Complements of $a \varepsilon \mathfrak{S}$ will be denoted by $a', a'', \cdots$ and have the property that $(a, a')=i$, $[a, a']=z$. We assume that each $a \neq z$ divides a point p, and that each $b \neq i$ is divisible by a divisor-free element q. It follows that $a \supset b$, $a \neq b$, implies the existence of a point $p \subset [a, b']$ and of a divisor-free element $q \supset (b, a')$ such that $a \supset p$, $b \not\supset p$ and $q \supset b$, $q \not\supset a$. Hence each element of $\mathfrak{S}$ is the union of the points which it divides and the crosscut of its divisor-free divisors.

If S is a set of elements of $\mathfrak{S}$, $u(S)$ $(k(S))$ will denote the union (crosscut) of the elements of S. If $a \varepsilon \mathfrak{S}$ we denote the set of points p (divisor-free elements q) such that $a \supset p$ $(q \supset a)$ by P_a (Q_a). If $\mathfrak{S}$ is the direct product of the sublattices $\mathfrak{S}_1$ and $\mathfrak{S}_2$, we write $\mathfrak{S}=\mathfrak{S}_1 \times \mathfrak{S}_2$.

An element a of $\mathfrak{S}$ is said to be ***neutral*** if $(a, [b, c]) = [(a, b), (a, c)]$ all $b, c \varepsilon \mathfrak{S}$. It is easily shown[4] that a is neutral if and only if $[a, (b, c)] = ([a, b], [a, c])$ all $b, c \varepsilon \mathfrak{S}$.

3. **Properties of elements with unique complements.** We need the following lemmas:

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[1] See §2.

[2] Added in proof: Theorems 3.1 and 4.1 are given by J. von Neumann in his *Continuous Geometrics* (Princeton).

[3] Garrett Birkhoff, Annals of Mathematics, (2), vol. 36 (1935), pp. 743–748; K. Menger, Annals of Mathematics, (2), vol. 37 (1936), pp. 456–481. Professor Birkhoff has informed the author that he has also obtained Theorem 4.2.

[4] O. Ore, Annals of Mathematics, (2), vol. 36 (1935), pp. 406–437.

LEMMA 3.1. *If an element a of $\mathfrak{S}$ has a unique complement a' and $a \supset b$, then $a' = [b', (b, a')]$.*

PROOF. It suffices to show that $[b', (b, a')]$ is a complement of a. Now

$$[a, [b', (b, a')]] = [b', [a, (b, a')]] = [b', (b, [a, a'])] = [b, b'] = z.$$

Also

$$(a, [b', (b, a')]) \supset (b, [b', (b, a')]) = [(b, b'), (b, a')] = (b, a') \supset a'.$$

But $(a, [b', (b, a')]) \supset a$. Hence $(a, [b', (b, a')]) \supset (a, a') = i$.

LEMMA 3.2. *If an element a of $\mathfrak{S}$ has a unique complement a' and if p is a point of $\mathfrak{S}$, then either $a \supset p$ or $a' \supset p$.*

PROOF. Suppose that $a \not\supset p$ and $a' \not\supset p$. Let $p' = [a, (a', p)]$. Then p' is a point and $p' \neq p$ since otherwise $a \supset p$ contrary to assumption. Now let q be a divisor-free element such that $q \not\supset p'$ but $q \supset p$. (Such a q always exists since $p = k(Q_p)$, and if $q \supset p'$ all $q \varepsilon Q_p$, then $p \supset p'$ and $p = p'$, which is impossible.) Now since q is a complement of p', we have $a' = [q, (p', a')]$ by Lemma 3.1. But then

$$\begin{aligned} a' &= [q, (p', a')] = [q, (a', [a, (a', p)])] \\ &= [q, (a', a), (a', p)] = [q, (a', p)] = (p, [q, q']) \supset p, \end{aligned}$$

contradicting $a' \not\supset p$.

THEOREM 3.1. *Let a have a unique complement a' and let $\mathfrak{a}$ $(\mathfrak{a}')$ denote the sublattice of all elements x (y) such that $a \supset x$ $(a' \supset y)$. Then $\mathfrak{S} = \mathfrak{a} \times \mathfrak{a}'$.*

PROOF. We show first that $x = ([a, x], [a', x])$ any $x \varepsilon \mathfrak{S}$. By Lemma 3.2, $x = (u(S), u(S'))$ where S (S') is the set of points divisible by x and a (a'). Hence $S = P_{[a,x]}$ and $S' = P_{[a',x]}$. Thus $x = ([a, x], [a', x])$. If $x = (x_1, x_2)$ where $a \supset x_1$, $a' \supset x_2$, then $x_1 = [a, x]$ and $x_2 = [a', x]$ by the modular condition. Hence every element of $\mathfrak{S}$ is uniquely expressible as a union of elements of $\mathfrak{a}$ and $\mathfrak{a}'$. Clearly $(x, y) = ((x_1, y_1), (x_2, y_2))$ and $[x, y] = ([a, x, y], [a', x, y]) = ([x_1, y_1], [x_2, y_2])$.

If p_1 and p_2 are distinct points, the element (p_1, p_2) is called the "line" joining p_1 and p_2. We have then the following result:

THEOREM 3.2. *An element $a \varepsilon \mathfrak{S}$ has a unique complement a' if and only if every line joining points of a and a' contains no other points.*

PROOF. Let a have a unique complement a'. Then $\mathfrak{S} = \mathfrak{a} \times \mathfrak{a}'$ by

Theorem 3.1. Let $a \supset p$ and $a' \supset p'$. If $(p, p') \supset p_1$ where p_1 is distinct from p and p', then $a \not\supset p_1$ since otherwise $a \supset (p, p_1) \supset p'$ and $[a, a'] \supset p'$, which is impossible. Similarly $a' \not\supset p_1$. But this contradicts Lemma 3.2. On the other hand let a have a second complement a'' and let $a' \supset p'$, $a'' \not\supset p'$, $a \supset p$ and $(a'', p') \supset p$. Then $p_1 = [a'', (p', p)]$ is a point distinct from p and p'. Clearly $(p, p') \supset p_1$.

The relationship of unique complements to projective geometry is clearly brought out by the following corollary to Theorem 3.2:

COROLLARY. *Every line of $\mathfrak{S}$ contains at least three points if and only if each element not the null or unit element of $\mathfrak{S}$ has at least two complements.*

If $\mathfrak{S}$ is of finite dimension, the Birkhoff-Menger representation theorem follows immediately from Theorems 3.1 and 3.2.

4. **Distributivity.** In this section we show that the elements with unique complements are the neutral elements of the lattice. We prove first a lemma:

LEMMA 4.1. *If a has a unique complement a', then a is the unique complement of a'.*

PROOF. Suppose that a' has a second complement a_1. Then $a \supset a_1$ by Theorem 3.1, and hence $a = a_1$ by the modular condition.

THEOREM 4.1. *An element a has a unique complement if and only if it is neutral.*

PROOF. Let a have a unique complement a'. Then $[a, (b, c)] = [a_1, (b_1, c_1)] = [a, (b_1, c_1)] = (b_1, c_1) = ([a, b], [a, c])$ by Theorem 3.1. Hence a is neutral. Conversely let a be neutral and let a' be a complement of a. Suppose that a' has a second complement a_1. Then $a = (a, [a_1, a']) = [(a, a_1), (a, a')] = (a, a_1)$ and hence $a \supset a_1$. Also $a = [a, (a_1, a')] = ([a, a_1], [a, a']) = [a, a_1]$ so that $a_1 \supset a$. Hence $a = a_1$. Thus a is the unique complement of a' and hence a' is the unique complement of a by Lemma 4.1.

It is interesting that Theorem 3.1 gives almost trivially the theorem that *if $\mathfrak{S}$ is finite dimensional and complements are unique, then $\mathfrak{S}$ is a Boolean algebra.*

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DEFINITIONS AND PROPERTIES OF MONOTONE FUNCTIONS[1]

G. BALEY PRICE

1. **Introduction.** We shall consider functions which are monotone in the following sense: $x(t)$ is monotone if and only if $x(t)$ is between $x(t_1)$ and $x(t_2)$ whenever t is between t_1 and t_2. This definition has content only after betweenness has been defined in the domain and range spaces. It is our purpose to consider several definitions of betweenness and the properties of the corresponding monotone functions.

2. **Order-monotone functions.** In this section we shall consider functions $x(t)$ defined on an interval of real numbers with values in a linear partially ordered space X or a partially ordered topological group X in the sense of Kantorovitch[2] [1]. We shall say that $x(t)$ is order-monotone if it is monotone according to the definition in the introduction with betweenness defined as follows: t is between t_1, t_2 if and only if $t_1 \leqq t \leqq t_2$; $x(t)$ is between $x(t_1)$, $x(t_2)$ if and only if $x(t_1) \leqq x(t) \leqq x(t_2)$. Throughout the remainder of this section, $x(t)$ is assumed to be order-monotone unless there is a statement to the contrary.

If $t_1 < t_2 < \cdots$ is a sequence with t_0 as a limit, it can be shown that $\lim x(t_n)$ exists or is infinite; similarly for a monotone decreasing sequence, $t_n' \to t_0$. If $\lim x(t_n) = \lim x(t_n') = x(t_0)$, we say that $x(t)$ is continuous at t_0; otherwise, $x(t)$ is discontinuous there. If $\lim x(t_n)$, $\lim x(t_n')$ both exist but are unequal, we say that $x(t)$ has a jump equal to their difference. It follows from a theorem of Kantorovitch [1, p. 130] that when $x(t)$ is order-monotone on a closed interval, the number of points of discontinuity at which the jump is equal to or greater than a fixed element in X is finite.[3] Furthermore, if $x_1(t)$, $x_2(t)$ are order-monotone in the same sense, their sum $x_1(t)+x_2(t)$ is order-monotone in that sense also.

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[1] Presented to the Society, September 6, 1938.

[2] Let X be a class of elements x which form an additive abelian group. Furthermore, let there be a relation $>$ defined so that for some of the elements $x \in X$ the relation $x > 0$ holds. We assume that this relation satisfies the following postulates: I. The relation $x > 0$ excludes the relation $x = 0$. II. If $x_1 > 0$ and $x_2 > 0$, then $x_1 + x_2 > 0$. III. To each element $x \in X$ there corresponds at least one element $x_1 \in X$ such that $x_1 \geqq 0$ and $x_1 - x \geqq 0$. IV. If $x > 0$ and $\lambda > 0$ is a real number, then $\lambda x > 0$. V. For every set E bounded above there exists a least upper bound sup E.

If I, II, III, V are satisfied in X, it is called a partially ordered topological group. If in addition IV is satisfied in X, it is called a linear partially ordered space.

If $x_2 - x_1 > 0$, we say $x_2 > x_1$. In a partially ordered space it is possible to define an absolute value $|x|$ of x; the absolute value of x is an element in the space and has the formal properties of the absolute value of a real number. For the definition of $|x|$, the definitions and properties of limits, and other results, the reader is referred to the paper of Kantorovitch.

Let $x(t)$ be any function from $a\leqq t\leqq b$ to a partially ordered topological group X; $x(t)$ is not assumed to be order-monotone. Subdivide $a\leqq t\leqq b$ by points t_i such that $a=t_0<t_1<\cdots<t_n=b$, and form the sums $\sum_{i=0}^{n-1}|x(t_{i+1})-x(t_i)|$. If the set of such sums formed for all subdivisions of $a\leqq t\leqq b$ has an upper bound (a finite element of X, and not an improper upper bound of the entire space), we say $x(t)$ has bounded variation. Furthermore,

$$\sup\left\{\sum_{i=0}^{n-1}|x(t_{i+1})-x(t_i)|\right\}$$

is called the total variation of $x(t)$ on $a\leqq t\leqq b$ and is denoted by $V_a^b[x(t)]$. If $x(t)$ is order-monotone, it can be shown that it has total variation equal to $|x(b)-x(a)|$. Let $x(t)$ be a function of bounded variation from $a\leqq t\leqq b$ to a partially ordered topological group X; it can be expressed as the difference of two order-monotone increasing functions which are bounded and greater than zero for each value of t. In particular, we can show that

$$x(t)=\{V_a^t[x(s)]+x_0\}-\{V_a^t[x(s)]+x_0-x(t)\},$$

where x_0 is a properly chosen constant, is a representation of the form stated. Furthermore, it is easily shown that the linear extension of the set of order-monotone functions $x(t)$ from $a\leqq t\leqq b$ to a linear partially ordered space X is the class of functions of bounded variation.

Finally, let $x_1(t)$, $x_2(t)$, $\cdots$ be a sequence of order-monotone functions defined on $a\leqq t\leqq b$ such that $\lim x_n(t)$ exists for each t and equals $x(t)$. Then $x(t)$ likewise is order-monotone.

3. **Metric-monotone functions.** We shall now consider functions $x(t)$ defined on an interval of real numbers with values in a complete metric space X with elements x_1, x_2 and distance function $\rho(x_1, x_2)$. We shall say that $x(t)$ is metric-monotone if it is monotone according to the definition in the introduction with betweenness defined as follows: t is between t_1, t_2 if and only if $t_1\leqq t\leqq t_2$; $x(t)$ is between $x(t_1)$,

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[3] Let nx denote the sum of n elements x; Kantorovitch has shown that the set of elements nx, $(n=1, 2, \cdots)$, is unbounded (see postulate V). If $x(t)$ is order-monotone increasing, for example, on $a\leqq t\leqq b$ and has n points of discontinuity at which the jump exceeds x, then $x(b)-x(a)>nx$, from which the result stated follows.

$x(t_2)$ if and only if $\rho[x(t_1), x(t_2)] = \rho[x(t_1), x(t)] + \rho[x(t), x(t_2)]$ (compare a definition of betweenness given by Menger [**2**, pp. 77–81]).

If $x(t)$ is metric-monotone on an interval, its discontinuities are either simple discontinuities or infinities, and the latter can occur only at the ends of the interval of definition. Let the jump at a simple discontinuity be defined as the distance between the two limits. If the interval of definition of a metric-monotone function is closed, the number of discontinuitities at which the jump exceeds a given constant is finite, and the set of all discontinuities is denumerable. The variation of a function $x(t)$ with values in a metric space is defined as a number in the usual way. The decomposition of a function $x(t)$ of bounded variation with values in a normed vector space into a sum of metric-monotone functions is lacking; it appears that this decomposition depends essentially on relations of order. If $x_1(t), x_2(t), \cdots$ is a sequence of metric-monotone functions defined on $a \leqq t \leqq b$, and if $\lim x_n(t)$ exists for each t and equals $x(t)$, then $x(t)$ is also metric monotone.

4. **Sphere-monotone functions.** In this section we shall consider functions $x(t)$ defined on an interval with values in a metric space X. We shall say that $x(t)$ is sphere-monotone if and only if it is monotone according to the definition in the introduction with betweenness defined as follows: t is between t_1, t_2 if and only if $t_1 \leqq t \leqq t_2$; $x(t)$ is between $x(t_1)$, $x(t_2)$ if and only if every open sphere which contains $x(t_1)$, $x(t_2)$ also contains $x(t)$.

The discontinuities of a sphere-monotone function are of certain simple types only. Let $t_1 < t_2 < \cdots$ be a sequence of numbers on the interval of definition of $x(t)$, and let $t_n \to t_0$. Then either $x(t_n)$ tends to a limit and $x(t)$ has a limit at t_0 on the left equal to it, or the set $\{x(t_n)\}$ has no point of accumulation in X. If t_0 is an interior point of the interval of definition of $x(t)$, similar results hold at t_0 on the right. Thus besides simple discontinuities and infinities, a sphere-monotone function has a third type of discontinuity that may be described as wandering. As in all previous cases, an infinity can occur only at an end point of an open interval on which $x(t)$ is defined. There are no wandering discontinuities at least when the set of values of $x(t)$ is compact.

In all previous cases, a monotone function has been a function of bounded variation; a sphere-monotone function, however, need not have bounded variation (see an example given by Graves [**3**, p. 166]). If $x_1(t), x_2(t), \cdots$ is a sequence of sphere-monotone functions, and if $\lim x_n(t)$ exists for each t and equals $x(t)$, then $x(t)$ is also sphere-monotone.

A function $x(t)$ defined on an interval with values in a metric space X is said to be measurable if and only if for every open sphere S in X the set $E_t[x(t) \varepsilon S]$ is measurable. Every sphere-monotone function $x(t)$ is measurable in this sense. Bochner [4, pp. 263–265] has also given a definition of measurable function. A necessary and sufficient condition that $x(t)$ be measurable in the sense of Bochner is that it be measurable, and that the set of values of $x(t)$, omitting at most those corresponding to t on a set of measure zero, be separable. The function in the example of the last paragraph (Graves [3, p. 166]) is sphere-monotone and hence measurable, but since the distance between each two of its values is one, it does not satisfy the condition for Bochner measurability. There are thus measurable functions which are not measurable in the sense of Bochner.

5. **Other monotone functions.** It is possible to define still other types of monotone functions; all that is required is that betweenness be defined in both the domain and range spaces of $x(t)$. In particular, let $x(t)$ be a function defined on an "interval" $a \leqq t \leqq b$ of a partially ordered topological group with values in a space of the same kind. The number of points of discontinuity at which the jump exceeds a given constant may not be finite; with this exception, the results given in §2 hold for this monotone function also. It should be observed that the Arzelà real-valued monotone functions of several variables are special cases of these monotone functions.

In a linear space, the following definition of betweenness is a natural one: x is between x_1, x_2 if and only if $x = \theta x_1 + (1-\theta)x_2$, where θ is a real number such that $0 \leqq \theta \leqq 1$. Corresponding to this definition of betweenness we have linear-monotone functions; apparently they have not been studied previously. An account of them will be given elsewhere.

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FACTORIZATION AND SIGNATURES OF LORENTZ MATRICES[1]

WALLACE GIVENS

The linear transformation $x \rightarrow \bar{x} = Lx$ (that is, $\bar{x}^i = L_j^i x^j$ with $i, j = 1, \cdots, n$) is called Lorentz if $(\bar{x}, \bar{x}) = (x, x)$, where

$$(1) \quad (x, x) = (x^1)^2 + (x^2)^2 + \cdots + (x^t)^2 - (x^{t+1})^2 - \cdots - (x^n)^2.$$

The matrices of all such transformations make up a group which we shall call the Lorentz group $\mathfrak{L}_{t,n-t}$. For $t=3$, $n=4$ it is usually called the extended Lorentz group.

In this paper we give extremely elementary proofs of two theorems. The first theorem has to do with the expression of a Lorentz matrix as a product of Lorentz matrices of simple type. For this it is sufficient that the elements of our matrices be chosen from a field of characteristic different from two. When the field is that of complex numbers, the signature of the quadratic form (1) is unimportant. The second theorem describes certain subgroups of the Lorentz group and for it we need an ordered (hence not a finite) field.

Each vector v for which $(v, v) \neq 0$ determines a transformation T_v with the equations

$$(2) \qquad \bar{x} = x - 2\frac{(v, x)}{(v, v)} v,$$

where

$$(3) \qquad (v, x) = v^1 x^1 + \cdots + v^t x^t - v^{t+1} x^{t+1} - \cdots - v^n x^n.$$

It is easy to verify the following:

(i) T_v is Lorentz.

(ii) The result of performing T_v twice is the identity.

(iii) Every multiple of v is a solution of the equations $\bar{x} = -x$, and conversely every solution is a multiple of v.

(iv) If $(v, y) = 0$ then $\bar{y} = y$ and conversely.

(v) For $v = e_j \equiv (0, \cdots, 0, 1, 0, \cdots, 0)$, where the one is in the jth place, T_{e_j} has the equations

$$(4) \quad \bar{x}^1 = x^1, \cdots, \bar{x}^{j-1} = x^{j-1}, \bar{x}^j = -x^j, \bar{x}^{j+1} = x^{j+1}, \cdots, \bar{x}^n = x^n.$$

(vi) If $v^1 = 0$, T_v has the equations

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[1] Presented to the Society, November 25, 1938, under the title *Signatures of Lorentz matrices.*

$$(5) \qquad \bar{x}^1 = x^1, \qquad \bar{x}^\alpha = M_\beta^\alpha x^\beta, \qquad \alpha, \beta = 2, \cdots, n,$$

where $\|M_\beta^\alpha\|$ is a matrix of $\mathfrak{L}_{t-1,n-t}$.

We call a transformation given by (2) a symmetry. Properties (iii) and (iv) show that each symmetry is a reflection in its associated hyperplane $(v, x) = 0$ in the same way that (4) is a reflection in the coordinate hyperplane $x^i = 0$.

THEOREM 1. *Every Lorentz transformation can be expressed as a product of symmetries.*[3]

PROOF. We use induction on n, the number of variables. Evidently the theorem is true for $n = 1$. Moreover, by property (vi), the assumption that the theorem is true for $n-1$ variables enables us to conclude that it is also true for transformations of the form (5) on n variables.

We distinguish two cases according as $L_1^1 \neq 1$ or $L_1^1 = 1$ for the given Lorentz matrix L of order n.

In the first case the vector $v = e_1 - Le_1$ determines a symmetry T_v (with matrix I_v), for $(v, v) = (e_1, e_1) - 2(e_1, Le_1) + (Le_1, Le_1) = \pm 2(1 - L_1^1) \neq 0$, the minus sign occurring if $t = 0$ in (1). Moreover, a computation with (2) shows that $I_v Le_1 = e_1$; that is, $I_v L \equiv L^*$ is of such a form that $L_1^{*i} = 0$ for $i > 1$ and $L_1^{*1} = 1$. Since L^* is Lorentz, $(e_1, e_j) = (L^*e_1, L^*e_j) = (e_1, L^*e_j) = \pm L_j^{*1}$, the minus sign occurring if $t = 0$, and hence $L_j^{*1} = 0$ for $j > 1$. The transformation associated with L^* is therefore of the form (5). By the hypothesis of the induction we have $I_v L = L^* = I_1 I_2 \cdots I_r$, where $I_1, \cdots,$ and I_r are the matrices of suitably chosen symmetries. Since $(I_v)^2 = 1$, $L = I_v I_1 \cdots I_r$ and we have the factorization of L.

In the second case we set $w = e_1 + Le_1$ and observe that $(w, w) = \pm 2(1 + L_1^1) \neq 0$. We find that $I_w Le_1 = -e_1$ and hence $L^* \equiv I_{e_1} I_w L$ (cf. (4)) is of the form (5). As before, $L^* = I_1 \cdots I_r$ and $L = I_w I_{e_1} I_1 \cdots I_r$. This completes the proof of Theorem 1.

We now assume that the field over which we are working is ordered and suppose that $0 < t < n$, where t is the number of plus signs in (1).

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[3] P. F. Smith, *On the linear transformations of a quadratic form into itself*, Transactions of this Society, vol. 6 (1905), pp. 1–16, working with the complex field, proved this theorem and showed that n symmetries are sufficient. We referred to this paper in presenting this theorem to the Society on November 25, 1938. Since then we have seen a proof by E. Cartan (*Leçons sur la Théorie des Spineurs* I, pp. 13–17) that n symmetries suffice in the real case as well. Our proof is shorter and applies for all fields not of characteristic two, but gives a factorization into not more than $2n$ symmetries. L. Autonne in a long memoir, *Sur la décomposition d'une substitution linéaire* . . . , Annales de la Université de Lyons, 1903, solved the problem in the real definite case.

It will be evident that the following considerations do not give significant results in the definite cases $t=0$ and $t=n$.

Breaking up a Lorentz matrix L into partial matrices

$$(6) \qquad L = \begin{Vmatrix} A & B \\ C & D \end{Vmatrix},$$

where A has t rows and columns, we find that the determinants $|A|$ and $|D|$ are different from zero. For, if $|A|=0$, there exists a vector $v=(v^1, \cdots, v^t, 0, \cdots, 0) \neq 0$ such that $Lv=\bar{v}=(0, \cdots, 0, \bar{v}^{t+1}, \cdots, \bar{v}^n)$. This is impossible for $(v, v)=(\bar{v}, \bar{v})$ and $(v, v)>0$ while $(\bar{v}, \bar{v})<0$. Similarly, $|D| \neq 0$.

We may therefore define for every Lorentz matrix the quantity

$$(7) \qquad \sigma_+(L) = \begin{cases} +1 & \text{if } |A| > 0, \\ -1 & \text{if } |A| < 0. \end{cases}$$

Similarly, $\sigma_-(L)$ is defined in terms of $|D|$. Brauer and Weyl[8] have called these quantities the temporal and spatial signatures of L.

THEOREM 2. *For every pair of Lorentz matrices L and M,*

$$(8) \qquad \sigma_+(L)\sigma_+(M) = \sigma_+(LM).$$

It will be sufficient to prove (8) only in the form

$$(9) \qquad \sigma_+(L)\sigma_+(I) = \sigma_+(LI),$$

where I is the matrix of a symmetry. For, M can be written as a product of matrices of symmetries, and successive applications of (9) will give (8). Similar statements of course apply to $\sigma_-(L)$.

We shall need the following lemma:

LEMMA. *If $|A| \neq 0$ and rank $N=1$, then $|A-N|=|A|[1-\text{trace}(A^{-1}N)]$.*

PROOF. Since rank of $(A^{-1}N)$ is 1, $|\lambda A-N|=|A||\lambda 1-A^{-1}N| = |A|[\lambda^n-\lambda^{n-1}\text{ trace }(A^{-1}N)]$, and we may take $\lambda=1$.

Let us call A, in (6), the spatial minor of L. Then the spatial minor of the matrix I_v of (2) is

$$(10) \qquad 1_t - \frac{2}{(v, v)} \|v^\alpha v^\beta\|,$$

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[8] *Spinors in n dimensions*, American Journal of Mathematics, vol. 57 (1935), pp. 425–449. This paper includes a proof of our Theorem 2 (for the real field), using, however, the spinor representation of the Lorentz group. The theorem is so much simpler than the representation theory that it seems desirable to give a direct algebraic proof.

where 1_t is the unit matrix of order t and α and β have the range $1, \cdots, t$. If we write $(v^1)^2+\cdots+(v^t)^2=(v, v)_+$ and $(v^{t+1})^2+\cdots+(v^n)^2=(v, v)_-$ so that $(v, v)=(v, v)_+-(v, v)_-$, the determinant of (10) is, by the lemma,

$$1-\frac{2(v, v)_+}{(v, v)}=-\frac{(v, v)_+ + (v, v)_-}{(v, v)}. \tag{11}$$

Since $(v, v)_+ + (v, v)_- > 0$,

$$\sigma_+(I_v)=\begin{cases}-1 & \text{if } (v, v)>0,\\ +1 & \text{if } (v, v)<0.\end{cases} \tag{12}$$

Breaking up L as in (6) and applying the lemma to the determinant of the spatial minor of LI_v, we find it to be

$$-\frac{|A|}{(v, v)}Q(v), \tag{13}$$

where Q is a quadratic form in the variables $v^1, \cdots, v^n$ with the matrix

$$S=\left\|\begin{matrix}1_t & A^{-1}B\\ (A^{-1}B)' & 1_{n-t}\end{matrix}\right\|, \tag{14}$$

where the prime denotes the tranpose matrix and the exponent -1 the inverse. To complete the proof it will be sufficient to show that $Q(v)$ is positive definite, for then the sign of $\sigma_+(LI_v)$ will be that of $-|A|/(v, v)$, and this, in virtue of (7) and (12), has the same sign as $\sigma_+(L)\sigma_+(I_v)$.

We prove that S is positive definite by showing that the matrix

$$T=\left\|\begin{matrix}1_t & -A^{-1}BD'\\ 0 & D'\end{matrix}\right\| \tag{15}$$

transforms S into the identity; that is, $T'ST=1$. To verify this we have recourse to the relations

$$A'A-C'C=1_t, \qquad B'A-D'C=0, \tag{16}$$

and

$$B'B-D'D=-1_{n-t},$$

which are necessary and sufficient in order that (6) be Lorentz. Computing $T'ST$ we find that it is 1_n if $K\equiv DD'-DB'(A')^{-1}A^{-1}BD'$ is equal to 1_{n-t}. This is proved by the computation

$$\begin{aligned} K &= DD' - DB'[1_t - (A')^{-1}C'CA^{-1}]BD' \\ &= DD' - DB'[1_t - BD^{-1}(D')^{-1}B']BD' \\ &= DD' - DB'BD' + DB'BD^{-1}(D')^{-1}B'BD' \\ &= DD' - D(D'D - 1_{n-t})D' + D(D'D - 1_{n-t})D^{-1}(D')^{-1}(D'D - 1_{n-t})D' \\ &= 2DD' - (DD')^2 + (DD' - 1_{n-t})(DD' - 1_{n-t}) \\ &= 1_{n-t}. \end{aligned}$$

Since $\sigma_+(1_n) = \sigma_-(1_n) = 1$, Theorem 2 gives $\sigma_+(L)\sigma_+(L^{-1}) = +1$ or $\sigma_+(L) = \sigma_+(L^{-1})$. Similarly, $\sigma_-(L) = \sigma_-(L^{-1})$. It follows that the set $\mathfrak{L}^{++}_{t,n-t}$ of Lorentz matrices satisfying the conditions

$$\mathfrak{L}^{++}_{t,n-t}: \qquad \sigma_+(L) = +1, \quad \sigma_-(L) = +1, \tag{17}$$

constitute a subgroup of $\mathfrak{L}_{t,n-t}$. Similarly, the Lorentz group contains the subgroups

$$\mathfrak{L}^{\pm+}_{t,n-t}: \qquad \sigma_+(L) = \pm 1, \quad \sigma_-(L) = +1, \tag{18}$$

$$\mathfrak{L}^{+\pm}_{t,n-t}: \qquad \sigma_+(L) = +1, \quad \sigma_-(L) = \pm 1. \tag{19}$$

The Lorentz matrices of determinant plus one of course also form a subgroup, say $\mathfrak{L}^{+}_{t,n-t}$. The matrices of this subgroup have the property that either

$$\mathfrak{L}^{+}_{t,n-t}: \quad \sigma_+(L) = \sigma_-(L) = +1 \text{ or } \sigma_+(L) = \sigma_-(L) = -1, \tag{20}$$

and every Lorentz matrix satisfying either of these conditions has determinant plus one. To prove this, we use (16) to get

$$\left\| \begin{matrix} A' & -C' \\ 0 & 1_{n-t} \end{matrix} \right\| \cdot \left\| \begin{matrix} A & B \\ C & D \end{matrix} \right\| = \left\| \begin{matrix} 1_t & 0 \\ C & D \end{matrix} \right\| . \tag{21}$$

Taking determinants gives $|A|\,|L| = |D|$. Since $|L| = \pm 1$, we have in all cases

$$|L| = \sigma_+(L)\sigma_-(L), \tag{22}$$

which justifies (20).

CORNELL UNIVERSITY

ON THE ABSOLUTE SUMMABILITY OF FOURIER SERIES. II[1]

W. C. RANDELS

Bosanquet[2] has developed conditions for the absolute summability, $C(\alpha)$ of a Fourier series. An immediate consequence of these conditions is that absolute summability is a local property for $\alpha>1$. The purpose of this paper is to show by means of an example that absolute summability is not a local property for[3] $\alpha=1$.

A Fourier series is absolutely summable $C(1)$ if $\sum_{m=1}^{\infty}|\sigma_m-\sigma_{m-1}|<\infty$. We have

$$\sigma_m-\sigma_{m-1}=\frac{1}{m+1}\sum_{n=0}^{m}S_n-\frac{1}{m}\sum_{n=0}^{m-1}S_n=\frac{\sigma_{m-1}}{m+1}-\frac{S_m}{m+1},$$

and, if $f(x)$ vanishes for $x\leqq x_0>0$, then at $x=0$,

$$\begin{aligned}\sum_{m=0}^{\infty}\frac{|\sigma_{m-1}|}{m+1}&\leqq\frac{1}{4\pi}\sum_{m=0}^{\infty}\frac{1}{m+1}\int_0^{\pi}|\phi(t)|\frac{\sin^2(mt/2)}{m\sin^2 t/2}dt\\&\leqq\frac{1}{4\pi}\sum_{m=0}^{\infty}\frac{1}{m^2\sin^2 x_0/2}\int_0^{\pi}|\phi(t)|\,dt\\&<\infty,\end{aligned}$$

so that it is only necessary to consider

$$\sum_{m=0}^{\infty}|A_m(f,x)|=\sum_{m=0}^{\infty}\frac{1}{2\pi(m+1)}\left|\int_0^{\pi}\phi(f,t)\frac{\sin(m+1/2)t}{\sin t/2}dt\right|.$$

We define the functions

$$f_n(x)=\begin{cases}(n+1)|\sin x/2|, & \pi-\pi/3(n+1)\leqq|x|\leqq\pi,\\ 0, \text{ elsewhere}.\end{cases}$$

Then at $x=0$, $\phi(f_n,t)=2f_n(t)$ and, since

$$(-1)^m\sin(m+1/2)t\geqq 1/2,\qquad \pi-\pi/3(m+1/2)\leqq t\leqq\pi,$$

we have

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[1] Presented to the Society, April 8, 1939.

[2] L. S. Bosanquet, *The absolute summability of a Fourier series*, Proceedings of the London Mathematical Society, (2), vol. 41 (1936), pp. 517–528.

[3] This result has recently been proved by a different method by Bosanquet and Kestleman, *The absolute convergence of series and integrals*, ibid., vol. 45 (1939), pp. 88–97.

$$(1)\quad |A_m(f_n, 0)| > \frac{1}{4\pi(m+1)}\cdot\frac{(n+1)\pi}{3(n+1)} = \frac{1}{12(m+1)}, \qquad n > m,$$

and

$$A_m(f_n, 0) = \frac{(n+1)}{2\pi(m+1)(m+1/2)}\{\cos(m+1/2)t_0 - \cos(m+1/2)\pi\}, \quad t_0 = \pi - \pi/3(n+1),$$

so that

$$(2)\qquad |A_m(f_n, 0)| < (n+1)/\pi m^2.$$

By (1) it is possible to choose a sequence of integers $\{n_i\}$ in such a way that

$$(3)\qquad \sum_{m=0}^{n_i-1} |A_m(f_{n_i}, 0)| > \sum_{m=0}^{n_i-1} \frac{1}{24(m+1)} > 2^i.$$

The function $f(x)$ is then defined as $f(x) = \sum_{i=0}^{\infty} 2^{-i} f_{n_i}(x)$. For this function

$$\begin{aligned}\sum_{m=0}^{\infty} |A_m(f, 0)| &= \sum_{m=0}^{\infty}\left|\sum_{i=0}^{\infty} 2^{-i} A_m(f_{n_i}, 0)\right| \\ &\geqq \sum_{m=0}^{\infty}\sum_{n_i>m} 2^{-i}|A_m(f_{n_i}, 0)| \\ &\quad - \sum_{m=0}^{\infty}\sum_{n_i\leqq m} 2^{-i}|A_m(f_{n_i}, 0)|,\end{aligned}$$

since $|A_m(f_{n_i}, 0)| = (-1)^m A_m(f_{n_i}, 0)$ for $n_i > m$. By (2)

$$\begin{aligned}\sum_{m=0}^{\infty}\sum_{n_i\leqq m} 2^{-i}|A_m(f_{n_i}, 0)| &= \sum_{i=0}^{\infty} 2^{-i}\sum_{m\geqq n_i} |A_m(f_{n_i}, 0)| \\ &\leqq \sum_{i=0}^{\infty} 2^{-i}\sum_{m\geqq n_i}\frac{n_i}{m^2} < 2\sum_{i=0}^{\infty} 2^{-i} = 4,\end{aligned}$$

and by (3)

$$\begin{aligned}\sum_{m=0}^{\infty}\sum_{n_i>m} 2^{-i}|A_m(f_{n_i}, 0)| &= \sum_{i=0}^{\infty} 2^{-i}\sum_{m<n_i} |A_m(f_{n_i}, 0)| \\ &> \sum_{i=0}^{\infty} 2^{-i}2^i = \infty,\end{aligned}$$

so that $\sum_{m=}^{\infty} |A_m(f, 0)| = \infty$. It remains to show that $f(x) \subset L$ which is easily seen since

$$\int_{-\pi}^{\pi} | f(x) | \, dx = \sum_{i=0}^{\infty} 2^{-i} \int_{-\pi}^{\pi} | f_{n_i}(x) | \, dx$$

$$\leqq \sum_{i=0}^{\infty} 2^{-i} 2(n+1) \frac{\pi}{3(n+1)} < \infty .$$

We notice that, since this function vanishes in the neighborhood of the origin, it coincides with a function having an absolutely summable Fourier series in the neighborhood of the origin, and therefore absolute summability $C(1)$ is not a local property.

UNIVERSITY OF OKLAHOMA

COMPLETE REDUCIBILITY OF FORMS[1]

RUFUS OLDENBURGER

1. **Introduction.** We shall say that F is a form in *r essential variables* with respect to a field K if F cannot be brought by means of a nonsingular linear transformation in the field K to a form with less variables. Let F be a form of degree p written as $a_{ij\cdots k}x_ix_j \cdots x_k$, $(i, j, \cdots, k = 1, 2, \cdots, n)$. We arrange the coefficients of F in a matrix A whose n^{p-1} columns are of the form

$$\left\| \begin{matrix} a_{1j\cdots k} \\ a_{2j\cdots k} \\ \cdot \\ \cdot \\ \cdot \\ a_{nj\cdots k} \end{matrix} \right\| .$$

The index i is associated with the rows of A and the $p-1$ indices $j, \cdots, k$ are associated with the columns of A. We assume that the coefficients in F are so chosen that A is *symmetric* in the sense that the value of an element $a_{ij\cdots k}$ is unchanged under permutation of the subscripts. It can be shown[2] that F is *a form in r essential variables if and only if the rank of A is r.*

A form F is said to be *completely reducible* in a field K if F splits in K into a product of linear factors. Hočevar proved[3] that a form F with no repeated factors is completely reducible in the *complex field* if and only if F divides each third order minor of its Hessian. It is obvious that this result of Hočevar is not valid for each field of numbers. A form F of degree p is said to be ***nonsingular with respect to*** K if F can be written as a linear combination of pth powers of linearly independent linear forms with coefficients in K. Elsewhere the author proved[4] that the Hessian of a cubic form nonsingular with respect to K factors in K into linearly independent factors. For a field K with characteristic different from 2, 3, and element $a\neq 0$, the product $ax_1x_2 \cdots x_n$ in n independent variables $x_1, x_2, \cdots, x_n$ is the Hessian of the nonsingular cubic $C(a)$ where $6C(a)=ax_1^3+x_2^3+ \cdots +x_n^3$. We let $L_i=b_{ij}y_j$, $(i, j=1, 2, \cdots, n)$, denote an arbitrary set of n linear forms linearly independent with respect to K. We write Δ for the determinant of the matrix (b_{ij}). Applying the nonsingular linear transformation $x_1=L_1$, $x_2=L_2, \cdots, x_n=L_n$ to $C(1/\Delta^2)$ we obtain a form whose Hessian is $L_1L_2 \cdots L_n$. Hence each product of linearly independent linear forms is the Hessian of a nonsingular cubic form. We have proved the theorem which follows.

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[1] Presented to the Society, April 7, 1939.

[2] Oldenburger, *Composition and rank of n-way matrices and multilinear forms*, Annals of Mathematics, (2), vol. 35 (1934), pp. 622–653.

THEOREM 1. *Let K be a field with characteristic not 2 or 3. A form F of degree n in n essential variables is completely reducible in K if and only if F can be written as the Hessian of a cubic form nonsingular with respect to K.*

If F of Theorem 1 is completely reducible and F is the Hessian of a nonsingular cubic form C, then $C=a_iL_i^3$, $(i=1, 2, \cdots, n)$, and the linear forms $L_1, \cdots, L_n$ are the factors of F.

The utility of Theorem 1 is limited by the fact that the problem of representability of a form as the Hessian of a nonsingular cubic is unsolved. In the present paper we prove that a certain integer, called "minimal number," associated with a completely reducible form F of degree n is not greater than 2^{n-1}. From this property we obtain a *solution of the problem of complete reducibility of cubic forms* for a field K with characteristic not 2 or 3.

2. Minimal numbers and representations.

Elsewhere[5] the author proved that each symmetric form F of degree p can be written for a field K of order p or more as a linear combination of pth powers of linear forms. Such a linear combination with ρ terms we call a ρ-representation of F with respect to K. A representation of F with respect to K with a minimum number of terms is called a ***minimal representation*** of F with respect to K. The number of terms in such a representation we term the ***minimal number*** of F with respect to K, and denote this number by $m(F)$.

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[3] Hočevar, *Sur les formes décomposables en facteurs linéaires*, Comptes Rendus de l'Académie des Sciences, vol. 138 (1904), pp. 745–747.

[4] Oldenburger, *Rational equivalence of a form to a sum of pth powers*, Transactions of this Society, vol. 44 (1938), pp. 219–249; in particular p. 233.

[5] Oldenburger, *Representation and equivalence of forms*, Proceedings of the National Academy of Sciences, vol. 24 (1938), pp. 193–198.

THEOREM 2. *Let K be a field with characteristic[6] greater than n, and let F be a form of degree n completely reducible in K. Then $m(F) \leqq 2^{n-1}$.*

We write $\rho = 2^{n-1}$. Let $L_1, L_2, \cdots, L_\rho$ denote the different possible forms of the type $(x_1 \pm x_2 \pm x_3 \pm \cdots \pm x_n)$. Let $k_i = +1$ if L_i contains an even number of minus coefficients, and $k_i = -1$ if L_i contains an odd number of such coefficients. We consider the sum

$$\frac{1}{2^{n-1}}\left[\sum_{i=1}^{\rho} k_i L_i^n\right]. \tag{1}$$

Simple computation reveals that (1) is symmetric in the x's. We consider a product $\Pi = \pm x_1^a \cdots x_r^d$ of degree n with $r<n$ arising from the expansion of a term $k_iL_i^n$ in (1). Corresponding to the linear form L_i there is a unique form L_j, $(j \neq i)$, in (1) obtainable from L_i by changing the sign of x_n in L_i. Then $k_j = -k_i$. The product $P = x_1^a \cdots x_r^d$ arising from $k_jL_j^n$ has a coefficient the negative of that in Π. Thus the terms involving the product P, where these terms arise from $k_iL_i^p$ and $k_jL_j^p$, vanish. It follows that the coefficient of P in (1) is zero. It is obvious from the choice of the k_i that the coefficient of $x_1 \cdots x_n$ in (1) is $n!$, whence (1) is a ρ-representation of $n!x_1 \cdots x_n$. Since a completely reducible form F in n essential variables is equivalent to this product under nonsingular linear transformations in K, and *the minimal number is an invariant* of F, we have $m(F) \leqq 2^{n-1}$. It follows that if $F = L_1L_2 \cdots L_n$ where $L_1, L_2, \cdots, L_n$ are linearly dedependent linear forms, $m(F) \leqq 2^{n-1}$.

3. **Complete reducibility of cubic forms.** In the present section we assume that the underlying field K is such that when two forms are equal to each other for all values of the variables in K, corresponding coefficients of these forms are equal. In the case of cubic forms this means that the characteristic of K is different from 2, 3. Evidently, *a completely reducible cubic form is a form in not more than* 3 *essential variables*. Since the minimal number of a binary cubic is not greater

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[6] Restricting the characteristic of K to be greater than n is equivalent to assuming that the characteristic of K does not divide $n!$.

than 3, the theory of complete reducibility of binary forms may readily be supplied by the reader. In what follows we therefore consider cubic forms in 3 essential variables only.

THEOREM 3. *A cubic form F in 3 essential variables is completely reducible with respect to a field K if and only if*

(a) *The minimal number of F with respect to K is 4.*

(b) *If $\mu_i R_i^3$ is a minimal representation of F with respect to K, then roots $\sigma_i = (\mu_i/\mu_1)^{1/3}$ are in K for each i, and for some choice of the roots σ_i we have $\sum_{i=1}^{4}\sigma_i R_i \equiv 0$.*

A completely reducible cubic form F in 3 essential variables is equivalent under nonsingular linear transformations in the given field to $T=xyz$. By Theorem 2, $m(T)\leqq 4$. If $m(T)$ were 3, the form T would be equivalent to $C=au^3+bv^3+cw^3$ in the variables u, v, w, whence T is nonsingular. For T to be nonsingular it is necessary and sufficient[7] that the Hessian H of T split into linearly independent linear factors L, M, and N and under reduction of H to canonical form uvw, T transform covariantly to a reduced form C. Since the Hessian of T is already in canonical form and $T\not\equiv ax^3+by^3+cz^3$, we have $m(T)\neq 3$. The minimal number of a form cannot be less than the number of essential variables in the form, whence $m(T)=4$. Hence $m(F)=4$.

It is easy to prove that if $\sum_{i=1}^{r}\lambda_i(x+\alpha_i y)^n\equiv 0$, where the λ's are not zero, and $r\leqq n+1$, the α's can be grouped into sets $S_1, S_2, \cdots, S_\rho$ each of order 2 at least, where the α's in each set are equal; and if we let λ_i correspond to α_i, the sum of the λ's corresponding to the α's in S_i vanishes for each i in the range $1, 2, \cdots, \rho$. From this it follows rather immediately that if

$$6xyz \equiv \sum_{i=1}^{4}\lambda_i(x+\alpha_i y+\beta_i z)^3, \tag{2}$$

the right member of (2) is

$$\begin{aligned}(1/4ab)\{&(x+ay+bz)^3-(x+ay-bz)^3\\ &-(x-ay+bz)^3+(x-ay-bz)^3\}.\end{aligned} \tag{3}$$

It is readily verified that the coefficients of x, y, and z in a representation $\lambda_i L_i^3$, $(i=1, 2, 3, 4)$, of $6xyz$ are different from zero, whence any representation of $6xyz$ can be written as the right member of (2). Thus each representation of $6xyz$ is of the type (3), and (3) is a representation of $6xyz$ for each choice of a, b not zero. Since the representations of each form equivalent to $6xyz$ under nonsingular transformations can be obtained from $6xyz$ by substitutions $x=L$, $y=M$, $z=N$ where L, M, N are linearly independent linear forms, *a cubic form F in 3 essential variables is completely reducible if and only if each 4-representation of F is of the type*

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[7] Oldenburger, *Rational equivalence of a form to a sum of pth powers*, Transactions of this Society, vol. 44 (1938), pp. 219–249.

$$
\begin{aligned}
k\{(L+aM+bN)^3-(L+aM-bN)^3 & \\
-(L-aM+bN)^3+(L-aM-bN)^3\}, & \qquad (4)
\end{aligned}
$$

where k, a, $b\neq 0$, and L, M, N are linearly independent.

Let a cubic form F in three essential variable be given by a minimal representation $\sum_{i=1}^{4}\mu_i R_i^3$. If F is completely reducible, the forms $\mu_i R_i^3$ (i not summed; $i=1, 2, 3, 4$) are identically equal to the forms $\pm k[L\pm aM\pm bN]^3$ in some order and for some choice of k, a, b, L, M, and N. Then there exists an element c in the given field K such that $\rho_i=(c\mu_i)^{1/3}$ are in K, and an ordering of the values of i so that

$$
\begin{aligned}
L+aM+bN &\equiv \rho_1 R_1, & L+aM-bN &\equiv -\rho_2 R_2, \\
L-aM+bN &\equiv -\rho_3 R_3, & L-aM-bN &\equiv \rho_4 R_4.
\end{aligned}
\qquad (5)
$$

Equations (5) are solvable for L, M, N if and only if $\sum_{i=1}^{4}\rho_i R_i\equiv 0$. Evidently there exists an element c in K so that roots ρ_i in K exist if and only if there exist roots $\sigma_i=(\mu_i/\mu_1)^{1/3}$ in K. Theorem 3 is now proved.

ARMOUR INSTITUTE OF TECHNOLOGY

TRANSFORMATION THEORY OF INTEGRABLE DOUBLE-SERIES OF LINEAL ELEMENTS[1]

EDWARD KASNER AND JOHN DE CICCO

1. Introduction. We shall begin by giving some of the concepts in the geometry of lineal elements of three-space. By a *lineal element*, we mean simply a point and a direction (of a straight line) through that point. A lineal element may be defined by the five numbers (x, y, z, p, q) where (x, y, z) are the cartesian coordinates of the point and $(1, p, q)$ are the direction numbers of the direction of the element. We shall call the ∞^5 lineal elements of space a *plenum*.

A set of ∞^1 lineal elements of space is termed a *series*. A series may usually be pictured as the configuration obtained by attaching to each point of a curve a single direction (usually not the tangent direction). However, there is a degenerate type of series called the *point-union*, or *conical-union*. This consists of ∞^1 lineal elements through a fixed point. Thus the lineal elements of a point-union form a cone with the fixed point as vertex. A series may be given, in general, by the four equations $y=y(x)$, $z=z(x)$, $p=p(x)$, $q=q(x)$ where y, z, p, q are arbitrary functions of x only.

A *union* is a series which either consists of a curve together with the tangent directions of the curve, or as a special case, is a point-union. *The necessary and sufficient conditions that the series* $y=y(x)$, $z=z(x)$, $p=p(x)$, $q=q(x)$ *be a union are* $dy/dx=p$, $dz/dx=q$.

A collection of ∞^2 lineal elements of space is called a *double-series*. A double-series is the gemetric object obtained by attaching to each point of a surface a single direction (usually not a tangent direction). But there is a degenerate type of double-series, called the *point-union double-series*. This consists of ∞^1 point-unions. That is, if at each point of a curve, we construct ∞^1 lineal elements (a cone of lineal elements), the resulting configuration is a point-union double-series. There is another type of point-union double-series which is called a *star* or *bundle*. In this case, all the point-unions are concurrent so that a star or bundle consists of all the ∞^2 lineal elements through a given point. A double-series may be given by the three equations $z=z(x, y)$, $p=p(x, y)$, $q=q(x, y)$, where z, p, q are arbitrary functions of (x, y) only.

If we can find ∞^1 unions whose lineal elements coincide exactly with the ∞^2 lineal elements of a given double-series, then the double-series is said to be an *integrable double-series*. This means that an integrable double-series is either the configuration obtained by constructing at each point of a surface a single tangent direction to the surface, or as a special case, a point-union double-series. (The unions of a point-union double-series are the ∞^1 point-unions which determine the double-series.) *The necessary and sufficient condition that the double-series* $z=z(x, y)$, $p=p(x, y)$, $q=q(x, y)$ *be an integrable double-series is that the functions* z, p, q *of* (x, y) *satisfy the partial differential equation of first order* $q=z_x+pz_y$.

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[1] Presented to the Society, February 24, 1940.

A set of ∞^3 lineal elements of space is said to be a *field*. A field is the geometric object obtained by constructing at each point of space a single direction. However, there is a degenerate type of field, called the *point-union field*. This consists of ∞^2 point-unions. If at each point of a surface we construct ∞^1 lineal elements (a cone of lineal elements), the resulting object is a point-union field. Another type of point-union field is the *bundle-field*. This consists of ∞^1 stars or bundles. That is, if at each point of a curve, we construct all the ∞^2 lineal elements through the point, the result is a bundle-field. Of course, each of the ∞^1 bundles of a bundle-field contains ∞^1 concurrent point-unions so that a bundle-field consists of ∞^2 point-unions. A field may be given by the two equations $p=p(x, y, z)$, $q=q(x, y, z)$ where p and q are arbitrary functions of (x, y, z) only.

A field corresponds to two ordinary differential equations of the first order in the unknowns y and z considered as functions of x only. In other words, a field is determined by the two ordinary equations of first order $dy/dx=p(x, y, z)$, $dz/dx=q(x, y, z)$, where p and q are arbitrary functions of (x, y, z) only. By this, we find that *every field is an integrable field*. That is, by integrating our equations, we can always find ∞^2 unions whose lineal elements coincide exactly with the ∞^3 lineal elements of the given field. (Thus the unions of a point-union field are the ∞^2 point-unions which determine the field.)

A collection of ∞^4 lineal elements of space is termed a *conical-field*. A conical-field is the configuration obtained by constructing at each point of space ∞^1 lineal elements (a cone of lineal elements). But there is a degenerate type of conical-field called the *point-union conical-field*. This is the result of constructing at each point of a surface the bundle of ∞^2 lineal elements through that point. Thus a point-union conical-field consists of ∞^2 bundles. Since a bundle consists of ∞^1 concurrent point-unions, we find that a point-union conical-field consists of ∞^3 point-unions. A conical-field may be given by the equation $q=q(x, y, z, p)$ where q is an arbitrary function of (x, y, z, p) only.

A conical-field corresponds to a single Monge equation of the first order in the unknowns y and z considered as functions of x only.

In other words, a conical-field is determined by the Monge equation of first order $dz/dx=q(x, y, z, dy/dx)$. *Thus every conical-field is an inintegrable conical-field.* That is, by setting y equal to an arbitrary function of x and upon integrating the resulting ordinary differential equation of the first order in the unknown z, we can find ∞^∞ unions, whose lineal elements coincide exactly with the ∞^4 lineal elements of the given conical-field. Also every conical-field contains ∞^3 point-unions. (Thus the unions of a point-union conical-field are the ∞^∞ curves of the basic surface of the point-union conical-field and also the ∞^3 point-unions of the ∞^2 bundles which determine the conical-field.)

A transformation between the lineal elements of space is called a *lineal element transformation.* Any lineal element transformation is given by the equations

$$
\begin{gathered}
X = X(x, y, z, p, q), \quad Y = Y(x, y, z, p, q), \quad Z = Z(x, y, z, p, q), \\
P = P(x, y, z, p, q), \quad Q = Q(x, y, z, p, q),
\end{gathered}
\tag{1}
$$

where X, Y, Z, P, Q are arbitrary functions of (x, y, z, p, q) only such that the jacobian is not identically zero:

$$
J = \begin{vmatrix}
X_q & X_p & X_z & X_y & X_x \\
Y_q & Y_p & Y_z & Y_y & Y_x \\
Z_q & Z_p & Z_z & Z_y & Z_x \\
P_q & P_p & P_z & P_y & P_x \\
Q_q & Q_p & Q_z & Q_y & Q_x
\end{vmatrix} \not\equiv 0.
\tag{2}
$$

A lineal element transformation converts every series into a series, but it does not in general convert every union into a union. In the paper by Kasner, *General transformation theory of differential elements,* American Journal of Mathematics, vol. 32 (1904), pp. 391–401, the following fundamental theorem is proved.

All lineal element transformations of spaces may be classified with respect to the number of unions preserved into three distinct types:

TYPE 1. *The infinite group of extended point transformations. Any lineal element transformation which converts every union into a union must be an extended point transformation of the form*

$$
\begin{gathered}
X = X(x, y, z), \qquad Y = Y(x, y, z), \qquad Z = Z(x, y, z), \\
P = \frac{Y_x + pY_y + qY_z}{X_x + pX_y + qX_z}, \qquad Q = \frac{Z_x + pZ_y + qZ_z}{X_x + pX_y + qX_z},
\end{gathered}
\tag{3}
$$

where X, Y, Z are arbitrary functions of (x, y, z) only.

TYPE 2. *The set of lineal element transformations, not extended point transformations, fulfilling the conditions*

$$\tag{4} \begin{aligned} \frac{Z_q - QX_q}{Y_q - PX_q} &= \frac{Z_p - QX_p}{Y_p - PX_p} \\ &= \frac{Z_x + pZ_y + qZ_z - Q(X_x + pX_y + qX_z)}{Y_x + pY_y + qY_z - P(X_x + pX_y + qX_z)} . \end{aligned}$$

Any lineal element transformation of this set converts ∞^∞ *unions into unions, but not every union into a union. That is, the family of unions preserved involves an arbitrary function. This family is defined by a Monge equation of the second order of the form*

$$\tag{5} E\frac{d^2y}{dx^2} + F\frac{d^2z}{dx^2} + G = 0,$$

where E, F, G *are arbitrary functions of* $(x, y, z, dy/dx, dz/dx)$ *only. Any Monge equation of the above form is characterized by the possession of the Meusnier property.* (See Kasner, *The inverse of Meusnier's theorem*, this Bulletin, vol. 14 (1908), pp. 461–465.)

TYPE 3. *The set of lineal element transformations not fulfilling the conditions* (4). *Any lineal element transformation of this set converts exactly* ∞^4 *unions into unions.* (This is the most general case.)

A field is said to be a *normal field* if we can construct for the doubly infinite family of integral unions of the field ∞^1 orthogonal surfaces. Of course not every field is a normal field. Under an arbitrary lineal element transformation, every field is converted into a field, but every normal field is not converted in general into a normal field. In the paper by Kasner, *Lineal element transformations of space for which normal congruences of curves are converted into normal congruences*, Duke Mathematical Journal, vol. 5 (1939), pp. 72–83, the following result is derived.

The infinite group of lineal element transformations which convert every normal field into a normal field is isomorphic with the Lie group of contact transformations of surface elements.

Next we come to the *new problem* of the present paper. This will complete the list of fundamental problems in the theory of transformations of differential elements.

A lineal element transformation converts every double-series into a double series, but it does not in general transform every integrable double-series into an integrable double-series. The problem of this paper is *to find the group of lineal element transformations which carry every integrable double-series into an integrable double-series.* Our result is that *our group of lineal element transformations is the group of extended point transformations.* Thus the solution of our problem gives us a new characteristic property of the group of extended point transformations. (See the paper by Kasner and De Cicco, *Curvature element transformations which preserve integrable fields*, Proceedings of the National Academy of Sciences, vol. 25 (1939), pp. 104–111, where an analogous characterization of the group of extended contact transformations of lineal elements of the plane is given.)

We note that we do *not* assume that the individual unions in the family of ∞^1 unions are converted into unions. But from our proof it *does* result that if *every* integrable double-series becomes such a double-series, then the individual unions are actually converted into individual unions, and *therefore*, the result is our extended point transformation.

2. Integrable double-series into integrable double-series.

In this section, we shall state and prove our result.

FUNDAMENTAL THEOREM. *The group of lineal element transformations which convert every integrable double-series into an integrable double-series is the group of extended point transformations.*

The sufficiency of our theorem is obvious. The remainder of the paper is concerned with the proof of the necessity of the theorem.

Let the lineal element transformation T as given by equations (1) carry every integrable double-series into an integrable double-series. Then there are ∞^∞ integrable double-series of the form $z=z(x, y)$, $p=p(x, y)$, $q=q(x, y)$ which are carried into the integrable double-series of the form $Z=Z(X, Y)$, $P=P(X, Y)$, $Q=Q(X, Y)$.

Therefore for these integrable double-series, the condition $q=z_x+pz_y$ must be transformed by T into the condition

$$Q = Z_x + PZ_y. \tag{6}$$

The equation (6) may be written, where $z_x=q-pz_y$, in the form

$$\begin{aligned} &Q\begin{vmatrix} X_x + X_z z_x + X_p p_x + X_q q_x & X_y + X_z z_y + X_p p_y + X_q q_y \\ Y_x + Y_z z_x + Y_p p_x + Y_q q_x & Y_y + Y_z z_y + Y_p p_y + Y_q q_y \end{vmatrix} \\ (7)\quad &= \begin{vmatrix} Z_x + Z_z z_x + Z_p p_x + Z_q q_x & Z_y + Z_z z_y + Z_p p_y + Z_q q_y \\ Y_x + Y_z z_x + Y_p p_x + Y_q q_x & Y_y + Y_z z_y + Y_p p_y + Y_q q_y \end{vmatrix} \\ &+ P\begin{vmatrix} X_x + X_z z_x + X_p p_x + X_q q_x & X_y + X_z z_y + X_p p_y + X_q q_y \\ Z_x + Z_z z_x + Z_p p_x + Z_q q_x & Z_y + Z_z z_y + Z_p p_y + Z_q q_y \end{vmatrix}. \end{aligned}$$

The equation (7) must be an identity in z_y, p_x, p_y, q_x, q_y, after we eliminate z_x. Upon setting the coefficients equal to zero and simplifying, we obtain the *nine* partial differential equations of first order

$$
\begin{vmatrix} Z_q - QX_q & Z_p - QX_p \\ Y_q - PX_q & Y_p - PX_p \end{vmatrix} = 0, \quad
\begin{vmatrix} Z_p - QX_p & Z_z - QX_z \\ Y_p - PX_p & Y_z - PX_z \end{vmatrix} = 0,
$$

$$
\begin{vmatrix} Z_p - QX_p & Z_y - QX_y \\ Y_p - PX_p & Y_y - PX_y \end{vmatrix} = 0, \quad
\begin{vmatrix} Z_q - QX_q & Z_z - QX_z \\ Y_q - PX_q & Y_z - PX_z \end{vmatrix} = 0,
$$

$$
\begin{vmatrix} Z_q - QX_q & Z_y - QX_y \\ Y_q - PX_q & Y_y - PX_y \end{vmatrix} = 0, \quad
\begin{vmatrix} Z_q - QX_q & Z_x - QX_x \\ Y_q - PX_q & Y_x - PX_x \end{vmatrix} = 0,
$$

$$
(8) \qquad \begin{vmatrix} Z_p - QX_p & Z_x - QX_x \\ Y_p - PX_p & Y_x - PX_x \end{vmatrix} = 0,
$$

$$
\begin{vmatrix} Z_x + pZ_y - Q(X_x + pX_y) & Z_z - QX_z \\ Y_x + pY_y - P(X_x + pX_y) & Y_z - PX_z \end{vmatrix} = 0,
$$

$$
\begin{vmatrix} Z_x + qZ_z - Q(X_x + qX_z) & Z_y - QX_y \\ Y_x + qY_z - P(X_x + qX_z) & Y_y - PX_y \end{vmatrix} = 0.
$$

In the rest of the paper, we shall concern ourselves with the complete solution of these equations. We shall show that the only possible solution for the five functions X, Y, Z, P, Q with nonvanishing jacobian is furnished by the equations (3), and therefore our transformation is an extended point transformation.

We shall prove that

$$
(9) \quad Z_q - QX_q = 0, \; Z_p - QX_p = 0, \; Y_q - PX_q = 0, \; Y_p - PX_p = 0.
$$

Let us suppose that this is not the case. That is, let us assume that at least one of the quantities of the left-hand sides of the preceding equations is different from zero. From the first seven of the equations (8), we obtain

$$
(10) \qquad \begin{aligned} \frac{Z_q - QX_q}{Y_q - PX_q} = \frac{Z_p - QX_p}{Y_p - PX_p} &= \frac{Z_z - QX_z}{Y_z - PX_z} \\ &= \frac{Z_y - QX_y}{Y_y - PX_y} = \frac{Z_x - QX_x}{Y_x - PX_x}. \end{aligned}
$$

But these equations make the jacobian (2) of our transformation zero. This contradiction establishes the equations (9).

Since the equations (9) are satisfied, we find that the first seven of equations (8) are zero. Therefore the equations (8) by means of equations (9) reduce to the *two* partial differential equations of first order

$$(11)\qquad \begin{aligned} &\begin{vmatrix} Z_x + pZ_y - Q(X_x + pX_y) & Z_z - QX_z \\ Y_x + pY_y - P(X_x + pX_y) & Y_z - PX_z \end{vmatrix} = 0, \\ &\begin{vmatrix} Z_x + qZ_z - Q(X_x + qX_z) & Z_y - QX_y \\ Y_x + qY_z - P(X_x + qX_z) & Y_y - PX_y \end{vmatrix} = 0. \end{aligned}$$

We shall prove that

$$(12)\qquad \begin{aligned} Z_x + pZ_y + qZ_z - Q(X_x + pX_y + qX_z) &= 0, \\ Y_x + pY_y + qY_z - P(X_x + pX_y + qX_z) &= 0. \end{aligned}$$

Let us suppose that this is not the case. That is, let us assume that at least one of the quantities of the left-hand sides of the preceding equations is not zero. From equations (11), we find after simplification

$$(13)\qquad \frac{Z_z - QX_z}{Y_z - PX_z} = \frac{Z_y - QX_y}{Y_y - PX_y} = \frac{Z_x - QX_x}{Y_x - PX_x}.$$

These equations together with the equations (9) make the jacobian of our transformation zero. This contradiction establishes the equations (12).

By means of (9) and (12), we have found that the complete solution of equations (8) is given by the equations

$$(14)\qquad \begin{gathered} Z_x + pZ_y + qZ_z = Q(X_x + pX_y + qX_z), \\ Y_x + pY_y + qY_z = P(X_x + pX_y + qX_z), \\ Z_p = QX_p, \qquad Z_q = QX_q, \qquad Y_p = PX_p, \qquad Y_q = PX_q. \end{gathered}$$

We shall show that $X_q = X_p = 0$. Let us suppose that at least one of these quantities is not zero. Upon taking the partial derivative with respect to q of the third and the fifth of the preceding equations, we obtain

$$(15)\qquad Z_{pq} = QX_{pq} + Q_qX_p, \qquad Y_{pq} = PX_{pq} + P_qX_p.$$

Upon taking the partial derivative with respect to p of the fourth and the sixth of the equations (14), we find

$$(16)\qquad Z_{pq} = QX_{pq} + Q_pX_q, \qquad Y_{pq} = PX_{pq} + P_pX_q.$$

Subtracting the corresponding equations of (15) and (16) and comparing the results with the last four of equations (14), we obtain

$$(17)\qquad Q_q/Q_p = P_q/P_p = Z_q/Z_p = Y_q/Y_p = X_q/X_p.$$

But these equations make the jacobian (2) of our transformation zero. This contradiction proves our assertion that $X_p = X_q = 0$.

Since $X_q = X_p = 0$, we find from equations (14) that $Z_q = Z_p = Y_q = Y_p = X_q = X_p = 0$. Therefore the equations (14) become the equations (3). This proves that our lineal element transformation is an extended point transformation. The proof of our fundamental theorem is complete.

3. **General lineal element transformations.** A lineal element transformation, not an extended point transformation, will convert *some* particular integrable double-series into integrable double-series. This subject will be considered elsewhere.

COLUMBIA UNIVERSITY AND
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NOTE ON THE CURVATURE OF ORTHOGONAL TRAJECTORIES OF LEVEL CURVES OF GREEN'S FUNCTION. III

J. L. WALSH

If R is a simply connected region of the extended (x, y)-plane with boundary B, and if Green's function $G(x, y)$ exists for R with pole in the finite point O, we denote by $\{T\}$ the set of orthogonal trajectories to the level curves $G(x, y) = \log r$, $0 < r < 1$, in R. The totality of circles each osculating at O one of the set of curves T passing through O consists precisely of the set of circles through O and through another fixed point D, depending on O and R. The point D is called the *conjugate of O with respect to R*. The term "circle" is here and below used in the extended sense, to include straight line, unless otherwise noted.

In a series of papers[1] the writer has recently studied some of the properties of the point D, notably (in M and I) that every circle through O and D cuts B; and (in II) that every point exterior to R is the conjugate with respect to R of a suitably chosen point O interior to R. It is the object of the present note to establish still further properties of the conjugate, namely the following theorems:

THEOREM 1. *Let R be a simply connected region of the w-plane with at least two boundary points. Let C be a circle intersecting the boundary of R in the finite point α. Let C be the boundary of a circular region R' (a half-plane, interior of a circle, or exterior of a circle, boundary points not included) whose points lie in R, and let T be a triangle contained in R', with the vertex α. Let the sequence of points $w_1, w_2, \cdots$ lie in T and approach α. Then the conjugate of w_n with respect to R also approaches α as n becomes infinite.*

THEOREM 2. *Let R be a simply connected region of the w-plane with at least two boundary points, and let w_0 be a boundary point of R. Then there exists a sequence of points $w_1, w_2, \cdots$ in R approaching w_0 such that the conjugate of w_n with respect to R approaches w_0.*

THEOREM 3. *There exists a limited Jordan region R of the w-plane, a boundary point w_0 of R, and a sequence $w_1, w_2, \cdots$ of points of R approaching w_0 such that the conjugate of w_n with respect to R becomes infinite with n.*

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[1] American Mathematical Monthly, vol. 42 (1935), pp. 1–17; Proceedings of the National Academy of Sciences, vol. 23 (1937), pp. 166–169; this Bulletin, vol. 44 (1938), pp. 520–523. We shall refer to these papers as M, I, II respectively.

Let $w=f(z)$ map $|z|<1$ onto R. Then the conjugate of w with respect to R is the point

$$w' = f(z) - \frac{2[f'(z)]^2(1 - z\bar{z})}{f''(z)(1 - z\bar{z}) - 2\bar{z}f'(z)};$$

this is the function whose continuity (as a function of w) we are studying.

1. **Proof of Lemma** 1. As a step in the proof of Theorem 1 we establish a preliminary result.

LEMMA 1. *Let the finite point O lie interior to the simply connected region R whose Green's function with respect to O exists, and let each point of the boundary B of R lie on or to the right of the vertical line Oy. Let at least one point A of B lie to the right of Oy. Then the conjugate D of O with respect to R is finite and lies to the right*[2] *of Oy.*

In the proof it is convenient to transform O to infinity by a linear transformation of the complex variable, so the boundary B of R can now be supposed finite. We preserve the original notation. The conjugate of O is a finite point D, the conformal center of gravity[3] of R. There exists a line L with (for suitable orientation of the plane) each point of the boundary B of R lying on or to the right of L, and with at least one point A of B lying to the right of L. For definiteness let A be the point (or one of the points) of B farthest from L.

If no boundary point of R lies on L, a suitably chosen level curve of Green's function $G(x, y)$ for R with pole in O lies to the right of L. The center of gravity of this level curve with positive mass distribution defined by the equation

$$d\sigma = \frac{\partial G}{\partial n} ds$$

is the conjugate of O with respect to R, and lies to the right of L. The conclusion of Lemma 1 follows in this case. Henceforth we suppose at least one point of B to lie on L.

An arbitrary cut in the region R (assumed to lie in the w-plane) corresponds under the conformal map $\zeta=\zeta(w)$ of R onto $|\zeta|>1$ to a cut in the region[4] $|\zeta|>1$. Let a circle K be drawn with center A and radius one-half the distance from A to L. A certain simply connected region S consisting of points of R and containing some points of R lying to the right of A, lies interior to K and is bounded entirely by points of B and of K. That is to say, we define S as the set of points P each of which can be joined to a fixed point H in K and R to the right of A, by a Jordan arc PH not cutting B or K. The boundary of S contains at least one arc Q of K whose end-points are points of B but which otherwise lies in R; all such arcs Q (by hypothesis belonging to the boundary of S) are finite or denumerably infinite in number. Under the conformal map each arc Q corresponds to a Jordan arc Q_1 in the region $|\zeta|>1$ whose end-points are distinct points of $|\zeta|=1$, whether or not the two end-points of Q are distinct.[5] Under the conformal map the region S corresponds to a region S_1 bounded in part by arcs Q_1. We now prove that *the boundary of S_1 must contain at least one arc of the circle* $|\zeta|=1$.

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[2] It follows (see II) that O cannot be a maximum or even a critical point of the function $r(a)$.

[3] See M and I.

[4] See for instance Carathéodory, *Conformal Representation* (Cambridge Tract, no. 28), p. 83.

Two distinct arcs Q_1 may have a terminal point T_1 in common, but if so the corresponding arcs Q have a terminal point T in common, and a suitably chosen neighborhood of T contains no boundary points of S interior to K. For there exists a sequence of Jordan arcs J_1 in S_1 with their end-points on the respective arcs Q_1; and the end-points of the J_1 approach T_1. Let a point M_1 of S_1, whose transform is the point M of S, be separated by each of these Jordan arcs plus the boundary of S_1 from the neighborhood of T_1 in S_1. The corresponding sequence of Jordan arcs J in S have their end-points on the respective arcs Q, and each arc J together with the boundary of S separates M from the neighborhood of at least one boundary point of S. If each of the Jordan arcs J together with the boundary of S separates a neighborhood in S of more than one boundary point from M, let E denote the totality of such boundary points. No point of E not an end-point of an arc Q can be a limit point of a sequence of boundary points of S not lying on E; for two nonintersecting arcs J together with the two subarcs of arcs Q intercepted between their terminal points bound a Jordan region in S which contains then no point of the boundary of S in its interior; of course S is simply connected, and no boundary point of S lies exterior to K. If E contains more than one point, it contains either a point interior to K or a point on K not an end-point of arcs Q (hence lying on an arc of K belonging to B), and in any case contains a point N with the property that the mapping function $\zeta=\zeta(w)$ for R is continuous in some neighborhood of N in S and constant (equal to the value of ζ in the point T_1) at every boundary point of S in the neighborhood, which is impossible.[6] It follows that E consists of a single point T; a suitably chosen neighborhood of T contains no boundary point of S interior to K.

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[5] Carathéodory, op. cit., p. 85.

[6] Carathéodory, op. cit., p. 82.

The method of proof just given shows also that a point T_1 of $|\zeta|=1$, not an end-point of an arc of $|\zeta|=1$ which is part of the boundary of S_1 but which is a limit point of an infinity of Jordan arcs Q_1, corresponds in the sense just considered to a single point T on B and K; a suitably chosen neighborhood of T contains no boundary point of S interior to K. The point A is not an interior point of S, so it follows that there exists some point of K which is a limit point of boundary points of S interior to K, whence not all points of B on K can be points T as considered above, and the boundary of S_1 must contain at least one arc of the circle $|\zeta|=1$, say of length $\sigma>0$, as we desired to prove.

Denote by d the distance from A to L. Choose r, $(0<r<1)$, so near unity that an arc of the Jordan curve B_r: $G(x, y)=\log r$ interior to K and S corresponds to an arc of $|\zeta|=1/r$ in S_1 of length greater than $\sigma/2r$, which is possible by inspection of the situation in the ζ-plane; and also choose r so near to unity that no point of the curve B_r lies at a greater distance from B than some positive number

$$d' < \sigma d/(8\pi - 2\sigma).$$

Then no point of the curve B_r lies at a distance greater than d' to the left of L. The weight of that part of B_r interior to K is at least $\sigma/2$. The center of gravity of the weighted locus B_r lies to the right of L at a distance not less than

$$\left\{\frac{\sigma}{2}\frac{d}{2} - \left(2\pi - \frac{\sigma}{2}\right)d'\right\}\Big/2\pi,$$

which is positive. That is to say, the center of gravity of the weighted locus B_r (that is, the conformal center of gravity of R, or the conjugate of O with respect to R) lies to the right of L; so Lemma 1 follows.

2. **Proof of Theorem 1.** For convenience suppose the plane oriented so that α lies at the right-hand extremity of a horizontal diameter of C. Let R_n be the region of the w-plane obtained from R by stretching without rotation so that final and initial lengths are in the ratio $1:r(w_n)$, with the new transform of w_n corresponding to the origin. Here (as in II) we use $r(w_n)$ to denote the ***inner radius*** of R with respect to the point w_n. From each sequence R_n can be extracted a subsequence of regions converging to a kernel in the sense of Carathéodory. For the corresponding functions $w=f_n(z)$ which map $|z|<1$ onto R_n are univalent with $f_n(0)=0$ and $|f_n'(0)|=1$, hence

form a normal family in $|z|<1$. Let us suppose a subsequence of the R_n to converge to the kernel R_0, with the corresponding mapping functions $w=f_n(z)$ converging continuously in $|z|<1$ to the mapping function $w=f_0(z)$ for R_0, with $f_0(0)=0$, $|f_0'(0)|=1$. The kernel R_0 is obviously not a single point. Then we have for the subsequence

$$\lim_{n\to\infty} -2\frac{[f_n'(0)]^2}{f_n''(0)} = -2\frac{[f_0'(0)]^2}{f_0''(0)}, \tag{1}$$

or in other words (compare I) the conjugate of the origin with respect to R_n approaches the conjugate of the origin with respect to R_0.

We shall henceforth assume that the sequence R_n itself converges to R_0 and that (1) is valid. This assumption involves no loss of generality, for if the conclusion of the theorem is false it is false for a sequence of indices such that R_n converges to a kernel and such that (1) is valid for that sequence.

We shall now assume that the angle (argument, amplitude) $\angle(\alpha-w_n)$ approaches a limit γ as n becomes infinite; this assumption likewise involves no loss of generality, for if the theorem is false there exists a subsequence of the w_n for which this angle approaches a limit and whose conjugates do not approach α. Of course γ is not the angle $\pi/2$ or $-\pi/2$.

Denote by t_n the boundary point of R nearest to w_n; then t_n lies on or exterior to C:

$$|w_n - t_n| \geqq a - |w_n|, \tag{2}$$

where C is the circle $|w|=a$; here we assume also that C is a proper circle in whose interior T lies. This assumption involves no loss of generality. The inequality

$$|w_n - t_n| \leqq r(w_n) \tag{3}$$

is well known.[7] Moreover for suitably chosen $\delta>0$ independent of n we have

$$\delta|w_n-\alpha| \leqq a-|w_n| \tag{4}$$

by virtue of the fact that w_n lies in the triangle T. We obviously have, by (2), (3), and (4),

$$|w_n-\alpha|/r(w_n) \leqq 1/\delta.$$

There is no boundary point of R_n or of R_0 closer to O than the dis-

---

[7] See for instance Pólya-Szegö, *Aufgaben und Lehrsätze aus der Analysis*, vol. 2, Berlin, 1925, p. 21, exercise 121.

tance 1/4, by the distortion theorem (Verzerrungssatz). But by the inequality just established there is a boundary point of R_n (namely the transform of the point α) on the line segment from O in the direction $\angle(\alpha - w_n)$ of length $1/\delta$, so there is a boundary point of the kernel R_0 on the line segment from O in the direction γ of length $1/\delta$. Consequently R_0 has at least one boundary point which lies *to the right* of the vertical line through O.

It follows from Lemma 1 that the conjugate of O with respect to R_0 is not infinite, follows from (1) that the conjugate of O with respect to R_n approaches a finite limit, and follows from the relation $r(w_n) \to 0$ (proved in II) that the distance from w_n to its conjugate with respect to R (which is $r(w_n)$ multiplied by the distance from O to its conjugate with respect to R_n) approaches zero. Theorem 1 is established.

It is clear from the proof of Theorem 1 that the essential part of the proof is that every kernel R_0 of a subsequence of the regions R_n should have the property that the conjugate of the origin with respect to R_0 shall be finite; whenever the sequence R_n has this property, the condition $w_n \to \alpha$ implies that the conjugate of w_n with respect to R also approaches α.

3. **Proof of Theorem** 2. Thanks to Theorem 1, the proof of Theorem 2 is extremely simple. For definiteness suppose w_0 finite (the contrary case can be reduced to this by a linear transformation). Let ω_k be a point of R whose distance from w_0 is less than $1/2^k$. Let γ_k be the circle whose center is ω_k and whose radius is the distance from ω_k to the boundary of R; this distance is necessarily less than $1/2^k$. The interior of this circle lies in R, but at least one point α of the circumference is a boundary point of R. A triangle T satisfying the requirements of Theorem 1 can be constructed. By Theorem 1 there exists a point w_k interior to γ_k whose conjugate with respect to R lies in the circle $|w - w_0| = 1/2^{k-1}$, and we have $|w_k - w_0| \leq 1/2^{k-1}$. The sequence w_k satisfies the requirements of Theorem 2.

4. **Proof of Theorem** 3. The region R whose existence is asserted in Theorem 3 is now to be constructed by the following method.[8] We consider the sequence of circles C_n in the $w(=x+iy)$-plane, each tangent to its predecessor:

$$C_0\colon\quad x^2 + y^2 = 1, \qquad C_1\colon\quad (x - 3/2)^2 + y^2 = 1/4,$$

$$C_n\colon\quad (x - 3(2^n - 1)/2^n)^2 + y^2 = 1/2^{2n}.$$

---

[8] This method is quite similar to one employed for a somewhat different purpose in a forthcoming paper by Seidel and Walsh, of which an abstract was published in Proceedings of the National Academy of Sciences, vol. 24 (1938), pp. 337–340.

The interiors of these circles are to be joined by canals in the neighborhoods of the points

$$1,\ 2,\ 5/2,\ \cdots,\ \alpha_n{:}\,3-1/2^{n-1},\ \cdots$$

so as to form a Jordan region; the banks of the canals are short segments of lines parallel to the axes of reals, a short distance above and below that axis near the points α_n of tangency of successive circles; the arcs of the original circles intercepted between those lines are to be suppressed, and the banks of the canals are to be terminated by successive circumferences. We proceed to indicate in more detail the construction of the canals.

Let the canal in the neighborhood of the point α_n be bounded by the lines $y=\pm\delta_n$, with $\delta_n>0$. Let R be the region formed by the interior of the circles C_n together with the canals, and let the point $w=0$ correspond to the point $\zeta=0$ when R is mapped onto $|\zeta|<1$ with directions at the origins unaltered. When δ_1 approaches zero, the *kernel* of the variable region R in the sense of Carathéodory (op. cit., p. 75) is precisely the interior of C_0, *independently of the values of* δ_2, δ_3, $\cdots$. The function $w=f_0(\zeta)$ which maps $|\zeta|<1$ onto R with $f_0(0)=0$, $f_0'(0)>0$, approaches the function $w=\phi_0(\zeta)\equiv\zeta$ which maps $|\zeta|<1$ onto R with $\phi_0(0)=0$, $\phi_0'(0)>0$; convergence is uniform in every $|\zeta|\leqq r<1$. The conjugate of the origin with respect to R is $-2[f_0'(0)]^2/f_0''(0)$, which approaches the conjugate of the origin with respect to the interior of C_0, namely the point at infinity. Consequently it is possible to choose δ_1 so small independently of δ_2, δ_3, $\cdots$ that the conjugate of the point $w_0=0$ with respect to R lies exterior to the circle $|w|=1$.

Introduce the notation for the center of C_n: $w_n=3(2^n-1)/2^n$. When δ_1 and δ_2 approach zero, the kernel of the variable region R is the interior of C_1 independently of δ_3, δ_4, $\cdots$, if the region $|\zeta|<1$ is mapped onto R by the function $w=f_1(\zeta)$ with $f_1(0)=w_1$, $f_1'(0)>0$. Then the function $w=f_1(\zeta)$ approaches the function $w=\phi_1(\zeta)\equiv w_1+\zeta/2$ which maps $|\zeta|<1$ onto the interior of C_1 with $\phi_1(0)=w_1$, $\phi_1'(0)>0$; convergence is uniform in every $|\zeta|\leqq r<1$. The conjugate of w_1 with respect to R is $-2[f_1'(0)]^2/f_1''(0)$, which approaches the conjugate of w_1 with respect to the interior of C_1, namely the point at infinity. Consequently it is possible to choose δ_1 and δ_2 so small independently of δ_3, δ_4, $\cdots$ that the conjugate of the point w_1 with respect to R lies exterior to the circle $|w|=2$. The number δ_1 has been subjected to a similar restriction in connection with the conjugate of the point w_0, and is to be subjected to no further condition.

We continue now in the way thus commenced. The numbers δ_n and

δ_{n+1} are to be chosen independently of $\delta_{n+2}, \delta_{n+3}, \cdots$ in such manner that the conjugate of the point w_n with respect to R lies exterior to the circle $|w| = 2^{n+1}$; each δ_n (for $n > 1$) is subjected then to two conditions, and the numbers δ_n can be determined in succession. The resulting region R is a Jordan region. The sequence w_n approaches the boundary point $w = 3$ of R, and the conjugate of w_n with respect to R becomes infinite with n. Theorem 3 is established.

HARVARD UNIVERSITY

ON THE ORDER OF THE PARTIAL SUMS OF FOURIER POWER SERIES[1]

OTTO SZÁSZ

Dedicated to L. Fejér on his sixtieth birthday.

Let $f(x)$ be a Lebesgue integrable function, and denote the partial sums of its Fourier series by $s_n(f; x)$. It is well known that $s_n = o(n)$ uniformly[2] in x. Recently W. C. Randels[3] gave an example showing that this estimate cannot be improved. The same conclusion can be drawn from a note by E. C. Titchmarsh;[4] and A. Zygmund in his review of Randels' article (Zentralblatt für Mathematik, vol. 18, p. 353) pointed to another device, using convex coefficient sequences, which would establish the same fact.

In this note a simple construction is given, using a sequence of polynomials in the complex variable z. This leads to a sharper result showing that even for Fourier power series (that is, a power series considered on its circle of convergence and integrable) the estimate cannot be improved. Moreover, an example $F(z) = \sum_{n=0}^{\infty} c_n z^n$ is given which has the additional property that $F(z)/(1-z)$ is a generalized Fourier power series on $|z| = 1$.

We start with a sequence of polynomials of increasing degree $P_n(z) = (\sum_{\nu=0}^{m} c_{n\nu} z^\nu)^2 = \sum_{\nu=0}^{2m} a_{n\nu} z^\nu$ having the following properties:

$$(1) \qquad \frac{1}{2\pi} \int_{-\pi}^{\pi} | P_n(e^{ix}) | \, dx = \sum_{\nu=0}^{m} | c_{n\nu} |^2 = 1,$$

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[1] Presented to the Society, April 15, 1939.

[2] In fact, if $c_0, c_1, \cdots$ are the Fourier coefficients, then $c_n \to 0$. Hence $\sum_0^n |c_\nu| = o(n)$.

[3] W. C. Randels, *On the order of the partial sums of a Fourier series*, this Bulletin, vol. 44 (1938), pp. 286–288.

[4] E. C. Titchmarsh, *Principal value Fourier series*, Proceedings of the London Mathematical Society, (2), vol. 23 (1925), pp. xli–xliii.

$$(2) \qquad \max_{k\leqq 2m}\left|\sum_{\nu=0}^{k} a_{n\nu}\right| \geqq \gamma n, \qquad \gamma \text{ an absolute constant.}$$

It is sufficient to consider two special cases,[5] given in formulas (3) and (4)

$$(3) \qquad P_n(z) = (n+1)^{-1}\left(\sum_{0}^{n} z^{\nu}\right)^2, \qquad n = 1, 2, \cdots; m = n.$$

In this case $\sum_{\nu=0}^{n}|c_{n\nu}|^2=1$, $\sum_{\nu=0}^{2n}a_{n\nu}=P_n(1)=n+1$, and $\gamma=1$;

$$(4) \quad P_n(z) = (2n)^{-1}(1-z^n)^4(1-z)^{-2}, \qquad n = 1, 2, \cdots; m = 2n-1.$$

In this case

$$\begin{aligned} P_n(z) &= (2n)^{-1}\{(1-z^n)^2(1-z)^{-1}\}^2 \\ &= (2n)^{-1}\{(1+\cdots+z^{n-1})(1-z^n)\}^2 \\ &= (2n)^{-1}(1+\cdots+z^{n-1}-z^n-\cdots-z^{2n-1})^2. \end{aligned}$$

Hence $\sum_{\nu=0}^{2n-1}|c_{n\nu}|^2=2n/2n=1$. Furthermore from (4)

$$P_n(z) = (2n)^{-1}(1+2z+\cdots+nz^{n-1}+\cdots).$$

Hence

$$\sum_{\nu=0}^{n-1} a_{n\nu} = \frac{1}{2n}\cdot\frac{n(n+1)}{2} > \frac{n}{4}; \qquad \gamma = 1/4.$$

Note that in the second example $P_n(1)=0$. This property is essential for the last section.

Let $d_1 \geqq d_2 \geqq d_3 \geqq \cdots$ be an arbitrarily given monotone sequence of positive numbers tending to zero, and let $n_1 < n_2 < n_3 < \cdots$ be integers such that $\sum_{\nu=1}^{\infty} d_{n_\nu} < \infty$. Denote the degree of the polynomial $P_{n_\nu}(z)$ by $2m_\nu$, and let, for example, $\lambda_1=1$,

$$(5) \qquad \lambda_{k+1} = 2m_k + \lambda_k + 1, \qquad k = 1, 2, \cdots,$$

which gives

$$(6) \qquad \lambda_{k+1} = k + 1 + 2\sum_{\nu=1}^{k} m_\nu, \qquad k = 1, 2, 3, \cdots.$$

It is then clear that in the totality of polynomials $z^{\lambda_\nu}P_{n_\nu}(z)$,

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[5] The polynomials given in (3) have been used by F. Riesz to construct a Fourier power series for which $\limsup_{n\to\infty}\int_{-\pi}^{\pi}|s_n(e^{ix})|dx=\infty$. Cf. A. Zygmund, Proceedings of the London Mathematical Society, (2), vol. 34 (1932), pp. 392–400, and Zygmund, *Trigonometrical Series*, p. 165.

($\nu=1, 2, \cdots$), no power of z is repeated. Hence the series of polynomials

$$F(z) = \sum_{\nu=1}^{\infty} d_{n_\nu} z^{\lambda_\nu} P_{n_\nu}(z) \tag{7}$$

with each term written out separately is a power series convergent for $|z| < 1$. Moreover, when $0 < r < 1$,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |F(re^{ix})|\, dx \leqq \frac{1}{2\pi}\sum_{\nu=1}^{\infty} d_{n_\nu} \int_{-\pi}^{\pi} |P_{n_\nu}(re^{ix})|\, dx = \sum_{\nu=1}^{\infty} d_{n_\nu} < \infty;$$

whence $F(e^{ix})$ exists and is Lebesgue integrable. Note that *for both examples the series* (7) *converges uniformly in any closed region on* $|z| \leqq 1$ *excluding the point* $z=1$. *Thus* $F(z)$ *is continuous in that region.*

For example (3) we now have $s_{2n_k+\lambda_k}(F;\,1) = \sum_{\nu=1}^{k}(1+n_\nu)d_{n_\nu}$. Since d_n decreases as n increases we get, using (5) and (6),

$$\begin{aligned} s_{2n_k+\lambda_k}(F;\,1) &\geqq d_{n_k}\left(k + \sum_{\nu=1}^{k} n_\nu\right) > (1/2)d_{n_k}\lambda_{k+1} \\ &> (1/2)d_{n_k}(2n_k + \lambda_k). \end{aligned} \tag{8}$$

Thus $s_{\lambda_{k+1}-1} > (1/2)(\lambda_{k+1}-1)d_{\lambda_{k+1}-1}$, giving

$$\limsup_{n\to\infty} \frac{s_n}{nd_n} \geqq 1/2, \tag{9}$$

where d_n has been chosen approaching 0 as slowly as we please.

In the case of example (4) we consider

$$s_{2m_k+\lambda_k+n_{k+1}+1}(F;\,1) = (1/4)d_{n_{k+1}}(1 + n_{k+1}),$$

where $m_k = 2n_k - 1$. Thus

$$2m_k + 1 = 4n_k - 1 = \lambda_{k+1} - \lambda_k, \quad 1 + n_{k+1} = 1 + (\lambda_{k+2} - \lambda_{k+1} + 1)/4.$$

Also $2m_k + \lambda_k + n_{k+1} + 1 = \lambda_{k+1} + (\lambda_{k+2} - \lambda_{k+1} + 1)/4$. Denoting this number by μ_k, we have $s_{\mu_k} > (1/4)(\mu_k - \lambda_{k+1})d_{\mu_k}$. Hence

$$\frac{s_{\mu_k}}{\mu_k d_{\mu_k}} > \frac{1}{4}\,\frac{\mu_k - \lambda_{k+1}}{\mu_k} = \frac{1}{4}\,\frac{\lambda_{k+2} - \lambda_{k+1} + 1}{\lambda_{k+2} + 3\lambda_{k+1} + 1}.$$

Assuming, as we may, $\lambda_{k+1}/\lambda_k \to \infty$, we get

$$\limsup_{n\to\infty} \frac{s_n}{nd_n} \geqq 1/4. \tag{10}$$

Let us rewrite the function defined by (7) as

$$F(z) = \sum_{n=1}^{\infty} c_n z^n, \tag{7'}$$

and consider

$$\int_0^z F(t)dt = \sum_{n=1}^{\infty} \frac{1}{n+1} c_n z^{n+1} = F_1(z).$$

$F_1(z)$ is absolutely continuous. Thus by a theorem of Hardy and Littlewood[6] $\sum_{n=1}^{\infty} |c_n|/(n+1) < \infty$. On the other hand

$$\begin{aligned} \sum_{\nu=1}^{n} c_\nu &= n \sum_{\nu=1}^{n} \frac{c_\nu}{\nu+1} - \sum_{\nu=1}^{n} (n-\nu+1)\frac{c_\nu}{\nu+1} \\ &= n s_n(F_1; 1) - \sum_{\nu=1}^{n} s_\nu(F_1; 1) \qquad (11) \\ &= n\{s_n - F_1(1)\} - \sum_{\nu=1}^{n} (s_\nu - F_1(1)). \end{aligned}$$

It now follows that there does not exist a sequence (δ_n) tending to zero, such that for every absolutely continuous Fourier power series $F(z)$ we have $s_n - F_1(1) = O(\delta_n)$. For then for a positive number ρ depending on F we should have $|s_n - F_1(1)| < \rho\delta_n$, $(n = 1, 2, 3, \cdots)$, and hence, from (11),

$$\left|\sum_{\nu=1}^{n} c_\nu\right| < \rho n \delta_n + \rho \sum_{\nu=1}^{n} \delta_\nu, \qquad \frac{1}{n}\left|\sum_{\nu=1}^{n} c_\nu\right| < \rho\delta_n + \frac{\rho}{n}\sum_{\nu=1}^{n} \delta_\nu.$$

Thus if $\sigma_n = 2\max\,(\delta_n, (1/n)\sum_{\nu=1}^{n}\delta_\nu)$, then $\sigma_n \to 0$, and $(1/n)|\sum_{\nu=1}^{n} c_\nu| < \rho\sigma_n$. Let now $d_n^2 = \max_{\nu \geq n} \sigma_\nu$; then $d_n \downarrow 0$, and $(1/nd_n)|s_n| < \rho d_n \downarrow 0$, which contradicts the result in (9) and (10).

Now consider again the function (7′) corresponding to the example in (4). Then for $|z| < 1$, $(1-z)^{-1}F(z) = \sum_{n=1}^{\infty} s_n z^n$, where $s_n = \sum_{\nu=1}^{n} c_\nu$. From (7)

$$(1-z)^{-1}F(z) = \sum_{\nu=1}^{\infty} d_{n_\nu} z^{\lambda_\nu} P_{n_\nu}(z)(1-z)^{-1},$$

where the series converges uniformly in any closed region on $|z| \leq 1$ excluding the point $z = 1$. Now for $0 < \epsilon < \pi$

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[6] G. H. Hardy and J. E. Littlewood, *Some new properties of Fourier constants*, Mathematische Annalen, vol. 97 (1926), pp. 159–209, in particular Theorem 16, p. 208.

$$\int_{\epsilon}^{\pi}(1-e^{ix})^{-1}F(e^{ix})dx=\sum_{\nu=1}^{\infty}d_{n_\nu}\int_{\epsilon}^{\pi}e^{i\lambda_\nu x}P_{n_\nu}(e^{ix})(1-e^{ix})^{-1}dx$$
$$=\frac{1}{2}\sum_{\nu=1}^{\infty}d_{n_\nu}\frac{1}{n_\nu}\int_{\epsilon}^{\pi}e^{i\lambda_\nu x}(1-e^{in_\nu x})^{4}(1-e^{ix})^{-3}dx,$$

and

$$\left(\int_{\epsilon}^{\pi}+\int_{-\pi}^{-\epsilon}\right)e^{i\lambda_\nu x}(1-e^{in_\nu x})^{4}(1-e^{ix})^{-3}dx\rightarrow 0,\qquad \epsilon\rightarrow 0.$$

Thus $\int_{-\pi}^{\pi}(1-e^{ix})^{-1}F(e^{ix})dx$ exists as a generalized integral and

$$\int_{-\pi}^{\pi}F(e^{ix})(1-e^{ix})^{-1}dx=0.$$

On the other hand $s_1, s_2, s_3, \cdots$ are the generalized Fourier coefficients of this function. Thus $o(n)$ is the sharpest estimate for the Fourier coefficients of a generalized Fourier power series. For generalized Fourier series Titchmarsh[7] constructed an example along essentially different lines.

UNIVERSITY OF CINCINNATI

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[7] E. C. Titchmarsh, *The order of magnitude of the coefficients in a generalized Fourier series*, Proceedings of the London Mathematical Society, (2), vol. 22 (1924), pp. xxv–xxvi.

A THEOREM ON CONTINUOUS FUNCTIONS IN ABSTRACT SPACES[1]

WILLIAM T. REID

In this note the following Theorem A concerning continuous functions in a very general abstract space is established, and from this theorem are deduced certain results concerning semi-metric spaces. In particular, Theorems 2.2 and 2.3 below generalize a theorem proved by Montgomery[2] concerning the behavior of the distances between points of a metric space under transformations of the space into itself.

The space S to be considered is a collection of "points" such that to each nonvacuous subset M of S there is defined a unique nonvacuous enclosure set $\overline{M}$ such that if M_0 and M_1 are nonvacuous subsets then $(\overline{M_0+M_1})=\overline{M}_0+\overline{M}_1$. We shall not even require that M be a subset of $\overline{M}$, although unquestionably this latter condition is desirable for a far-reaching topological study of abstract spaces in general. Two sets M and N are *mutually separated* if $N\overline{M}+\overline{N}M=0$. A point set X is said to be *connected* if it is not the sum of two nonvacuous mutually separated point sets. Let $\phi(p)$ be a single-real-valued function defined on S. If M is a nonvacuous subset of S, we shall denote by $\phi(M)$ the set of real numbers $\phi(p)$ determined as p ranges over M. Using the usual absolute value as the metric in the space of real numbers, and defining the enclosure of a set of real numbers as the set, together with all its limit points, the function $\phi(p)$ will be said to be *continuous* on S if, for every subset M of $S, \phi(\overline{M})$ is contained in $\overline{\phi(M)}$.

THEOREM A. *Suppose S is a space of the above described sort which is connected, and $f(p, q)$ is a single-real-valued function defined for each pair (p, q) of S and such that:* (i) $f(p, p)=0$; (ii) $f(p, q)=f(q, p)$; (iii) *$f(p, q)$ is continuous in its arguments separately on S. Then either $f(p, q)$ is of constant sign (that is, for all (p, q) either $f(p, q)\leqq 0$ or $f(p, q)\geqq 0$), or there exists a pair of points $p\neq q$ such that $f(p, q)=0$.*

1. **Proof of Theorem A.** The conclusion of this theorem will be established by indirect argument. For suppose that this conclusion is not true. Then $f(p, q)\neq 0$ for $p\neq q$, and there exist points p_1, q_1, p_2, q_2

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[1] Presented to the Society, September 8, 1939.

[2] D. Montgomery, *A metrical property of point-set transformations*, this Bulletin, vol. 40 (1934), pp. 620–624.

such that $f(p_1, q_1) > 0$, $f(p_2, q_2) < 0$. It will now be proved that this condition is in contradiction to the hypotheses of the theorem.

We shall first show that this latter condition implies that there exist points x, y_1, y_2 such that $f(x, y_1) > 0$, $f(x, y_2) < 0$. If $q_2 = q_1$, we may take $x = q_1$, $y_1 = p_1$, $y_2 = p_2$; if $q_2 = p_1$ we may take $x = p_1$, $y_1 = q_1$, $y_2 = p_2$. There remains the case $q_2 \neq q_1$ and $q_2 \neq p_1$; in this case if $f(p_1, q_2) > 0$, take $x = q_2$, $y_1 = p_1$, $y_2 = p_2$, whereas, if $f(p_1, q_2) < 0$, take $x = p_1$, $y_1 = q_1$, $y_2 = q_2$. It is to be remarked that this part of the proof does not utilize the connectedness of S. The original proof of the author of this result involved the use of connectedness. The above simple proof was suggested by L. M. Graves.

Now for a point x let A_x denote the set of all points p such that $f(x, p) < 0$, and B_x the set of all points p such that $f(x, p) > 0$. On the assumption that the theorem is false it has been shown that there exists an x such that neither A_x nor B_x is vacuous. For each such x we have $x + A_x + B_x = S$. From the continuity of $f(x, p)$ as a function of p it follows readily that A_x and B_x are mutually separated; moreover, since $f(x, x) = 0$, $f(x, p) \neq 0$ for $p \neq x$, we have that

$$\bar{x} = x, \qquad \overline{(x + A_x)} \cdot B_x = 0, \qquad \overline{(x + B_x)} \cdot A_x = 0.$$

Finally, the connectedness of S implies $x \cdot \bar{A}_x = x$, $x \cdot \bar{B}_x = x$, and that $x + A_x$ and $x + B_x$ are connected sets.

Now consider an arbitrary point q of A_x. Then x is a point of A_q since $f(q, x) = f(x, q) < 0$. Suppose $B_q \cdot B_x \neq 0$. Then $q + B_q = (q + B_q) \cdot A_x + (q + B_q) \cdot B_x$, and the sets $(q + B_q) \cdot A_x$, $(q + B_q) \cdot B_x$ are mutually separated since A_x and B_x are mutually separated. But by the above argument the set $q + B_q$ is connected. Therefore, $B_q \cdot B_x = 0$, and B_x is a subset of A_q. Since q was an arbitrary point of A_x, we have for each q of A_x and each p of B_x that $f(q, p) < 0$. For p a fixed point of B_x, it follows from the fact that $x \cdot \bar{A}_x = x$ and the continuity of $f(q, p)$ as a function of q that $f(x, p) \leqq 0$. This, however, is impossible in view of the definition of B_x. Hence Theorem A is established.

2. **Applications of Theorem A.** We shall now consider a space S which is topologized by means of a real symmetric distance function $d(p, q)$. That is, $d(p, q)$ is defined for each pair of points (p, q) of S and: (a) $d(p, p) = 0$; (b) $d(p, q) = d(q, p)$; (c) $d(p, q) \neq 0$ if $p \neq q$. If M is a subset of S, the enclosure of M shall be defined as the set of all points p satisfying the condition that there is a corresponding sequence $\{p_n\}$ belonging to M such that $\lim_{n \to \infty} |d(p, p_n)| = 0$. The enclosure function for such a space is clearly additive, and hence such a space is of the kind considered above. For a space of this latter sort it is readily

seen that a subset M is contained in its enclosure $\overline{M}$; moreover, if M consists of a single point p, then $\overline{M}=\bar{p}=p$. Such a space is a semi-metric space in the sense of Fréchet if in addition $d(p, q) \geqq 0$ for all points p, q. Clearly a space S which is topologized by means of a real symmetric distance function is a semi-metric space with respect to the new distance function $|d(p, q)|$. In view of the above Theorem A we have, however, the further result:

THEOREM 2.1. *Suppose S is topologized by means of a real symmetric distance function $d(p, q)$. If S is connected and $d(p, q)$ is continuous in its arguments separately on S, then either $d(p, q)>0$ or $d(p, q)<0$ for all distinct points p and q; that is, either S is a semi-metric space with respect to $d(p, q)$, or S is a semi-metric space with respect to the distance function $-d(p, q)$.*

In view of this theorem, there is no loss of generality in stating the following result for a semi-metric space instead of for a space topologized by means of a real symmetric distance function.

THEOREM 2.2. *Suppose S is a connected semi-metric space whose distance function $d(p, q)$ is continuous in its arguments separately. If T is a continuous transformation on S to S, and there exist pairs of points (p_1, q_1), (p_2, q_2) such that $d(Tp_1, Tq_1)>d(p_1, q_1)$, $d(Tp_2, Tq_2)<d(p_2, q_2)$, then there exists a point pair (p, q), $(p \neq q)$, such that $d(Tp, Tq)=d(p, q)$.*

It is to be emphasized that we do not require that the image of S under T be the whole of S; moreover, if S_1 is the image of S under T, it is not supposed that there is a one-to-one correspondence between the points of S and S_1.

If $f(p, q)$ is defined as $d(Tp, Tq)-d(p, q)$, Theorem 2.2 is an immediate consequence of Theorem A. This latter theorem extends in several directions a result proved by Montgomery (loc. cit.). In particular, Montgomery assumed the space which he considered to be arc-wise connected.

In the above cited paper, Montgomery proved as a preliminary result that if S is a conditionally compact metric space and T is any one-to-one transformation of S into the whole of itself which does not leave all distances invariant, then T must increase at least one distance and decrease at least one distance. Actually, his proof applies to a semi-metric space whose distance function is continuous in the arguments (p, q) jointly. Morover, for his proof of an auxiliary lemma to be valid it is not necessary to assume that S itself is conditionally compact; it is sufficient to assume that if p is an arbitrary point of S, then the subset of S consisting of the points $[p, Tp, T^2p, \cdots]$ is conditionally compact. For convenience, we shall term such a transformation a *conditionally compact transformation.* If S itself is conditionally compact, then clearly every one-to-one transformation of S into itself is conditionally compact; it is readily seen, however, that there are non-trivial examples of conditionally compact transformations of a space S into itself for which the given space S is as a whole not conditionally compact. Montgomery's proof of the lemma in the above cited paper establishes the following result: *Suppose S is a semi-metric space whose distance function $d(p, q)$ is continuous in (p, q) jointly, and that T is a one-to-one conditionally compact transformation of S into the whole of itself. Then if T increases the distance between some two points of S, there also exist two points whose distance is decreased under T.* Hence we have the following conclusion:

LEMMA. *Suppose S is a semi-metric space whose distance function $d(p, q)$ is continuous in (p, q) jointly. If T is a one-to-one transformation of S into the whole of itself such that T and its inverse T^{-1} are each conditionally compact transformations, and T does not leave all distances invariant, then T must increase at least one distance and decrease at least one distance.*

The following result is then a consequence of this lemma and Theorem 2.2.

THEOREM 2.3. *Suppose S is a connected semi-metric space whose distance function $d(p, q)$ is continuous in its arguments (p, q) jointly. If T is a continuous one-to-one transformation of S into the whole of itself such that T and its inverse T^{-1} are each conditionally compact transformations, then there are two distinct points of S whose distance is invariant under T.*

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A THEOREM ON SURFACES[1]

ERNEST P. LANE

It is a fact which is familiar[2] to projective differential geometers that the non-rectilinear asymptotic curves on an analytic non-developable ruled surface, not a quadric, in ordinary three-dimensional space belong to linear complexes if, and only if, the ruled surface belongs to a linear congruence, that is, if, and only if, the ruled surface has two distinct or coincident rectilinear directrices. It is furthermore known[3] that in this case the asymptotic curves on the ruled surface are projectively equivalent. It has also been demonstrated[4] that if the asymptotic curves on an analytic non-ruled surface S in ordinary space belong to linear complexes, then the asymptotic ruled surfaces of S, that is, the ruled surfaces composed of the tangents of the asymptotic curves of either family, constructed at the points of a fixed asymptotic curve of the other family, on S have rectilinear directrices and therefore are such that their asymptotic curves belong to linear complexes. The converse of this theorem is also true, as will be shown below. Moreover, it will be proved below, by the aid of some formulas computed[5] by MacQueen and the author, that in this case the asymptotic curves of one family on the surface S are projectively equivalent, as are also the asymptotic curves of the other family on S.

In attempting to determine whether the asymptotic curves on the asymptotic ruled surfaces of a non-ruled surface S are twisted cubics if the asymptotic curves on the surface S are twisted cubics, the author answered this question in the affirmative and discovered the following general theorem, which seems to have escaped notice hitherto and which it is the purpose of this note to put on record and demonstrate:

If the asymptotic curves on an analytic non-ruled surface S in ordinary space belong to linear complexes, then the asymptotic curves of each family on S are projectively equivalent, not only to each other, but also

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[1] Presented to the Society, December 2, 1939.

[2] A. Peter, *Die Flächen deren Haupttangentenkurven linearen Komplexen angehören*, Leipzig, dissertation, 1895.

[3] C. T. Sullivan, *Properties of surfaces whose asymptotic curves belong to linear complexes*, Transactions of this Society, vol. 15 (1914), pp. 167–196. See p. 171.

[4] Sullivan, loc. cit., p. 178; also Peter, loc. cit.

[5] E. P. Lane and M. L. MacQueen, *Surfaces whose asymptotic curves are twisted cubics*, American Journal of Mathematics, vol. 60 (1938), pp. 337–344. See p. 339.

to all the non-rectilinear asymptotic curves on all the asymptotic ruled surfaces of S which are composed of the tangents of the asymptotic curves of the other family on S, constructed at the points of the various asymptotic curves of the first family on S.

In order to prove this theorem it is convenient to collect here the notations and equations which form the analytic basis of a classical projective differential theory of non-ruled surfaces whose asymptotic curves belong to linear complexes. Let us consider in ordinary space an analytic non-ruled surface S whose parametric vector equation, referred to asymptotic parameters u, v, is

$$x = x(u, v). \tag{1}$$

The four coordinates x of a variable point x on the surface S satisfy two partial differential equations which can be reduced, by a suitably chosen transformation of proportionality factor, to Fubini's canonical form,

$$x_{uu} = px + \theta_u x_u + \beta x_v, \quad x_{vv} = qx + \gamma x_u + \theta_v x_v \quad (\theta = \log \beta\gamma), \tag{2}$$

subscripts indicating partial differentiation, and the coefficients being functions of u, v which satisfy three integrability conditions which need not be written here. It is known that if the parametric asymptotic curves on the surface S belong to linear complexes, the coefficients β, γ can, by a suitably chosen transformation of parameters, be specialized so that

$$\beta = \gamma = (U'V')^{1/2}/(U + V), \qquad U'V' \neq 0, \tag{3}$$

where U is an arbitrary function of u alone, and V of v alone, and the accent denotes differentiation with respect to the appropriate variables. Moreover, the coefficients p, q are in this case given by

$$\begin{aligned} p &= (1/2)(3l_{uu} - (3/2)l_u^2 - 3\beta l_v - U_1), \\ q &= (1/2)(3l_{vv} - (3/2)l_v^2 - 3\beta l_u - V_1), \end{aligned} \tag{4}$$

in which $l = \log \beta$ and U_1, V_1 are defined by

$$U_1 = (DU^2 + EU + F)/U', \quad V_1 = (DV^2 - EV + F)/V', \tag{5}$$

where D, E, F are arbitrary constants. The integrability conditions are identically satisfied by these expressions for the coefficients of equations (2).

It has been shown[6] that the fourth-order linear homogeneous differential equation for the u-curve through a point x on the surface S has its seminvariants P_2, P_3, P_4 given by the formulas

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[6] Lane and MacQueen, loc. cit., pp. 337–339.

$$(6) \qquad P_2 = (1/6)((1/2)S + U_1), \quad P_3 = (1/4)((1/2)S_1 + U_1'), \\ P_4 = (1/4)(U_1^2 + 3U_1''),$$

where S is the Schwarzian derivative of U, namely,

$$(7) \qquad S = (U''/U')' - (1/2)(U''/U')^2.$$

By means of the transformation of proportionality factor

$$(8) \qquad \bar{x} = \beta^{-3/2}x,$$

this differential equation for the u-curve through the point x can be written in the form

$$(9) \qquad \bar{x}_{uuuu} + 6P_2\bar{x}_{uu} + 4P_3\bar{x}_u + P_4\bar{x} = 0.$$

Since the coefficients of this equation are independent of v, it follows that the differential equation of all the u-curves on the surface S is this same equation and that these asymptotic u-curves are therefore projectively equivalent, and similarly for the v-curves on S, as previously stated.

It is now proposed to compute the differential equation of the same form for the non-rectilinear asymptotic curves on the asymptotic ruled surface R_v composed of v-tangents at the points of the u-curve through the point x on the surface S. Any point y on the v-tangent through the point x (except the point x itself) is given by the formula

$$(10) \qquad y = \beta^{-3/2}(hx + x_v),$$

where h is an arbitrary parameter. It is possible to determine h as a function of u, v so that the locus of the point y, when u varies and v=const., is an asymptotic curve on the ruled surface R_v. In fact, specializing a known[7] result, we find that the locus of the point y is an asymptotic curve on the surface R_v in case

$$(11) \qquad h_u = -(3/2)\beta^2.$$

But since $l_{uv} = \beta^2$, as can easily be verified, it follows that

$$(12) \qquad h = -(3/2)l_v + c + V_2,$$

where c is an arbitrary constant and V_2 is an arbitrary function of v

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[7] Lane, *Projective Differential Geometry*, University of Chicago Press, 1932, p. 114, ex. 6.

alone. By the aid of this expression for h, and the transformation (8), we find

$$y = (c + V_2)\bar{x} + \bar{x}_v, \tag{13}$$

and besides differentiation of equation (9) with respect to v yields

$$(\bar{x}_v)_{uuuu} + 6P_2(\bar{x}_v)_{uu} + 4P_3(\bar{x}_v)_u + P_4(\bar{x}_v) = 0. \tag{14}$$

It is now evident that y also satisfies equation (9), identically in c and v. Therefore the differential equation for all the non-rectilinear asymptotic curves on all the asymptotic ruled surfaces R_v is the same as the differential equation (9) for the asymptotic u-curves on the surface S. It follows that all these curves are projectively equivalent. A similar argument can be made with u and v interchanged, and so the theorem is proved.

As a corollary, the theorem that the u-curves on the surface S belong to linear complexes if, and only if, the asymptotic curves on the surface R_v do so can be deduced at once, since these curves are now known to be projectively equivalent. In like manner, the theorem that the u-curves on the surface S are twisted cubics if, and only if, the asymptotic curves on the surface R_v are twisted cubics can be deduced.

UNIVERSITY OF CHICAGO

ON GENERAL METHODS FOR OBTAINING CONGRUENCES INVOLVING BERNOULLI NUMBERS

H. S. VANDIVER

In a previous article[1] the writer proved a theorem concerning congruences in rings. This was employed to obtain various congruences involving the Bernoulli numbers, in particular the relation[2]

$$h_1^{n_1}h_2^{n_2}\cdots h_s^{n_s}(\beta_1 h_1^{p-1}+\beta_2 h_2^{p-1}+\cdots+\beta_s h_s^{p-1})^j \equiv 0 \pmod{p^j, p^{n_1-1}, p^{n_2-1}, \cdots, p^{n_s-1}},$$

$n_i \not\equiv 0 \pmod{p-1}$; p a prime; $i=1, 2, \cdots, s$; $\beta_1, \beta_2, \cdots, \beta_s$ are integers such that

$$\sum_{i=1}^{s}\beta_i \equiv 0 \pmod{p},$$

and the left-hand member is expanded in full employing the multinomial theorem, and b_t/t is substituted for h_i^t in the result, $i=1, 2, \cdots, s$, with the b's defined by the following recursion formula:

$$(b+1)^n = b_n;\ n > 1;\ b_0 = 1.$$

I shall now discuss other methods for obtaining congruences involving the b's. The proofs will be indicated only. We have the following result.

THEOREM I. *If $a_1, a_2, \cdots, a_s$; $x_1, x_2, \cdots, x_s$ are integers with the x's positive and the congruences*

$$a_1C_{x_1,i}+a_2C_{x_2,i}+\cdots+a_sC_{x_s,i} \equiv 0 \pmod{p^{k-i}},$$

$i=0, 1, \cdots, k-1$; $C_{x,a}=0$ when $a>x$, are all satisfied, then

$$\sum_{i=1}^{s}\frac{a_i b_{n+(p-1)x_i}}{n+(p-1)x_i} \equiv 0 \pmod{p^k, p^{n-1}}.$$

The proof of this depends on the formula

$$\frac{n^{2i}-1}{2i}b_{2i} \equiv \sum_{a=1}^{p^\alpha-1} y_a a^{2i-1} \pmod{p^\alpha}$$

for $p>3$, and noting that we can write

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[1] This Bulletin, vol. 43 (1937), pp. 418–423.

[2] Here (mod $p^j, \cdots, p^{n_s-1}$) means (mod $(p^j, \cdots, p^{n_s-1})$).

$$a^{n+(p-1)t} = a^n(1 + pq(a))^t,$$

where $q(a) = (a^{p-1} - 1)/p$, and expanding the right-hand member.

By means of this theorem we may set up numerous congruences such as the following, for $p > 3$, $n \not\equiv 0 \pmod{\overline{p-1}}$:

$$B'_{n+p\mu} - pB'_{n+\mu} + (p-1)B'_n \equiv 0 \pmod{p^3},$$

$$xB'_{n+py\mu} - yB'_{n+px\mu} + (y-x)B'_n \equiv 0 \pmod{p^4},$$

where $B_a = (-1)^{a-1}b_{2a}$; $B'_a = (-1)^a B_a/a$; and x and y are any positive or zero integers with $\mu = (p-1)/2$.

Beeger[3] proved the interesting congruence

$$(-1)^{i-1}\frac{B_{n+m\mu}}{2n + 2m\mu} \equiv \sum_{s=1}^{i}(-1)^{s-1+(m-s+1)\mu} \cdot C_{m,s-1}C_{m-s,i-s}\frac{B_{n+(s-1)\mu}}{2n+2(s-1)\mu}$$

modulo p^i, where $m \geqq i$, $i < 2n-1$, $2n \not\equiv 0 \pmod{\overline{p-1}}$. Another proof of this, from Theorem I, follows if we can show that

$$\sum_{s=1}^{i}(-1)^{s-1}C_{m,s-1}C_{m-s,i-s}C_{s-1,k} + (-1)^i C_{m,k} \equiv 0 \pmod{p^{i-k}},$$

$k = 0, 1, 2, \cdots, i-1$. This is possible, as is seen on expanding the expression

$$(y - z)^n = (x + y - (x + z))^n,$$

in which x, y and z are indeterminates, and equating the coefficients of each of the terms involving x to zero, and then applying the known relation $C_{n+1,a} = C_{n,a} + C_{n,a-1}$. Let

$$n_1, n_2, \cdots, n_s, \qquad n_i < m, \tag{1}$$

form a repetitive set modulo m; that is, there exists an n ($\neq 1$) called a multiplier of the set, such that

$$nn_1, nn_2, \cdots, nn_s$$

are congruent modulo m to the set (1) in some order.[4] Consider the set

$$(y_{ak}m - n_a k)/n \tag{2}$$

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[3] This Bulletin, vol. 44 (1938), pp. 684–686.

[4] The writer employed the term conjugate set for this type of set in the Annals of Mathematics, (2), vol. 18 (1917), p. 106, but this does not agree with the terminology of group theory. Concerning the concept of repetitive sets in a semi-group see Proceedings of the National Academy of Sciences, vol. 23 (1937), pp. 554–555.

where $a=1, 2, \cdots, s$; m, n and k are prime each to each, n and k are multipliers of (1),

$$y_{ak} \equiv -n_a k/m \pmod{n},$$

and y_{ak} is further selected so that it is the least integer satisfying the above congruence which also makes

$$y_{ak}m - n_a k > 0.$$

Then the set (2) may be shown to give the set (1) in some order. For $k=1$ this was obtained in a bit different form on page 110 of the article just cited, and proceeding as in that paper we obtain generalizations of most of the results therein. In particular by using the repetitive set $1, 2, \cdots, m-1$, modulo m, we find, if $i<p-1$,

$$b_i \frac{n^i - (-1)^i}{ik^{i-1}} \equiv \sum_{a=1}^{m-1} y_{ak} a^{i-1} \pmod{m}.$$

The set (2) having the multiplicative property of repetitive sets, it is a bit curious then that we may extend some of these ideas to integers in arithmetic progression. Consider the expressions $r<m$; p prime;

$$\frac{y_t p + mt + r}{n}; \qquad t = 0, 1, \cdots, p-1; \tag{3}$$

where y_t is selected so that each of these is an integer and also so that each is of the form $r \pmod{m}$. Suppose that $p \equiv s \pmod{m}$. Then if $(m, n)=1$; $n>m$; $(nr-r+s, m)=1$; then the maximum value for y_t is $mn-n-1<m(n-1)$ so that each of these is less than pm; hence they are the integers $mt+r$; $t=0, 1, \cdots, p-1$, in some order. This property of the set (3) yields a number of results including the relation

$$\frac{n^i - 1}{i}(mb + r)^i \equiv \sum_{t=0}^{p-1} y_t(mt + r)^{i-1} \pmod{p},$$

with the restrictions on m, n and p mentioned above and also $i \not\equiv 1$ $\pmod{p-1}$.

University of Texas

APPROXIMATION TO REAL IRRATIONALS BY CERTAIN CLASSES OF RATIONAL FRACTIONS[1]

W. T. SCOTT

1. **Introduction.** Hurwitz proved that if ω is a real irrational number, then the inequality

$$|\omega - p/q| < k/q^2 \tag{1}$$

is satisfied by infinitely many rational fractions p/q when $k \geq 1/5^{1/2}$, and further, that there exist irrationals everywhere dense on the real axis for which (1) is satisfied by only a finite number of fractions[2] when $k < 1/5^{1/2}$. He used simple continued fractions to get this result. The same result has since been obtained in two different ways by Ford.[3]

If o denotes an odd integer and e an even integer, then all irreducible fractions p/q are of three classes $[o/e]$, $[e/o]$, and $[o/o]$. It will be shown that

If $k \geq 1$, *there are infinitely many fractions of each of the three classes satisfying* (1), *regardless of the value of the real irrational number* ω.

If $k < 1$, *there exist irrational numbers everywhere dense on the real axis for which* (1) *is satisfied by only a finite number of fractions of a given one of the three classes.*

The proof, like Ford's first proof of Hurwitz' theorem, will depend to a large extent on geometric properties of elliptic modular transformations.

2. **Proof of the first part of the theorem.** For each fraction p/q construct (see Fig. 1), in the upper half-plane an S-circle, $S(p/q;\, k)$ of radius k/q^2 and tangent to the real axis at $z = p/q$. Let L be a line in the upper half-plane perpendicular to the real axis at $z = \omega$. *Then* (1) *is satisfied by* p/q *if and only if* L *cuts* $S(p/q;\, k)$.

The group of elliptic modular transformations is the set of all transformations of the form

$$z' = \frac{\alpha z + \beta}{\gamma z + \delta}, \qquad \alpha\delta - \beta\gamma = 1, \tag{2}$$

where α, β, γ, δ are integers. These conformal transformations carry the upper half plane into itself and the real axis into itself. Circles (including straight lines) are carried into circles.[4]

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[1] Presented to the Society, February 24, 1940.

[2] A. Hurwitz, Mathematische Annalen, vol. 39 (1891), pp. 279–285.

[3] L. R. Ford, Proceedings of the Edinburgh Mathematical Society, vol. 35 (1916), pp. 59–65; American Mathematical Monthly, vol. 45 (1938), pp. 586–601.

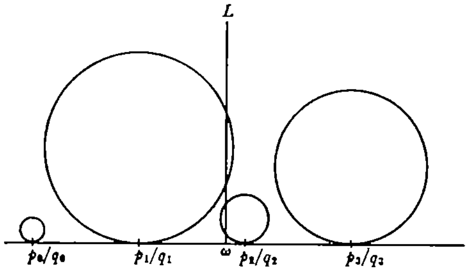

FIG. 1

Consider the subgroup

$$G(z) = \frac{az + 2b}{2cz + d}, \qquad ad - 4bc = 1,$$

a, b, c, d integers. It is well known that a fraction p/q is irreducible if and only if there exist integers p_0, q_0, such that $pq_0 - qp_0 = 1$. It follows that

$$G(p/q) = \frac{ap + 2bq}{2cp + dq}$$

is irreducible if p/q is, since

$$(ap + 2bq)(2cp_0 + dq_0) - (2cp + dq)(ap_0 + 2bq_0) = pq_0 - qp_0.$$

By inspection we see that $G(p/q)$ is a fraction of the same class as p/q. Hence, *the classes* $[o/e]$, $[e/o]$, $[o/o]$ *are invariant under a transformation by any member of* $G(z)$.

The set of S-circles of a given class is invariant under a transformation by any member of $G(z)$.

Because of the preceding result it will suffice to show that the set of all circles $S(p/q; k)$ is invariant under any modular transformation.

Consider the horizontal line $y = h$, or $z - \bar{z} = 2ih$ where h is a positive constant and $\bar{z}$ is the complex conjugate of z. When the values of z and $\bar{z}$ given by the modular transformation (2) are substituted in this equation, it is found that the line is carried into $S(\alpha/\gamma; 1/2h)$. Now choose $h = 1/2k$ and define $S(1/0; k)$ to be the line $y = h$. The desired result is immediate.

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[4] A detailed account of the elliptic modular group can be found in the early chapters of L. R. Ford, *Automorphic Functions*, 1929, New York.

There is a member of $G(z)$ *which carries* $S(p/q;\, k)$ *into any prescribed* S*-circle of the same class.*

Let p/q be a fraction of class $[o/e]$. There exist integers p_0, q_0, such that $pq_0 - qp_0 = 1$; and the transformation

$$z' = \frac{-(q_0 + nq)z + (p_0 + np)}{qz - p}$$

carries $z = p/q$ into $z' = \infty$. Since p is odd and q is even, the integer n can be chosen so that this transformation is a member of $G(z)$. Similarly, there exists a member of $G(z)$ which carries the prescribed fraction of class $[o/e]$ into $z' = \infty$. The first transformation followed by the inverse of the second transformation is a member of $G(z)$ since $G(z)$ is a group, and it is easily seen that this transformation has the desired properties.

The classes $[e/o]$ and $[o/o]$ are similarly treated, except that the $z' = \infty$ should be replaced by $z' = 0$ and $z' = 1$.

A simple calculation shows that $S(p/q;\, k)$ and $S(p'/q';\, k)$ are tangent if and only if

$$k^2 = (1/4)(pq' - qp')^2.$$

If p/q and p'/q' are different fractions of the same class and $k = 1$, the condition for tangency becomes

$$pq' - qp' = \pm 2.$$

Hence, when $k = 1$, *two circles of the same class are either tangent or else wholly external to each other.*

We note that the circles $S(1/2n;\, 1)$ are tangent for consecutive integral values of n; and the same is true for the circles $S([2n-1]/2n;\, 1)$. Even when $S(1/0;\, 1)$ is excluded these two sets of circles still have a common member; namely, $S(1/2;\, 1)$. These two sets of circles form a chain which extends from $-1/2$ to $3/2$, except for breaks at 0 and 1 (see Fig. 2). We apply $z' = z + 2b$, which are members of $G(z)$, and see that *there is a chain of tangent* S*-circles of class* $[o/e]$ *between every pair of consecutive integral points when* $k = 1$.

The modular transformations

$$z' = -1/z', \qquad z' = -z/(z-1)$$

interchange the classes $[o/e]$, $[e/o]$, and the classes $[o/e]$, $[o/o]$. The chain property for class $[o/e]$ has an analogue for the two remaining classes. Only fractions of the class $[o/e]$ will be treated here, but similar treatment would yield like results for the other two classes.

The line L must cut $S(1/0;\ 1)$. As L is traced toward $z=\omega$, an S-circle of a chain between two integral points must be encountered since ω is irrational. This S-circle is now transformed into $S(1/0;\ 1)$ by a suitable member of $G(z)$; and this same transformation carries L into a semicircle, one of whose ends is at an irrational point. As the semicircle is traced from its intersection with $S(1/0;\ 1)$ toward its irrational end an S-circle of a chain between two integral points must be encountered. This S-circle is not one which has been encountered before. It is now transformed into $S(1/0;\ 1)$ by a suitable member of $G(z)$ and the procedure is continued. The process cannot terminate; for otherwise some transform of $z=\omega$ by a member of $G(z)$ would be rational, and this is not possible. *This shows that L cuts infinitely many S-circles of class $[o/e]$ when $k=1$.*

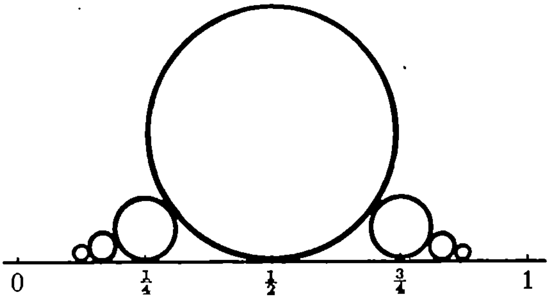

FIG. 2

3. **Proof of the second part of the theorem.** We shall show that there exist irrational numbers for which the inequality (1) is satisfied by only a finite number of fractions of class $[o/o]$ when $k<1$. The geometric properties used to show this are preserved by the modular transformations and hence this result will be valid for the other two classes. The proof will then be complete since the transforms of any point on the real axis by $G(z)$ are everywhere dense on the real axis.

Let K_n be a semicircle with center at $z=-n$, where n is a positive integer, and tangent to $S(-1/1;\ k_n)$ and $S(1/1;\ k_n)$. Because of the tangency conditions the radius of K_n is

$$r_n = ((n-1)^2 + k_n^2)^{1/2} + k_n, \qquad r_n = ((n+1)^2 + k_n^2)^{1/2} - k_n.$$

On solving these equations for r_n and k_n we get

$$r_n = (n^2+1)^{1/2}, \qquad k_n = n(n^2+1)^{-1/2}.$$

The two points at which K_n cuts the real axis are $-n \pm (n^2+1)^{1/2}$, and these are the roots of $z^2+2nz-1=0$. If the left member is regarded as a quadratic form in z and 1, Pell's equation is

$$t^2-(n^2+1)u^2=1.$$

A solution is obviously $t=2n^2+1$, $u=2n$, and this serves to determine the modular transformation

$$T_n(z)=\frac{z+2n}{2nz+(4n^2+1)}$$

whose fixed points are the ends of the semicircle[6] K_n. But any circle passing through the fixed points of a modular transformation is a fixed circle for the transformation. *Then any circle through the ends of K_n is a fixed circle for $T_n(z)$.* We note also that $T_n(z)$ is a member of $G(z)$.

The transformation $T_n(z)$ carries $S(-[2n+1]/1;\ k_n)$ into $S(1/[2n-1];\ k_n)$; and the arc of K_n lying between its points of tangency with these circles is carried into an arc of K_n which adjoins the original arc at its point of tangency with $S(1/[2n-1];\ k_n)$. Repeated application of $T_n(z)$ and its inverse will cover all of K_n, its end points excluded, by transforms of this arc. If no S-circles of class $[o/o]$ intersect the arc of K_n between $S(-[2n+1]/1;\ k_n)$ and $S(1/[2n-1];\ k_r)$, *it will follow that K_n is tangent to infinitely many S-circles of class $[o/o]$ and intersects none.*

It is clear that no S-circle of class $[o/o]$ can cut K_n in the arc between $S(-[2n+1]/1;\ k_n)$ and $S(1/1;\ k_n)$ since the S-circles are external to each other and have maximum diameter 2. The condition that no S-circle of class $[o/o]$ intersect K_n in the arc between $S(1/1;\ k_n)$ and $S(1/[2n-1];\ k_n)$ is found to be

$$p^2+2npq-q^2\geqq 2n,$$

where p and q are odd positive integers, $q<2n$. This can be written in the form

$$2n+(2nq+p+1)(p-1)+(2n-1-q)(q-1)\geqq 2n,$$

which obviously holds, since no terms are negative.

Consider now any value of $k<k_n$. The S-circles for k are smaller than those for k_n and none of class $[o/o]$ touch K_n. Semicircles C', C'', one inside and the other outside K_n, which pass through the ends of K_n and intersect no S-circles of class $[o/o]$ can be constructed (see

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[6] A discussion of Pell's equation can be found in G. B. Mathews, *Theory of Numbers*, part I, Cambridge, 1892.

Fig. 3) since all such semicircles are invariant under $T_n(z)$. There are no S-circles of class $[o/o]$ in the region between C' and C''. If L is a line terminating at one of the end points of K_n, then near the real axis it lies in the region between C' and C''. *Hence L cuts at most a finite number of S-circles of class $[o/o]$.*

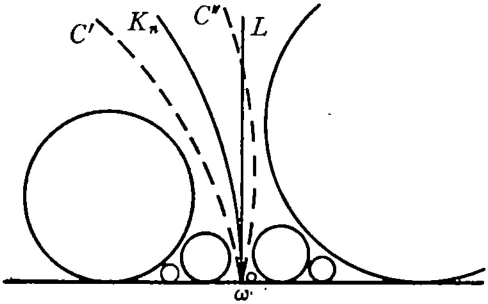

FIG. 3

Thus, for any $k<1$, we need only to choose a positive integer n such that $k<n(n^2+1)^{-1/2}$. The result obtained above shows that the irrationals $-n\pm(n^2+1)^{1/2}$ have only a finite number of fractions of class $[o/o]$ satisfying (1) for this value of k. This completes the proof of the theorem.

ARMOUR INSTITUTE OF TECHNOLOGY

ON THE CONCEPT OF A RANDOM SEQUENCE[1]

ALONZO CHURCH

Von Mises has based his frequency theory of probability on the notion of a *Kollektiv*,[2] that is, of an infinite sequence of trials of an event whose possible outcomes have each a definite probability but otherwise appear entirely at random. (Convenient illustrative examples are an infinite sequence of tosses of a coin, an infinite sequence of rolls of a die, and the like.)[3]

Abstractly the *Kollektiv* may be represented by an infinite sequence of points of an appropriate space, the *Merkmalraum*. Or if the number of possible outcomes of a trial is finite (and it may well be argued that this is always the case for any actual physical observation[4]), it is sufficient to employ an infinite sequence of natural numbers which are less than a fixed natural number. This infinite sequence—of points or of natural numbers—satisfies certain conditions which correspond to those appearing in the description of a *Kollektiv* as just given, and which we shall express by saying that it is a random sequence (*regellose Folge*).

For the present purpose it is largely sufficient to confine attention to the case that each trial has only two possible outcomes, as with the toss of a coin adjudged as falling heads or tails, or the roll of a die adjudged as showing or not showing an ace. The *Kollektiv* may then be represented abstractly by a random sequence of 0's and 1's: in the case of the coin, for instance, we may let 1 correspond to the fall of heads and 0 to tails.

The definition of a random sequence of 0's and 1's as given by von Mises may perhaps be put in the following form:

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[1] Presented to the Society, April 8, 1939.

[2] Richard von Mises, *Grundlagen der Wahrscheinlichkeitsrechnung*, Mathematische Zeitschrift, vol. 5 (1919), pp. 52–99; *Wahrscheinlichkeit, Statistik und Wahrheit*, Vienna, 1928; *Wahrscheinlichkeitsrechnung*, Leipzig and Vienna, 1931; and see especially the second edition of *Wahrscheinlichkeit, Statistik und Wahrheit*, Vienna, 1936, for its account of the objections which have been raised to von Mises's theory and the alternatives which have been proposed.

[3] The introduction of an *infinite* sequence of trials (tosses of a coin, and so on) is, of course, an abstraction from the realities of the situation, made for the sake of the mathematical theory. It is an instance of the familiar device of employing the infinite as being, for certain purposes, a convenient and useful approximation to the large finite.

[4] In cases where the number of possible outcomes of a trial is taken as infinite, either there is a further element of abstraction involved (the infinite again replacing the large finite), or else the problem considered has no direct physical application.

An infinite sequence $a_1, a_2, \cdots$ of 0's and 1's is a random sequence if the two following conditions are satisfied:

(1) If $f(r)$ is the number of 1's among the first r terms of $a_1, a_2, \cdots$, then $f(r)/r$ approaches a limit p as r approaches infinity.

(2) If $a_{n_1}, a_{n_2}, \cdots$ is any infinite sub-sequence of $a_1, a_2, \cdots$, formed by deleting some of the terms of the latter sequence according to a rule which makes the deletion or retention of a_n depend only on n and $a_1, a_2, \cdots, a_{n-1}$, and if $g(r)$ is the number of 1's among the first r terms of $a_{n_1}, a_{n_2}, \cdots$, then $g(r)/r$ approaches the same limit p as r approaches infinity.

The inclusion of condition (2) corresponds to the *Prinzip vom ausgeschlossenen Spielsystem* of von Mises.[5] If a fixed number of wagers of "heads" are to be made, at fixed odds and in fixed amount, on the tosses of a coin, no advantage is gained in the long run if the player, instead of betting at random, follows some system, such as betting on every seventh toss, or (more plausibly) betting on the next toss after the appearance of four tails in succession, or (still more plausibly) making his nth bet after the appearance of $n+4$ tails in succession. This is accepted by von Mises as a sufficiently familiar and uncontroverted empirical generalization to be made fundamental to his theory in this way.

However, this definition, as given by von Mises or as rephrased above, while clear as to general intent, is too inexact in form to serve satisfactorily as the basis of a mathematical theory.

A plausible attempt to state the definition more exactly is the following (the numbers b_i serve as a convenient device to represent a function of a variable number of variables as a function of one variable):

An infinite sequence $a_1, a_2, \cdots$ of 0's and 1's is a random sequence if the two following conditions are satisfied:

(1) If $f(r)$ is the number of 1's among the first r terms of $a_1, a_2, \cdots$, then $f(r)/r$ approaches a limit p as r approaches infinity.

(2) If ϕ is any function of positive integers, if $b_1 = 1$, $b_{n+1} = 2b_n + a_n$, $c_n = \phi(b_n)$, and the integers n such that $c_n = 1$ form in order of magnitude an infinite sequence $n_1, n_2, \cdots$, and if $g(r)$ is the number of 1's among the first r terms of $a_{n_1}, a_{n_2}, \cdots$, then $g(r)/r$ approaches the same limit p as r approaches infinity.

However it has been pointed out by various authors[6] that the definition in this form is self-contradictory, in the sense that it makes the class of random sequences associated[7] with any probability p other than 0 or 1 an empty class. For the failure of (2) may always be shown by taking $\phi(x) = a_{\mu(x)}$, where $\mu(x)$ is the least positive integer m such that $2^m > x$: the sequence $a_{n_1}, a_{n_2}, \cdots$ will then consist of those and only those terms of $a_1, a_2, \cdots$ which are 1's. This means that the definition in this form does not satisfactorily represent the requirement that the deletion or retention of a_n shall not depend on a_n or on the sequence $a_1, a_2, \cdots$ as a whole, but only on n and $a_1, a_2, \cdots, a_{n-1}$. Grave question is raised whether this requirement, made in vague terms by von Mises, can be satisfactorily represented in an exact definition at all.

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[5] Loc. cit.

[6] Erhard Tornier, *Wahrscheinlichkeitsrechnung und Zahlentheorie*, Journal für die reine und angewandte Mathematik, vol. 160 (1929), pp. 177–198; Hans Reichenbach, *Axiomatik der Wahrscheinlichkeitsrechnung*, Mathematische Zeitschrift, vol. 34

This difficulty may be avoided by abandoning the attempt to define a random sequence and substituting some less restricted class of sequences, such as the admissible numbers of Copeland[8] or the equivalent normal sequences of Reichenbach.[9]

These admissible numbers (to adopt Copeland's term) are closely related to the normal numbers of Borel[10]—indeed an admissible number associated with the probability 1/2 is the same as a number *entièrement normal* to the base 2. The definition may be stated as follows: An infinite sequence $a_1, a_2, \cdots$ of 0's and 1's is an admissible number if it is associated with a probability p and if, for every positive integer m and every set of distinct positive integers $r_1, r_2, \cdots, r_k$ which are all less than or equal to m, the sequence whose nth term is the product $a_{nm+r_1}a_{nm+r_2} \cdots a_{nm+r_k}$ is associated with the probability[11] p^k.

(1931–1932), pp. 568–619; E. Kamke, *Über neuere Begründungen der Wahrscheinlichkeitsrechnung*, Jahresbericht der deutschen Mathematiker-Vereinigung, vol. 42 (1932–1933), pp. 14–27; Arthur H. Copeland, *Point set theory applied to the random selection of the digits of an admissible number*, American Journal of Mathematics, vol. 58 (1936), pp. 181–192.

[7] An infinite sequence of 0's and 1's will be said to be associated with the probability p if $f(r)/r$ approaches p as r approaches infinity, where $f(r)$ is the number of 1's among the first r terms of the sequence.

[8] Arthur H. Copeland, *Admissible numbers in the theory of probability*, American Journal of Mathematics, vol. 50 (1928), pp. 535–552. The infinite sequences of 0's and 1's are there taken as binary fractional expansions of real numbers between zero and one.

[9] Loc. cit. (1931–1932). See also Hans Reichenbach, *Les fondements logiques du calcul des probabilités*, Annales de l'Institut Henri Poincaré, vol. 7 (1937), pp. 267–348.

[10] Émile Borel, *Les probabilités dénombrables et leurs applications arithmétiques*, Rendiconti del Circolo Matematico di Palermo, vol. 27 (1909), pp. 247–271, reprinted as Note V to Borel's *Leçons sur la Théorie des Fonctions*, 2d edition, 1914, and 3d edition, 1928.

[11] Copeland imposes the further conditions $p \neq 0$, $p \neq 1$.

The admissible numbers have properties which are sufficient to form a basis for a large part of the theory of probability, and they have the important advantage that their existence, for any assigned probability p, can be proved.[12] Their use for this purpose, however, is open to certain objections from the point of view of completeness of the theory, as has been forcibly urged by von Mises,[13] and it is therefore desirable to consider further the question of finding a satisfactory form for the definition of a random sequence.

The purpose of the present note is to call attention to the following possibility in this connection.

It may be held that the representation of a *Spielsystem* by an arbitrary function ϕ is too broad. To a player who would beat the wheel at roulette a system is unusable which corresponds to a mathematical function known to exist but not given by explicit definition; and even the explicit definition is of no use unless it provides a means of calculating the particular values of the function. As a less frivolous example, the scientist concerned with making predictions or probable predictions of some phenomenon must employ an effectively calculable function: if the law of the phenomenon is not approximable by such a function, prediction is impossible. Thus a *Spielsystem* should be represented mathematically, not as a function, or even as a definition of a function, but as an effective algorithm for the calculation of the values of a function.

Now a formal definition of effective calculability, for functions of positive integers, has been proposed by the author,[14] and the adequacy of this definition to represent the empirical notion of an effective calculation finds strong support in a recent result of Turing.[15]

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[12] Copeland, loc. cit. (1928). See also Borel, loc. cit. (1909).

[13] Mathematische Annalen, vol. 108 (1933), pp. 771–772; *Wahrscheinlichkeit, Statistik und Wahrheit*, 2d edition, pp. 116–117.

[14] Alonzo Church, *An unsolvable problem of elementary number theory*, American Journal of Mathematics, vol. 58 (1936), pp. 345–363. A function of positive integers is defined to be effectively calculable if it has either of the two equivalent properties of λ-definability (in the sense of Church-Kleene) or general recursiveness (in the sense of Herbrand-Gödel). Cf. S. C. Kleene, *λ-definability and recursiveness*, Duke Mathematical Journal, vol. 2 (1936), pp. 340–353, and *General recursive functions of natural numbers*, Mathematische Annalen, vol. 112 (1936), pp. 727–742.

[15] A. M. Turing, *On computable numbers, with an application to the Entscheidungsproblem*, Proceedings of the London Mathematical Society, (2), vol. 42 (1936–1937), pp. 230–265; *A correction*, ibid., vol. 43 (1937), pp. 544–546; and *Computability and λ-definability*, Journal of Symbolic Logic, vol. 2 (1937), pp. 153–163. Turing proves the equivalence of λ-definability and general recursiveness to a notion of computability whose definition, briefly stated, is as follows: A function ϕ is computable if it is possible to make a computing machine, with a finite number of parts of finite size, which will calculate $\phi(n)$ for any assigned n, printing intermediate calculations

It is therefore suggested that this definition of effective calculability be employed in order to define a random sequence as follows:

An infinite sequence $a_1, a_2, \cdots$ of 0's and 1's is a random sequence if the two following conditions are satisfied:

(1) If $f(r)$ is the number of 1's among the first r terms of $a_1, a_2, \cdots$, then $f(r)/r$ approaches a limit p as r approaches infinity.

(2) If ϕ is any effectively calculable function of positive integers, if $b_1=1$, $b_{n+1}=2b_n+a_n$, $c_n=\phi(b_n)$, and the integers n such that $c_n=1$ form in order of magnitude an infinite sequence $n_1, n_2, \cdots$, and if $g(r)$ is the number of 1's among the first r terms of $a_{n_1}, a_{n_2}, \cdots$, then $g(r)/r$ approaches the same limit p as r approaches infinity.

The existence of random sequences in this sense is an immediate consequence of a result of Doob,[16] or alternatively of a theorem of Wald,[17] if use is made of the fact that the set of effectively calculable functions can be represented as a subset of an effectively enumerable set and is therefore itself (noneffectively) enumerable. From Doob's theorem, taken in conjunction with Borel's result[18] that the infinite sequences of 0's and 1's associated with the probability 1/2 (regarded as binary fractional expansions of real numbers between zero and one) form a set of measure one, it follows that the random sequences associated with the probability 1/2 (similarly regarded) form a set of measure one; and from the existence of random sequences associated with the probability 1/2 the existence of random sequences associated with other probabilities is readily derived. From Wald's Theorem I it follows as a corollary that the set of random sequences associated with a fixed probability has the power of the continuum.

That every random sequence is an admissible number is also easily demonstrated. On the other hand, the set of random sequences is more restricted than the set of admissible numbers; this follows, for example, from the existence of admissible numbers $a_1, a_2, \cdots$ such that

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and the final result on a tape with which the machine must be supplied (no upper limit is placed on the time or on the length of tape required for a particular calculation). Actually, Turing imposes several further conditions on the computing machine, but these are more or less clearly nonessential.

[16] J. L. Doob, *Note on probability*, Annals of Mathematics, (2), vol. 37 (1936), pp. 363–367. The author is indebted to A. H. Copeland for calling his attention to the significance of Doob's theorem in this connection, as well as to the matter of effective constructibility of admissible numbers (footnote 19), and for other suggestions.

[17] Abraham Wald, *Sur la notion de collectif dans le calcul des probabilités*, Comptes Rendus des Séances de l'Académie des Sciences, vol. 202 (1936), pp. 180–183, and *Die Widerspruchsfreiheit des Kollektivbegriffes der Wahrscheinlichkeitsrechnung*, Ergebnisse eines mathematischen Kolloquiums, vol. 8 (1937), pp. 38–72.

[18] Loc. cit. (1909).

a_n is an effectively calculable function of n, a property[19] which clearly cannot be possessed by any random sequence.

Thus an existence proof for random sequences is necessarily non-constructive.

Use of the above proposed definition of a random sequence as fundamental to the theory of probability is consequently open to the objection that by its means such otherwise apparently combinatorial matters as elementary questions of probability in connection with the tossing of a coin are made to depend on the powerful (and dubious) non-constructive methods of analysis. It is clear, however, that any definition of a random sequence more stringent than this one would have the same disadvantage, and on the other hand that no definition in any respect less stringent could be regarded as even approximately representing von Mises's intention or as being free from such objections as those brought by him against the use of admissible numbers or normal sequences.

Nevertheless it would seem to be of interest to investigate criteria of randomness of intermediate strength, in particular the definition of a random sequence which results if the condition that ϕ be effectively calculable is replaced by the condition that ϕ be primitive recursive. Since the primitive recursive functions are effectively enumerable, sequences satisfying this criterion can be effectively constructed in accordance with Wald's Theorem V.[20]

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[19] The example of von Mises, Mathematische Annalen, vol. 108 (1933), p. 769, is readily specialized so as to have this property. A very neat effective construction of an admissible number is implied in a paper of D. G. Champernowne, *The construction of decimals normal in the scale of ten*, Journal of the London Mathematical Society, vol. 8 (1933), pp. 254–260.

[20] Loc. cit. (1937). Wald relies on the common notion of effectiveness and has no exact definition. His proof is entirely applicable here. Wald also remarks on a criterion of randomness—in general more stringent than that proposed in the present paper—which consists, in effect, in replacing the condition that ϕ be effectively calculable by the condition that ϕ be definable within a fixed system of symbolic logic L. There are, however, several objections to this criterion. It is unavoidably relative to the choice of the particular system L and thus has an element of arbitrariness which is artificial. If used within the system L, it requires the presence in L of the semantical relation of *denotation* (known to be problematical on account of the Richard paradox). If it is used outside of L, it becomes necessary to say more exactly what is meant by "definable in L," and the questions of consistency and completeness of L are likely to be raised in a peculiarly uncomfortable way.

TYPICALLY-REAL FUNCTIONS WITH $a_n = 0$ FOR $n \equiv 0 \pmod 4$[1]

M. S. ROBERTSON

1. **Introduction.** Let

$$f(z) = z + \sum_{2}^{\infty} a_n z^n \tag{1.1}$$

be typically-real for $|z| < 1$; that is, $f(z)$ within this circle is regular and takes on real values when and only when z is real. In particular, if $f(z)$ is univalent for $|z| < 1$ and has real coefficients, it is also typically-real. We suppose in addition that

$$a_n = 0 \qquad \text{for } n \equiv 0 \pmod 4. \tag{1.2}$$

In this paper we obtain sharp inequalities for the coefficients a_n.

Sharp inequalities for a_n are already well known[2] with the more restrictive condition

$$a_n = 0 \qquad \text{for } n \equiv 0 \pmod 2 \tag{1.3}$$

holding. In this case $|a_n| \leqq n$ with equality occurring for the odd function $(z+z^3)(1-z^2)^{-2}$. If besides, $f(z)$ is univalent and real on the real axis, the coefficients are bounded and satisfy[3] the inequalities

$$|a_{2n-1}| + |a_{2n+1}| \leqq 2, \qquad |a_3| \leqq 1. \tag{1.4}$$

With the less restrictive condition (1.2) replacing (1.3) the author obtains the following new and sharp inequalities:

$$|a_n| + 2^{-3/2}[(n-2)|a_{2m}| + n|a_2|] \leqq n, \qquad m, n \text{ odd}, n > 1; \tag{1.5}$$

$$|a_n| + 2^{-1/2}(n-1)|a_2| \leqq n, \qquad n \text{ odd}; \tag{1.6}$$

$$|a_n| + |a_2| \leqq 2^{3/2}, \qquad |a_2| \leqq 2^{1/2}, \qquad n \text{ even}. \tag{1.7}$$

In each case the equality sign holds for the typically-real function

$$z(1 - 2^{1/2}z + z^2)^{-1} = 2^{1/2}\sum_{1}^{\infty} \sin n\pi/4 \cdot z^n.$$

Since this function is also univalent for $|z| < 1$, the inequalities above are sharp also for the class of univalent functions with real coefficients for which (1.2) holds.

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[1] Presented to the Society, September 8, 1939.

[2] See W. Rogosinski, *Über positive harmonische Entwicklungen und typisch-reelle Potenzreihen*, Mathematische Zeitschrift, vol. 35 (1932), pp. 93–121.

[3] See J. Dieudonné, *Polynomes et fonctions bornées d'une variable complexe*, Annales de l'École Normale Supérieure, vol. 48 (1931), pp. 247–358.

Since (1.5) may be written in the form

$$|a_{2m}| + |a_2| \leqq 2^{3/2}\left[1 - \limsup_{n\to\infty} |a_n/n|\right], \tag{1.8}$$

(1.7) will follow at once as well as the following theorem.

THEOREM. *If within the unit circle the typically-real function*

$$f(z) = z + \sum_{2}^{\infty} a_n z^n, \qquad a_n = 0 \text{ for } n \equiv 0 \pmod 4,$$

has $\limsup_{n\to\infty} |a_n/n| = 1$, *then* $f(z)$ *is an odd function; that is to say,* $a_n = 0$ *for* $n \equiv 0 \pmod 2$.

In a recent paper[4] the author discussed a similar problem when $a_n = 0$ for $n \equiv 0 \pmod p$, p odd, and particularly for $p = 3$. The method used in that paper does not generalize completely to $p > 3$. Certain modifications in the method were necessary to take care of asymmetric phases which appear when $p > 3$, and these are given here for $p = 4$. The method appears to fail completely for $p > 4$.

2. **Proof of the inequalities.** Let $\Im f(re^{i\theta}) = v(r, \theta)$, for $r < 1$. Since $f(z)$ is typically-real for $|z| = r < 1$,

$$\begin{aligned} &v(r, \theta) > 0 \quad \text{for} \quad 0 < \theta < \pi, \qquad v(r, \theta) < 0 \quad \text{for} \quad \pi < \theta < 2\pi, \\ &v(r, \pi - \theta) = -v(r, \pi + \theta), \qquad v(r, \theta) = -v(r, -\theta). \end{aligned} \tag{2.1}$$

In what follows we shall write $v(r, \theta)$ as simply $v(\theta)$. Since also

$$a_n = 0 \qquad \text{for } n \equiv 0 \pmod 4, \tag{2.2}$$

it follows that

$$f(z) + f(ze^{\pi i/2}) + f(ze^{\pi i}) + f(ze^{3\pi i/2}) \equiv 0, \tag{2.3}$$

and in particular the imaginary part of the left-hand member is zero. We write this as

$$v(\theta) + v(\pi/2 + \theta) - v(\pi - \theta) - v(\pi/2 - \theta) \equiv 0. \tag{2.4}$$

The coefficients of $f(z)$ are given by

$$a_n = \frac{2}{\pi r^n}\int_0^\pi v(\theta) \sin n\theta d\theta. \tag{2.5}$$

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[4] See M. S. Robertson, *On certain power series having infinitely many zero coefficients*, Annals of Mathematics, (2), vol. 40 (1939), pp. 339–352.

Let

$$\int_0^\pi v(\theta)\sin n\theta d\theta = \int_0^{\pi/4} + \int_{\pi/4}^{\pi/2} + \int_{\pi/2}^{3\pi/4} + \int_{3\pi/4}^{\pi} \tag{2.6}$$
$$= I_1 + I_2 + I_3 + I_4.$$

In I_2 let $\theta = \pi/2 - \phi$ and obtain

$$I_2 = \int_0^{\pi/4} v(\pi/2 - \phi)\sin n(\pi/2 - \phi)d\phi. \tag{2.7}$$

In I_3 let $\theta = \pi/2 + \phi$ and obtain

$$I_3 = \int_0^{\pi/4} v(\pi/2 + \phi)\sin n(\pi/2 + \phi)d\phi. \tag{2.8}$$

In I_4 let $\theta = \pi - \phi$ and obtain

$$I_4 = \int_0^{\pi/4} v(\pi - \phi)\sin n(\pi - \phi)d\phi. \tag{2.9}$$

In I_1 substitute for $v(\theta)$ the value obtained from (2.4). Combining the new forms for I_1, I_2, I_3, and I_4 we have

$$\begin{aligned} &\int_0^\pi v(\phi)\sin n\phi d\phi \\ &\quad = \int_0^{\pi/4} \{Av(\pi - \phi) + Bv(\pi/2 - \phi) + Cv(\pi/2 + \phi)\} d\phi, \end{aligned} \tag{2.10}$$

where for brevity we write

$$\begin{aligned} A &= \sin n(\pi - \phi) + \sin n\phi = 2 \sin n\pi/2 \cos n(\pi/2 - \phi), \\ B &= \sin n(\pi/2 - \phi) + \sin n\phi = 2 \sin n\pi/4 \cos n(\pi/4 - \phi), \\ C &= \sin n(\pi/2 + \phi) - \sin n\phi = 2 \sin n\pi/4 \cos n(\pi/4 + \phi). \end{aligned} \tag{2.11}$$

Thus

$$\begin{aligned} \int_0^\pi v(\phi)\sin n\phi d\phi &= 2 \sin n\pi/2 \int_0^{\pi/4} v(\pi - \phi)\cos n(\pi/2 - \phi)d\phi \\ &\quad + 2 \sin n\pi/4 \int_0^{\pi/4} v(\pi/2 - \phi)\cos n(\pi/4 - \phi)d\phi \\ &\quad + 2 \sin n\pi/4 \int_0^{\pi/4} v(\pi/2 + \phi)\cos n(\pi/4 + \phi)d\phi \\ &= K_1 + K_2 + K_3. \end{aligned} \tag{2.12}$$

In K_1 let $\phi=\pi/2-\alpha$, in K_2 let $\phi=\pi/4-\alpha$, and in K_3 let $\phi=\alpha-\pi/4$. Then

$$\tag{2.13}\begin{aligned}\int_0^{\pi} v(\phi)\sin n\phi d\phi &= 2\sin n\pi/2\int_{\pi/4}^{\pi/2} v(\pi/2+\alpha)\cos n\alpha d\alpha\\ &\quad + 2\sin n\pi/4\int_0^{\pi/2} v(\pi/4+\alpha)\cos n\alpha d\alpha.\end{aligned}$$

Hence the formula (2.5) for the coefficients a_n may be replaced by

$$\tag{2.14}\begin{aligned}a_n = \frac{4}{\pi r^n}\Bigg[&\sin n\pi/2\int_{\pi/4}^{\pi/2} v(\pi/2+\phi)\cos n\phi d\phi\\ &+ \sin n\pi/4\int_0^{\pi/2} v(\pi/4+\phi)\cos n\phi d\phi\Bigg].\end{aligned}$$

In particular, since $a_1=1$ we have

$$\tag{2.15}\begin{aligned}1 = \frac{4}{\pi r}&\int_{\pi/4}^{\pi/2} v(\pi/2+\phi)\cos\phi d\phi\\ &+ \frac{2^{3/2}}{\pi r}\int_0^{\pi/2} v(\pi/4+\phi)\cos\phi d\phi.\end{aligned}$$

For even values of $n=2k$, k odd, we have

$$\tag{2.16} a_{2k} = \frac{4(-1)^{k-1}}{\pi r^{2k}}\int_0^{\pi/2} v(\pi/4+\phi)\cos 2k\phi d\phi,$$

whence follows the inequality (to be used later)

$$\tag{2.17} \frac{4}{\pi}\int_0^{\pi/2} v(\pi/4+\phi)d\phi \geqq r^{2m}\,|\,a_{2m}\,|,$$

and in addition the equality

$$\tag{2.18}\begin{aligned}&\frac{4}{\pi}\int_0^{\pi/2} v(\pi/4+\phi)d\phi\\ &\qquad = \frac{8}{\pi}\int_0^{\pi/2} v(\phi+\pi/4)\cos^2 k\phi d\phi + (-1)^k r^{2k} a_{2k}.\end{aligned}$$

From (2.14) we have for odd values of n

$$r^n|a_n| \leqq \frac{4n}{\pi}\int_{\pi/4}^{\pi/2} v(\phi+\pi/2)\cos\phi d\phi + \frac{2^{3/2}}{\pi}\int_0^{\pi/2} v(\phi+\pi/4)d\phi. \tag{2.19}$$

With the aid of (2.18) the last inequality becomes

$$\begin{aligned} r^n|a_n| &+(-1)^{k-1}2^{-1/2}r^{2k}a_{2k} \\ &\leqq \frac{4n}{\pi}\int_{\pi/4}^{\pi/2} v(\phi+\pi/2)\cos\phi d\phi + \frac{2^{5/2}}{\pi}\int_0^{\pi/2} v(\phi+\pi/4)\cos^2 k\phi d\phi \\ &\leqq \frac{4n}{\pi}\int_{\pi/4}^{\pi/2} v(\phi+\pi/2)\cos\phi d\phi + \frac{2^{5/2}k}{\pi}\int_0^{\pi/2} v(\phi+\pi/4)\cos\phi d\phi \\ &= (n-2k)\left[\frac{4}{\pi}\int_{\pi/4}^{\pi/2} v(\phi+\pi/2)\cos\phi d\phi + \frac{2^{3/2}}{\pi}\int_0^{\pi/2} v(\phi+\pi/4)\cos\phi d\phi\right] \\ &\quad + 2k\left[\frac{4}{\pi}\int_{\pi/4}^{\pi/2} v(\phi+\pi/2)\cos\phi d\phi + \frac{2^{3/2}}{\pi}\int_0^{\pi/2} v(\phi+\pi/4)\cos\phi d\phi\right] \\ &\quad - \frac{2^{3/2}(n-2k)}{\pi}\int_0^{\pi/2} v(\phi+\pi/4)\cos\phi d\phi, \end{aligned}$$

whence, on account of the equalities (2.15), (2.18) with $k=1$, and (2.17) for values of $2k<n$, we have

$$\begin{aligned} r^n|a_n| &+ (-1)^{k-1}2^{-1/2}r^{2k}a_{2k} \\ &\leqq rn - \frac{2^{3/2}(n-2k)}{\pi}\int_0^{\pi/2} v(\phi+\pi/4)\cos^2\phi d\phi \\ &= rn - \frac{2^{1/2}}{4}(n-2k)\left[\frac{4}{\pi}\int_0^{\pi/2} v(\phi+\pi/4)d\phi + r^2a_2\right] \\ &\leqq rn - (2^{1/2}/4)(n-2k)\left[r^{2m}|a_{2m}| + r^2a_2\right]. \end{aligned} \tag{2.20}$$

By considering the function $-f(-z)$, which is also typically-real, we obtain an inequality similar to this last one except that a_2 and a_{2k} have been replaced by $-a_2$ and $-a_{2k}$. Consequently, on combining both inequalities and letting r approach one we have for k and n odd

$$\begin{aligned} |a_n| &+ 2^{-3/2}\left[(n-2k)|a_{2m}| \right. \\ &\left. + |(n-2k)a_2+(-1)^{k-1}2a_{2k}|\right] \leqq n, \qquad 2k<n. \end{aligned} \tag{2.21}$$

In particular, for $k=1$ we derive for n odd

$$|a_n| + 2^{-3/2}[(n-2)|a_{2m}| + n|a_2|] \leqq n, \qquad n > 1. \tag{2.22}$$

If in addition $m=1$, then for n odd

$$|a_n| + 2^{-1/2}(n-1)|a_2| \leqq n. \tag{2.23}$$

From (2.22) on dividing by n and letting $n\to\infty$ we have

$$\begin{gathered} |a_{2m}| + |a_2| \leqq 2^{3/2}\left[1 - \limsup_{n\to\infty}\left|\frac{a_n}{n}\right|\right] \leqq 2^{3/2}, \\ |a_2| \leqq 2^{1/2}, \qquad \limsup_{n\to\infty}\left|\frac{a_n}{n}\right| \leqq 1 - 2^{-1/2}|a_2|. \end{gathered} \tag{2.24}$$

Though (2.22), (2.23), and (2.24) hold for m either even or odd, the interesting inequalities are for n and m both odd. In this case they are sharp, as is seen from an inspection of the coefficients of the univalent function

$$z(1 - 2^{1/2}z + z^2)^{-1} = 2^{1/2}\sum_{n=1}^{\infty} \sin n\pi/4 \cdot z^n.$$

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MINIMUM PROBLEMS IN THE FUNCTIONAL CALCULUS[1]

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In the year 1922 Hahn[2] published a proof of a multiplier rule for a problem more general than the well known problem of Bolza, and later, by quite different methods, Graves[3] established analogues of the multiplier rule and of the Weierstrass condition for the problem of Lagrange with integro-differential side conditions. Using Graves' method and considering a somewhat more general problem, the author[4] obtained analogues of the multiplier rule, the Clebsch condition, and the Jacobi condition. All these problems however are concerned with functionals defined over certain special function spaces. In 1934 Lusternik[5] attempted to formulate a multiplier rule for a general problem in a Banach space, but his proofs do not seem satisfactory. Two years later in a doctoral dissertation, written at the University of Kentucky, L. P. Hutchison[6] established a Lagrange rule for such a problem under natural hypotheses on the functions involved, and the author[7] proceeding in another direction generalized the statement and proof of the multiplier rule to abstract spaces.

In the present paper two problems of the functional calculus are considered in detail: the first is concerned with minimizing a functional defined on a region of a normed linear space; and the second restricts the so-called admissible points to satisfy an equation defined by a general operator. For the first of these problems it is shown that the theory of necessary and of sufficient conditions is quite like that for functions of several variables as well as for problems of the calculus of variations. In studying the second problem a multiplier rule is established under four different hypotheses, whose interrelations are studied, and the remaining necessary, as well as the sufficient conditions, are obtained under three of these hypotheses.

1. **The simple problem.** To describe the situation to be treated in

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[1] Presented to the Society, April 15, 1939.

[2] *Ueber die Lagrange'sche Multiplikatorenmethode*, Sitzungsberichte der Akademie der Wissenschaften, Vienna, vol. 131 (1922), pp. 531–550.

[3] *A transformation of the problem of Lagrange in the calculus of variations*, Transactions of this Society, vol. 35 (1933), pp. 675–682.

[4] *The minima of functionals with associated side conditions*, Duke Mathematical Journal, vol. 3 (1937), pp. 418–425.

[5] *Sur les extrêmes relatifs des fonctionnelles* (in Russian), Recueil Mathématique de la Société Mathématique de Moscou, vol. 41 (1934), pp. 390–401.

[6] *On Implicit Function and Lagrange Multiplier Theorems*, 1936.

[7] *A multiplier rule in abstract spaces*, this Bulletin, vol. 44 (1938), pp. 388–394.

this section we may consider a real-valued function f which is defined on a region X_0 of a normed linear space X. The problem of this section is then that of finding a point x in the region X_0 which minimizes the function f in the class of points in X_0. Throughout this paper it will often be convenient to refer to elements in X_0 as *admissible points* and to suppose that f is of class[8] C''.

It is evident from the definitions of the first and second variations of a functional that if x is a minimizing point, then the equation

$$\delta f(x, \xi) = 0$$

must be an identity in ξ, and the lower bound of the second variation $\delta^2 f(x, \xi)$ on the unit sphere must be non-negative. This last condition is, of course, an immediate consequence of the fact that the second variation $\delta^2 f(x, \xi)$ must be non-negative.

We may now ask whether it is possible by slightly strengthening the two necessary conditions stated above to obtain a sufficiency theorem. This question is answered affirmatively in the following theorem:

THEOREM 1.1. *If x_0 is an admissible point at which the first variation $\delta f(x_0, \xi)$ vanishes identically for ξ in the space X and at which the lower bound of the second variation $\delta^2 f(x_0, \xi)$ on the unit sphere is positive, then there is a neighborhood N of the point x_0 such that $f(x) > f(x_0)$ for every $x \neq x_0$ in N.*

To prove the theorem we recall that the function f is of class C'' and hence that the second variation $\delta^2 f(x, \xi)$ is continuous in x uniformly on the unit sphere.[9] Analytically this implies that for every positive e there is a neighborhood N_e of x_0 such that for every ξ on the unit sphere and x in N_e

$$\delta^2 f(x, \xi) > \delta^2 f(x_0, \xi) - e \geqq B - e,$$

where B is the lower bound of $\delta^2 f(x_0, \xi)$ for all points ξ on the unit sphere. The number B however is by hypothesis positive, and hence it is clear that the second variation will be positive at every point x in the neighborhood N_B. To complete the proof it suffices to expand $f(x)$ by means of Taylor's theorem which tells us that there is a positive number t less than unity such that

$$f(x) - f(x_0) = \delta^2 f[x_0 + t(x - x_0), (x - x_0)]/2,$$

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[8] For the definitions and some properties of terms commonly used in the functional calculus see, for example, L. M. Graves, *Topics in the functional calculus*, this Bulletin, vol. 41 (1935), pp. 641–662.

[9] See Graves, loc. cit., p. 651.

since the first variation vanishes identically. This equation, combined with the result reached immediately above, then yields the conclusion of the theorem.

It is interesting to notice that in order to apply the results stated above to the calculus of variations the space X should be taken to be the set of derivatives $y'(x)$ of admissible arcs $y(x)$ and the norm of an element $y'(x)$ to be $[\int_{x_1}^{x_2}(y^2(x)+y'^2(x))dx]^{1/2}$. It can then be shown that if the strengthened form of the Legendre condition holds, our condition on the lower bound of the second variation is equivalent to the Jacobi condition, and a similar remark applies to the strengthened forms of these conditions.

2. **The more general problem.** A problem more general than the one just discussed may be stated, in terms of a function g defined on the region X_0 and having functional values in a Banach space Y, as that of finding an admissible point x which minimizes f in the class of admissible points satisfying the equation

$$g(x) = 0_y. \tag{2.1}$$

For the purposes of the analysis it will be supposed hereafter that the space X is complete and that the function g is of class C'' on the region X_0. It will moreover be assumed that for each admissible x the contradomain, that is, the class of functional values, of $\delta g(x, \xi)$ is closed.

An admissible point x is said to satisfy the *multiplier rule* if there is a constant l and a linear, continuous, and real-valued function L defined on the space Y such that $(l, L) \neq (0, 0)$ and such that the equation

$$l\delta f(x, \xi) + L[\delta g(x, \xi)] = 0$$

is an identity in ξ; and the point x is *regular* in case the contradomain of the function $\delta g(x, \xi)$ is the entire space Y.

THEOREM 2.1. *Every admissible point x that is not regular must satisfy the multiplier rule with the constant l equal to zero.*

To prove this result we note that the contradomain of the variation $\delta g(x, \xi)$ is a proper closed subset of the space Y and hence, by a well known result concerning linear functionals,[10] there is a linear and continuous functional L defined on the space Y and not identically zero, which vanishes at each point in the contradomain of $\delta g(x, \xi)$, that is, $L[\delta g(x, \xi)] = 0$. This functional is, of course, effective in the theorem, and the result is thus proved.

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[10] See, for example, S. Banach, *Théorie des Opérations Linéaires*, Warsaw, 1932, p. 59.

We may now proceed to examine in some detail the case in which x is a regular point. A preliminary result on this case is embodied in the following lemma:

LEMMA 2.1. *Let x be an admissible point which is regular. Then the first variation $\delta f(x, \xi)$ of f at the point x vanishes for every ξ satisfying the equation*

$$\delta g(x, \xi) = 0_y, \tag{2.2}$$

if and only if the point x satisfies the multiplier rule with the constant l equal to unity and with a unique functional L.

To establish this result it is convenient to examine the operation adjoint[11] to $\delta g(x, \xi)$. This operation is defined to be the function $G(L) \equiv L[\delta g(x, \xi)]$, where L varies over the class of linear, continuous functionals on the space Y. Since the equation

$$\delta g(x, \xi) = \eta \tag{2.3}$$

has a solution ξ for every η in Y, a theorem on the relation of a function to its adjoint[12] tells us that to every real-valued, linear, and continuous function k defined on the space X and vanishing at every point ξ which satisfies the equation (2.2) there corresponds a unique L such that $G(L) = k$. But by hypothesis δf is a functional having the same properties as k, and thus the lemma is proved, since the sufficiency of the condition in the lemma is obvious.

We now proceed to seek conditions under which the first variation $\delta f(x, \xi)$ vanishes at the solutions of the equation (2.2). To do this it is convenient to consider the following hypotheses:

(a) For every ξ satisfying (2.2) there is a function $x(b)$ of class C' defined on an interval of the real axis containing the origin, having its functional values admissible solutions of the equation (2.1), and such that $x(0) = x$, $x_b(0) = \xi$.

(b) If M is the class of all solutions of (2.2), there is a manifold N complementary to [13]M.

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[11] Ibid., p. 99 ff.

[12] Ibid., p. 148.

[13] It is obvious that M is linear and closed; and hence, by definition, N is a linear, closed subset of X such that X is the direct sum of M and N, that is, every ξ is uniquely expressible as the sum of a μ in M and a ν in N. See F. J. Murray, *On complementary manifolds and projections in the spaces L_p and l_p*, Transactions of this Society, vol. 41 (1937), pp. 138–152.

(c) There is a linear and closed subset N of the space X in one-to-one correspondence with Y such that if η corresponds to ν then $\delta g(x, \nu) = \eta$.

(d) The space X is the composite of two Banach spaces M and N such that the partial differential $\delta_n g(x, \nu)$ has an inverse[14] at the point x.

LEMMA 2.2. *Let x be an admissible point satisfying the equation* (2.1) *and to which the hypothesis* (a) *is applicable. If then f is a minimum at x for the problem of this section, the first variation $\delta f(x, \xi)$ of f vanishes for every solution ξ of the equation* (2.2).

The proof of this lemma is obvious when one notices that the function $f[x(b)]$ must have a minimum at $b = 0$.

It is now our purpose to investigate certain relations between the hypotheses stated above. These results are contained in the following lemma:

LEMMA 2.3. *If x is an admissible solution of the equation* (2.1), *then the hypothesis* (c) *is equivalent to the statement that x is a regular point at which hypothesis* (b) *is valid, and this statement implies that the hypothesis* (a) *holds at x; in addition, hypothesis* (d) *implies* (a).

If hypothesis (c) holds at the point x, it is evident that x is regular. To prove that the linear closed set N of hypothesis (c) is effective in (b), consider an arbitrary point ξ in X, let $\eta \equiv \delta g(x, \xi)$, and let ν be the point in N corresponding to η. The point $\xi - \nu$ must then lie in the set M, and hence $\xi = \mu + \nu$. To show that this representation is unique it suffices to notice that since N is linear it contains the point 0_x and, by its definition, no other point in common with M. Conversely, it is clear that for every ν in the manifold N of hypothesis (b) there is a unique η such that $\delta g(x, \nu) = \eta$. However since x is regular, there corresponds to every η in Y a point ξ in X such that $\delta g(x, \xi) = \eta$. But ξ is uniquely expressible as $\mu + \nu$ and $\delta g(x, \mu) = 0_y$. It is therefore evident that $\delta g(x, \nu) = \eta$ and that ν is unique.

We may now show that hypothesis (c) implies (a) by considering a solution ξ of the equation (2.2). Let the point x be represented as $m_0 + n_0$ and consider the equation

$$g(m_0 + n + b\xi) = 0_y \tag{2.4}$$

in which it is understood that n is in a neighborhood of the value n_0 and b is so near to zero that the point $m_0 + n + b\xi$ is admissible. This equation has the initial value $(b, n) = (0, n_0)$ at which the differential with respect to n of g, $\delta g(x, \nu)$, has a continuous inverse. For, according to the hypothesis (c), the linear continuous function $\delta g(x, \nu)$ has a single-valued inverse which must be continuous since the spaces Y and N, a linear and closed subset of X, are complete.[15] There is then a unique solution $n(b)$ of (2.4) of class C' for b near zero and such that[16] $n(0) = n_0$. Differentiation of the equation (2.4) with n replaced by $n(b)$ then shows that $n_b(0) = 0_x$, since n_b is in N, and hence that the function $x(b) = m_0 + n(b) + b\xi$ is effective in hypothesis (a).

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[14] This is the assumption under which Hutchison established his multiplier rule.

If hypothesis (d) holds, a proof very much like the one just given suffices to show that (a) also holds.

Combining Lemmas 2.1, 2.2 and 2.3, we have the following multiplier rule:

THEOREM 2.2. *Let x be an admissible point which is regular and satisfies the equation* (2.1). *Then if hypothesis* (a), (b), (c), *or* (d) *holds at x and if the function f is a minimum at x for the problem of this section, the point x must satisfy the multiplier rule with the constant $l = 1$ and with a unique functional L.*

To obtain a further necessary condition we consider a regular point x satisfying (2.1) and suppose that hypothesis (a) is applicable. If f is a minimum at x for the problem being considered, then according to Theorem 2.2 there is a unique function

$$F(x) = f(x) + L[g(x)] \tag{2.5}$$

whose first variation $\delta F(x, \xi)$ vanishes identically, and a consideration of $F[x(b)]$, where $x(b)$ is described in hypothesis (a), serves to establish the following necessary condition:

THEOREM 2.3. *If the hypotheses of Theorem 2.2 hold at the point x, then the lower bound of the second variation $\delta^2 F(x, \xi)$ of the function* (2.5) *in the class of points ξ on the unit sphere satisfying the equation* (2.2) *is non-negative.*

3. **Sufficient conditions.** In order to establish conditions that are sufficient for a minimum it is desirable first to examine in more detail the hypotheses (c) and (d). For this purpose let x_0 be an admissible solution of the equation (2.1), which in case (c) is representable as $m_0 + n_0$ and in case (d) as (m_0, n_0). If we restrict m and n to lie near

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[15] See Banach, op. cit., p. 41.

[16] This result follows from the Hildebrandt-Graves existence theorem for implicit functions; see Graves, loc. cit., pp. 653–661.

to m_0 and n_0, then hypothesis (d) and existence theorems concerning implicit theorems assure us that the equation

$$g(m, n) = g[(m, n)] = 0_y \tag{3.1}$$

has a unique solution $n(m)$ of class C'' such that $n_0 = n(m_0)$. Consequently the point x_0 is a minimizing point for the function f in the class of admissible solutions of (2.1) if and only if the function

$$\phi(m) = f[m, n(m)] \tag{3.2}$$

has a minimum at m_0 in the class of all m in a neighborhood of m_0. In the equation (3.2) immediately above, it is, of course, understood that $f(m, n)$ is the value of f at the place $x = (m, n)$. Differentiation of equations (3.2) and (3.1) yields the relations

$$\begin{aligned}
\delta\phi(m_0, \mu) &= \delta_m f(x_0, \mu) + \delta_n f[x_0, \delta n(m_0, \mu)], \\
0_y &= \delta_m g(x_0, \mu) + \delta_n g[x_0, \delta n(m_0, \mu)], \\
\delta^2 \phi(m_0, \mu) &= \delta^2_{mm} f(x_0, \mu) + 2\delta^2_{mn} f[x_0; \mu, \delta n(m_0, \mu)] \\
&\quad + \delta^2_{nn} f[x_0, \delta n(m_0, \mu)] + \delta_n f[x_0, \delta^2 n(m_0, \mu)], \\
0_y &= \delta^2_{mm} g(x_0, \mu) + 2\delta^2_{mn} g[x_0; \mu, \delta n(m_0, \mu)] \\
&\quad + \delta^2_{nn} g[x_0, \delta n(m_0, \mu)] + \delta_n g[x_0, \delta^2 n(m_0, \mu)].
\end{aligned}$$

If the point x_0 satisfies the multiplier rule with $l = 1$ and if $F(m, n)$ is the value of the function $F[(m, n)]$ in (2.5), then evidently $\delta_m F(x_0, \mu) = \delta_n F(x_0, \nu) = 0$ for every μ and ν; and the equations above, combined in the obvious manner, imply that

$$\begin{aligned}
\delta\phi(x_0, \mu) &= 0, \\
\delta^2 \phi(x_0, \mu) &= \delta^2_{mm} F(x_0, \mu) + 2\delta^2_{mn} F[x_0; \mu, \delta n(m_0, \mu)] \\
&\quad + \delta^2_{nn} F[x_0, \delta n(m_0, \mu)] \\
&= \delta^2 F(x_0, \xi),
\end{aligned} \tag{3.3}$$

where $\xi \equiv [\mu, \delta n(m_0, \mu)]$. By means of these relations we may now state and prove the following sufficiency theorem:

THEOREM 3.1. *Let* x_0 *be an admissible and regular solution of the equation* (2.1) *which satisfies the multiplier rule with* $(l, L) = (1, L)$ *and let either of the hypotheses* (b), (c), *or* (d) *hold at* $x_0 = (m_0, n_0)$. *Furthermore let the lower bound of the second variation* $\delta^2 F(x_0, \xi)$ *of the function* $F(x) = f(x) + L[g(x)]$, *for all* ξ *on the unit sphere and satisfying* (2.2) *be positive. Then there is a neighborhood* N *of the point* x_0 *such that* $f(x) > f(x_0)$ *for every* $x \neq x_0$ *in* N *that satisfies the equation* (2.1).

If hypothesis (d) is valid, then the equations (3.3) established above and the hypotheses of the theorem show that the first variation $\delta\phi(m_0, \mu)$ vanishes for all μ and that there is a positive constant B such that

$$(3.4) \qquad \delta^2\phi(m_0, \mu) = \delta^2F(x_0, \xi) \geqq B\|\xi\|,$$

where $\xi = [\mu, \delta n(m_0, \mu)]$. Since however X is the composite of M and N, the value of $\|\xi\|$ is equal to the greater of $a\|\mu\|$ and $b\|\nu\|$, where a and b are positive constants,[17] and the relation (3.4) then implies that the hypotheses of Theorem 1.1 are satisfied at the point m_0. The theorem now follows for this case, as may be seen with the help of a remark in the first paragraph of this section.

The proof of the theorem when the hypothesis (b) or (c) holds at the point x_0 is similar to the one above but is somewhat more complicated. Since at x_0 hypothesis (c) is applicable every point ξ is uniquely expressible as $\mu(\xi)+\nu(\xi)$; and since both M and N are linear and closed, it is quite easy to see that μ and ν are linear in ξ. To show that they are also continuous we may consider a sequence ξ_n which converges to a point ξ_0 and is such that the sequence $\mu(\xi_n)$ converges to μ_0. It is then an immediate consequence of the closure of the set M and the uniqueness of the representation of each ξ that $\mu(\xi_0)=\mu_0$, and hence by a theorem on linear operators it follows that μ is continuous.[18] Similarly it may be shown that ν is continuous, and therefore since $\xi=\mu+\nu$, it is true that there is a positive constant $k\leqq 2$ such that

$$(3.5) \qquad \text{greater } [k\|\mu\|, k\|\nu\|] \leqq \|\xi\| \leqq \text{greater } [2\|\mu\|, 2\|\nu\|].$$

It now remains only to show that if we regard X as the composite of the spaces M and N and choose for the norm of (μ, ν) the greater of the numbers $a\|\mu\|$, $a\|\nu\|$, where $k\leqq a\leqq 2$, then the function g is of class C''. Having shown that g has this property, we see that the function $\phi(m)$ defined in (3.2) is well-defined and satisfies the relations (3.3) and (3.4), and therefore by the first of inequalities (3.5) that there is a positive number c such that

$$\delta^2\phi(m_0, \mu) = c\|\mu\|, \qquad \mu \text{ in } M.$$

The proof may then be completed by methods like those used before.

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[17] See Hildebrandt and Graves, *Implicit functions and their differentials in general analysis*, Transactions of this Society, vol. 29 (1927), pp. 127–153.

[18] See, Banach, op. cit., p. 41.

SOME THEOREMS ON CONTINUA[1]

EDWIN W. MILLER

The following theorem, proved by A. Mullikin in her thesis,[2] has been used extensively in certain point set theoretic investigations.

THEOREM. *If C is a continuum and F_1 and F_2 are closed, mutually exclusive and nonvacuous subsets of C, then there exists a component of $C-(F_1+F_2)$ which has a limit point in F_1 and a limit point*[3] *in F_2.*

In §1 we shall obtain a theorem (Theorem 1) which represents a strengthening of Mullikin's result. The theorem, in this stronger form, has numerous applications. We shall discuss some of these applications in §2 and §3.

1. A stronger form of Mullikin's theorem.

We prove the following theorem:

THEOREM 1. *Under the hypotheses of Mullikin's theorem, there exists a constituent*[4] *of $C-(F_1+F_2)$ which has a limit point in F_1 and a limit point in F_2.*

PROOF. Let $\{G_n\}$ and $\{H_n\}$ denote monotonic decreasing sequences of open sets which close down upon F_1 and F_2 respectively. We may suppose that G_1 and H_1 are so chosen that $\overline{G}_1\cdot\overline{H}_1=0$. Now, by Mullikin's theorem, there is a connected subset of $C-C\cdot(\overline{G}_n+\overline{H}_n)$ which has a limit point in $\overline{G}_n$ and a limit point in $\overline{H}_n$. The closure of such a connected set is a subcontinuum of C which "extends" from $\overline{G}_n$ to $\overline{H}_n$. If, then, we denote by Q_n the set of all points of $C-C\cdot(G_n+H_n)$ which lie on subcontinua of C which extend (in the above indicated sense) from $\overline{G}_n$ to $\overline{H}_n$, we have $Q_n\neq 0$.

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[1] Presented to the Society, April 10, 1937, under the title *On a theorem due to A. Mullikin.*

[2] *Certain theorems relating to plane connected point sets,* Transactions of this Society, vol. 24 (1922), pp. 144–162.

[3] In Mullikin's statement of this theorem, C is a bounded plane continuum. It is clear, however, that her proof applies if C is any continuum in a compact metric space. In the present paper, *all point sets under consideration are understood to be embedded in a compact metric space,* except when the contrary is expressly stated. If M is a point set and p is a point of M, then the maximal connected subset of M which contains p is called the component of M determined by p.

[4] A set of points K is said to be strongly connected, or to be a semicontinuum, if every pair of points of K lies on some continuum contained in K. If M is a point set and p is a point of M, then the maximal strongly connected subset of M which contains p is called the constituent of M determined by p.

By the same argument it follows that any subcontinuum of C which extends from $\overline{G}_n$ to $\overline{H}_n$ contains a continuum which extends from $\overline{G}_{n-1}$ to $\overline{H}_{n-1}$. This continuum, in turn, contains a continuum which extends from $\overline{G}_{n-2}$ to $\overline{H}_{n-2}$, and so on, so that for any n we have $\prod_{k=1}^{n} Q_k \neq 0$.

It will now be shown that for every n, the set Q_n is closed. Let p be a limit point of any sequence of points $p_1, p_2, \cdots, p_m, \cdots$ of Q_n. Each point p_m belongs to a subcontinuum of C extending from $\overline{G}_n$ to $\overline{H}_n$. If p is not a point of any of these continua, there will exist, according to a theorem due to Janiszewski,[5] a limit continuum, again a subcontinuum of C, which contains p and extends from $\overline{G}_n$ to $\overline{H}_n$.

Let $A_n = \prod_{k=1}^{n} Q_k$. It is clear that the sets $A_1, A_2, \cdots, A_n, \cdots$ form a monotonic decreasing sequence of closed, compact and nonvacuous sets. There will exist, according to a well known theorem due to Cantor, at least one point p common to all the sets A_n. Clearly, $p \varepsilon \prod_{n=1}^{\infty} Q_n$. From the definition of Q_n it is obvious that the constituent of $C-(F_1+F_2)$ which such a point p determines has a limit point in F_1 and a limit point in F_2.

2. **Applications to continua in the plane.** In this section we shall use Theorem 1 to obtain certain results regarding the common boundary of two plane domains.

THEOREM 2. *If B is a bounded plane continuum which is the common boundary of two or more domains, and if C is a closed proper subset of B which contains at least two points, then there is a constituent of $B-C$ which has at least two limit points in C.*

PROOF. If C is not a continuum, then $C=C_1+C_2$, where C_1 and C_2 are closed, nonvacuous sets, and $C_1 \cdot C_2 = 0$. Then, by Theorem 1, there is a constituent of $B-C$ which has a limit point in C_1 and a limit point in C_2, and therefore two limit points in C.

Suppose now that C is a continuum. We may distinguish two cases according as B is an indecomposable or a decomposable continuum.

Suppose that B is an indecomposable continuum. Then, since C is a proper subcontinuum of B, it lies entirely in some composant of B. Let K be any other composant of B. Then $K \subset B-C$. Now, it is well known that any composant of an indecomposable continuum is dense in that continuum.[6] Therefore every point of C is a limit point of K. But clearly, K is a constituent of $B-C$.

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[5] Z. Janiszewski, *Sur les continus irréductibles entre deux points*, Journal de l'École Polytechnique, (2), vol. 16 (1912), p. 98, Theorem 1.

[6] See Z. Janiszewski and C. Kuratowski, *Sur les continus indécomposables*, Fundamenta Mathematicae, vol. 1 (1920), p. 221, Theorem 8.

Suppose now that B is a decomposable continuum. In this case, according to a theorem proved independently by C. Kuratowski[7] and W. A. Wilson,[8] the continuum B is the sum of two continua K and L, such that $K \cdot L = M + N$, where M and N are closed, mutually exclusive and nonvacuous sets such that K and L are irreducible between M and N.

It is easily seen that there are just four essentially different cases here:

1. $C \subset K - (M+N)$,
2. $C \cdot M \neq 0$ and $C \cdot N = 0$,
3. $C = K$,
4. $C \supset K$ and $C \neq K$.

Case 1. Let K_1 be a constituent of $K - C$ determined by a point of M, and K_2 a constituent of $K - C$ determined by a point of N. Then, from a theorem due to Janiszewski,[9] the constituents K_1 and K_2 have each at least one limit point in C. Then obviously the set $K_1 + K_2$ has at least two limit points in C, for otherwise $\overline{K_1 + K_2}$ would be a proper subcontinuum of K which contains points of M and points of N. It follows that $L + K_1 + K_2$ is contained in a constituent of $B - C$ which has at least two limit points in C.

Case 2. Let x be any point of N. Denote by P the constituent of $K - C$ determined by x, and by Q the constituent of $L - C$ determined by x. If either P or Q has as many as two limit points in C, then a constituent of $B - C$ of the desired sort is determined. Assume that P has just one limit point, y, in C, and that Q has just one limit point, z, in C. Now if $y = z$, then y must be a point of M, and it follows easily that either $\overline{P}$ is a proper subcontinuum of K which contains a point of M and a point of N, or $\overline{Q}$ is a proper subcontinuum of L which contains a point of M and a point of N. Hence $y \neq z$, and $P + Q$ is contained in a constituent of $B - C$ which has at least two limit points in C.

Case 3. By Theorem 1, there is a constituent of $L - (M+N)$ which has a limit point in M and a limit point in N. This is the constituent sought.

Case 4. In this case $C \cdot L$ is a closed, nonvacuous and proper subset of L. It cannot be a continuum since L is irreducible between M and N, and $C \cdot L$ contains both M and N. Accordingly, $C \cdot L = C_1 + C_2$, where C_1 and C_2 are closed, mutually exclusive, nonvacuous subsets of L. Hence Theorem 1 applies and a constituent of the desired sort is obtained.

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[7] C. Kuratowski, *Sur la séparation des ensembles*, Fundamenta Mathematicae, vol. 12 (1928), p. 235, Theorem 7.

[8] W. A. Wilson, *On bounded regular frontiers in the plane*, this Bulletin, vol. 34 (1928), p. 86, Theorem 6.

[9] Z. Janiszewski, loc. cit., Theorem 4.

THEOREM 3. *If D is a bounded and connected plane domain and $M=\sum_{n=1}^{\infty}M_n$, where $M_m \cdot M_n=0$ if $m\neq n$, and each set M_n is a continuum which does not cut the plane and which lies, except for one point at most, entirely in D, then $D-D\cdot M$ is connected.*

PROOF. Let us assume that $W=D-D\cdot M$ is not connected. Then, according to a theorem of Knaster and Kuratowski,[10] there is a continuum K which cuts the plane between two points a and b of W and lies entirely in the complement of W. Now, if H denotes the boundary of D, it is easily proved that the closed set $H+K\cdot D$ cuts the plane between a and b. It follows by theorems due to Mazurkiewicz[11] and Kuratowski[12] that $H+K\cdot D$ contains a continuum L which is the common boundary of two domains which contain a and b respectively.

Now the set $L\cdot H$ must contain more than one point—in fact, it must contain a nondenumerable infinity of points. Otherwise, as follows easily from our hypotheses on the continua M_n, the equation $L=L\cdot H+\sum_{n=1}^{\infty}L\cdot M_n$ determines a decomposition of the bounded continuum L into a denumerable infinity of closed, mutually exclusive and nonvacuous sets. This is impossible as Sierpiński has shown.[13]

Since the closed set $L\cdot H$ contains more than one point and is a proper subset of L, we may apply Theorem 2. There exists, then, a constituent Q of $L-L\cdot H$ which has at least two limit points in H. Now Q must contain points of two different continua M_n, since each continuum M_n contains at most one point of H. Then Q, and therefore M, contains a continuum which contains points of two different sets M_n. But this contradicts Sierpiński's theorem.

3. **Extensions of Sierpiński's theorem.** The theorem of Knaster and Kuratowski employed in the proof of Theorem 3 is of considerable importance in establishing the existence of different sorts of con-

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[10] B. Knaster and C. Kuratowski, *Sur les ensembles connexes*, Fundamenta Mathematicae, vol. 2 (1921), p. 233, Theorem 37.

[11] S. Mazurkiewicz, *Sur un ensemble G_δ, punctiforme, qui n'est pas homéomorphe avec aucun ensemble linéaire*, Fundamenta Mathematicae, vol. 1 (1920), p. 63, Theorem 1.

[12] C. Kuratowski, *Sur les coupures irréductibles du plan*, Fundamenta Mathematicae, vol. 6 (1925), p. 133, Theorem 3.

[13] W. Sierpiński, *Un théorème sur les continus*, Tôhoku Mathematical Journal, vol. 13 (1918), pp. 300–303.

nected sets.[14] The content of the theorem is that if we construct a bounded plane set in such a way that no continuum in its complement separates any two points of the set, then we may be sure that the set constructed is connected. From this point of view the theorem of Sierpiński employed in the proof of Theorem 3 is of importance. From Sierpiński's theorem and the theorem of Knaster and Kuratowski it easily follows, for example, that if we delete from the interior of a circle a set which is the sum of a denumerable infinity of mutually exclusive closed sets, no one of which cuts the plane, then the set of points remaining is connected. It is clear that extensions of Sierpiński's theorem will yield us more general procedures for the construction of connected sets. In this section we shall accordingly establish a few theorems relating to the following general question. If A and M are point sets in a compact metric space and $M=\sum_{n=1}^{\infty}M_n$, where the M_n are mutually exclusive closed sets, under what circumstances will $A+M$ fail to contain an *M-join*, that is, a continuum which contains a point of each of two different sets M_n?

THEOREM 4. *Let P be a closed punctiform set and let $M=\sum_{n=1}^{\infty}M_n$, where* (1) *the sets M_n are mutually exclusive and closed, and* (2) *every component of every set M_n has at most one point in common with P. Then $M+P$ contains no M-join.*

PROOF. Suppose the contrary, and let C denote an M-join contained in $M+P$. Then from Sierpiński's theorem, $C\cdot P$ is nondenumerable. (For our purpose it is enough that $C\cdot P$ contains more than one point.) We may write $C\cdot P=C_1+C_2$, where C_1 and C_2 are closed, mutually exclusive and nonvacuous. By Theorem 1, there is a constituent Q of $C-C\cdot P$ which has a limit point in C_1 and a limit point in C_2. From condition (2), the semicontinuum Q must contain points of two different sets M_n. Therefore Q contains an M-join. But this contradicts Sierpiński's theorem.

THEOREM 5. *Let P be a closed punctiform set and $M=\sum_{n=1}^{\infty}M_n$, where* (1) *the sets M_n are mutually exclusive and closed, and* (2) *there exists an $\epsilon>0$ such that K is of diameter greater than ϵ if K contains more than one point of P and is a continuum contained in any M_n. Then $M+P$ contains no M-join.*

PROOF. Suppose the contrary, and let C denote an M-join contained in $M+P$. Then, by Sierpiński's theorem, $C\cdot P-C\cdot M\neq 0$. Let

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[14] See Knaster and Kuratowski, loc. cit. See also P. M. Swingle, *Two types of connected sets*, this Bulletin, vol. 37 (1931), pp. 254–258, and E. W. Miller, *Concerning biconnected sets*, Fundamenta Mathematicae, vol. 29 (1937), pp. 123–133.

x be a point of $C \cdot P - C \cdot M$. Then, by a theorem due to Janiszewski,[15] C contains a subcontinuum C_1 which contains x and is of diameter less than ϵ. Clearly, C_1 is an M-join. However, it is clear from the hypotheses of the present theorem that the sets $C_1 \cdot M_n$ satisfy the hypotheses of Theorem 4, so that the existence of C_1 contradicts Theorem 4.

THEOREM 6. *Let* $M = \sum_{n=1}^{\infty} M_n$, *where the sets* M_n *are mutually exclusive and closed. Let* $P = \sum_{k=1}^{\infty} P_k$, *where the sets* P_k *are punctiform and closed. If for every* k, *the set* $M + P_k$ *contains no M-join, then* $M + P$ *contains no M-join.*

We omit the proof of Theorem 6 since it closely follows the lines of the proof of Sierpiński's theorem.

In the present connection the following question is a natural one. Let $M = \sum_{n=1}^{\infty} M_n$, where the sets M_n are mutually exclusive and closed, and let $N = \sum_{k=1}^{\infty} N_k$, where the sets N_k are mutually exclusive and closed. If for every k the set $M + N_k$ contains no M-join, will it be true that $M + N$ contains no M-join? The following example shows that this is not necessarily the case.

On a linear interval (a, b) take a nowhere dense perfect set whose first point on (a, b) is a and whose last point is b. This will be the set N_1. Now on each interval (a', b') contiguous to N_1 take a nowhere dense perfect set whose first point on (a', b') is a' and whose last point is b'. These perfect sets will be our sets M_n. We take as our sets $N_2, N_3, N_4, \cdots$ the various closed intervals contiguous to the various sets M_n. It is easily seen that the conditions mentioned above are satisfied, and yet $M + N$ is the entire linear interval (a, b). For future reference it may be noticed here that N_1 has a point in common with infinitely many, in fact, with all of the sets M_n, and that there is a nondenumerable infinity of components of N_1 which have no point in common with M.

LEMMA. *If* (1) $M = \sum_{n=1}^{\infty} M_n$, *where the sets* M_n *are mutually exclusive, closed and nonvacuous, and* (2) $N = \sum_{k=1}^{\infty} N_k$, *where the sets* N_k *are mutually exclusive, closed and nonvacuous, and* (3) *for every* k *the set* $M + N_k$ *contains no M-join, then* $C \cdot M_n \neq 0$ *for infinitely many indices* n, *if* C *is an M-join contained in* $M + N$.

PROOF. Suppose that $C \cdot M_n \neq 0$ for only a finite number of indices n. To simplify the notation, let us suppose that $C \cdot M_n \neq 0$ for $n = 1, 2, \cdots, r$, and $C \cdot M_n = 0$ for $n > r$. By Theorem 1, there is a constituent Q of $C - C \cdot (M_1 + M_2 + \cdots + M_r)$ which has a limit point in M_1 and a limit point in $M_2 + \cdots + M_r$. We have, of course, $Q \cdot M = 0$. Therefore $Q \subset N$. By Sierpiński's theorem, Q must be a subset of some one set N_k. But this set N_k would then contain an M-join. As this contradicts condition (3), the result follows.

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[15] Z. Janiszewski, loc. cit., Theorem 4.

THEOREM 7. *If conditions* (1), (2) *and* (3) *of the preceding lemma hold, and if* (4) *for every* k, $M_n \cdot N_k \neq 0$ *for at most one* n,[16] *then* $M+N$ *contains no* M*-join.*

PROOF. Let us assume that $M+N$ contains an M-join C. We will first show that C must contain an M-join C_1 such that $C_1 \cdot (M_1 + N_1) = 0$.

By the preceding lemma and by condition (4), there is a positive integer j such that $C \cdot M_j \neq 0$ and $M_j \cdot (M_1 + N_1) = 0$. If $C \cdot (M_1 + N_1) \neq 0$, let K be a subcontinuum of C irreducible between[17] $M_1 + N_1$ and M_j.

Case 1. Suppose K is a decomposable continuum. Then $K = K_1 + K_2$, where K_1 and K_2 are proper subcontinua of K, $K_1 \supset K \cdot (M_1 + N_1)$, $K_1 \cdot M_j = 0$, $K_2 \supset K \cdot M_j$ and $K_2 \cdot (M_1 + N_1) = 0$. By Theorem 1 there is a constituent Q of $K - K \cdot (M_1 + N_1 + M_j)$ which has a limit point in M_j and a limit point in $M_1 + N_1$. From the conditions of the present theorem and from Sierpiński's theorem it follows easily that $Q \cdot M \neq 0$. Furthermore we have $Q \cdot K_2 \neq 0$. Then, clearly, $Q + K_2$ contains an M-join C_1 such that $C_1 \cdot (M_1 + N_1) = 0$.

Case 2. Suppose that K is an indecomposable continuum. Since K is irreducible between $M_1 + N_1$ and M_j, there exists a composant S_1 of K such that $S_1 \cdot (M_1 + N_1) \neq 0$ and $S_1 \cdot M_j = 0$. For the same reason there exists a composant S_2 of K such that $S_2 \cdot M_j \neq 0$ and $S_2 \cdot (M_1 + N_1) = 0$. If S_2 contains an M-join C_1, then $C_1 \cdot (M_1 + N_1) = 0$. We will suppose, then, that S_2 contains no M-join. Then, since S_2 is dense in K, the sets N_1 and M_n (for $n \neq j$) are all nowhere dense in K. Likewise M_j is nowhere dense in K since S_1 is dense in K. Therefore, in virtue of the theorem of Baire, there must exist a positive integer $h \neq 1$ such that N_h is dense in some region of K. It follows that there exists a continuum L in S_2 such that $L \cdot M_j \neq 0$ and $L \cdot N_h \neq 0$. Suppose $M_j \cdot N_h = 0$. Then, by Theorem 1, there is a constituent S of $L - L \cdot (N_h + M_j)$ which has a limit point in N_h and a limit point in M_j. It is easily shown that $S \cdot M \neq 0$. But then L, and therefore S_2, would contain an M-join. We may assume, then, that $M_j \cdot N_k \neq 0$. Now, there is a constituent Q of $K - K \cdot (M_1 + N_1 + M_j)$ which has a limit point in $M_1 + N_1$ and a limit point in M_j. We will have $Q \cdot M \neq 0$. Now Q must contain an M-join, for otherwise, $Q \cdot M = Q \cdot M_r$ where $r \neq j$, and since Q is dense in K, it would follow by the argument just used that $M_r \cdot N_k \neq 0$. But this is impossible by condition (4). This completes the proof that C must contain an M-join C_1 such that $C_1 \cdot (M_1 + N_1) = 0$.

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[16] Condition (4) can be replaced by (4'): For each given k, either (a) $M_n \cdot N_k \neq 0$ for at most a finite number of indices n, or (b) at most a countable infinity of components of N_k fail to have a point in common with M. In fact, it is not hard to show that (1), (2), (3), and (4') imply that N can be re-expressed as the sum of a countable infinity of sets in such a way that for these new sets N_k, conditions (2), (3) and (4) are satisfied.

[17] Z. Janiszewski, loc. cit., p. 109, Theorem 1.

In the same way as above, we now can prove that C_1 contains an M-join C_2 such that $C_2 \cdot (M_2 + N_2) = 0$. In general, we have $C_{k-1} \supset C_k$, where C_k is an M-join such that $C_k \cdot (M_k + N_k) = 0$. By the Cantor product theorem, there is at least one point p common to all the continua C_k. But this is impossible, since p must belong to $M + N$ and yet does not belong to $M_k + N_k$ for any k. This completes the proof of our theorem.

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ON INCIDENCE GEOMETRY

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The purpose of this note is to analyze the conditions needed in geometry to introduce ideal points without using order relations. Since only incidence relations are used, it is convenient to use the notation of lattice theory. The actual introduction of the ideal elements is a purely algebraic process belonging to the theory of ideal extension and will be given elsewhere. (See abstracts 44-5-201, 45-1-16, 45-1-17.) Beside conditions already familiar in lattice theory we need conditions for the existence of products (see definition of σ-lattice) and the obviously necessary condition for projectivization, Condition E. The conditions for the existence of products are needed because incidence geometry is not taken to be a lattice; in obtaining the projective extension it would be inconvenient to have to redefine the product of, say, two parallel lines; such a product, therefore, is left undefined. Condition E is not proved independent since our purpose is merely the elimination of considerations of order. Condition E has more force the greater the dimension of the space in which it operates. For dimension greater than 3 the development is consequently straightforward, so that we consider this case first. For dimension 3, however, Condition E appears to be a little too weak and we have a degenerate case requiring the use of the various forms of Desargues' theorem; the proof of these (D and D′) requires an axiom on the existence of transversals, Axiom T. The three-dimensional case is put last, but in it the connection with the classical theory (see Pasch-Dehn, Whitehead, and Baker) is most apparent.

1. **Geometric partial orderings.** For the case of any dimension greater than 2, we begin with "linear element" and $\leqq$ as undefined; the linear elements will later be classified according to their dimensions; also taken as undefined are the operations of joining: $a+b$, and intersecting: ab. For projective geometry these may be defined in terms of $\leqq$, but in general incidence geometry we wish to permit certain products ab to remain undefined for convenience in deriving extensions, hence axioms are added. "$\leqq$" is to be read as "on."

The following are the axioms for PARTIAL ORDERING:

1. $a \leqq a$ for every a.
2. $a \leqq b$ and $b \leqq c$ imply $a \leqq c$.

DEFINITIONS. $a = b$ *if* $a \leqq b$ *and* $b \leqq a$; $a < b$ *if* $\cdots$.

3. If A is a set of elements and $\sum(A)$ exists, then $a \leqq \sum(A)$ for all $a \varepsilon A$, and, if $a \leqq x$ for all $a \varepsilon A$, then $\sum(A) \leqq x$.

3′. If A is a set of elements and $\prod(A)$ exists, then $a \geqq \prod(A)$ for all $a \varepsilon A$, and, if $a \geqq x$ for all $a \varepsilon A$, then $\prod(A) \geqq x$.

4. $a \leqq b$ implies existence of $a+b$ (hence $=b$).

4′. $a \geqq b$ implies existence of ab (hence $=b$).

5. There is a $0 \leqq$ every a.

5′. There is a $1 \geqq$ every a.

MODULARITY. *If* $c \geqq a$ *and* bc, $a+bc$, $a+b$, $c(a+b)$ *exist, then* $c(a+b)=a+bc$.

This axiom leads to the classification of elements according to dimension.

ARCHIMEDEAN. *Every "chain"* $a_1 > a_2 > \cdots$ *or* $a_1 < a_2 < \cdots$ *is finite.*

DEFINITION. *a is "prime" over b if $a > b$ and no x exists with $a > x > b$. An element prime over* 0 *(under* 1*) is called a point (hyperplane). The Archimedean axiom assures the existence of both. The letter $p(h)$ is used to represent a point (hyperplane). The letter l is reserved for elements prime over a point (namely, lines).*

By a σ-lattice we mean an Archimedean, modular, partially-ordered set in which

1. if p is not on a, then pa exists (hence $=0$),
2. if $p \leqq a$, b, then ab exists,
3. $a+b$ always exists.

The essence of this definition is that the elements above a point form a modular lattice (that is, $a+b$ and ab always exist). Together with the complementation and irreducibility axioms given below, it implies that the elements above a point form a projective geometry.

In a σ-lattice one defines a "principal" chain as an ascending chain in which each element is prime over the preceding, the number of elements being called its length. It is then easy to show that the lengths of any two principal chains between a and b, where $a < b$, are equal. We then define dim a as the length of a principal chain between 0 and a minus 1, so that dim $0=-1$, dim $p=0$, and so on. Furthermore, $a < b$ implies dim $a <$ dim b, and, if ab exists, dim a $+$dim $b=$dim $ab+$dim $(a+b)$. To prove that $x=y$ by "counting dimensions" we show that $x \leqq y$ and dim $x=$dim y, usually by means of the above dimensional identity.

The following conditions are equivalent in any σ-lattice, so that any one may be used as an axiom, the property involved being called "complementation."

1. If $x \leqq b$, there is a y with $x+y=b$, and $\dim b = 1+\dim x+\dim y$.
2. If $a<b$, then $b=a+p_1+\cdots+p_r$, where p_i is not on $a+p_1+\cdots+p_{i-1}+p_{i+1}+\cdots+p_r$, and $\dim b=1+\dim a+\dim (p_1+\cdots+p_r)$.
3. If $0<b$, then $b=p_1+\cdots+p_r$, where p_i is not on $p_1+\cdots+p_{i-1}+p_{i+1}+\cdots+p_r$.
4. If $0<a$, then a is a sum of points.
5. If $a<b$, then there is a $p<b$ with p not on a (alternatively, if $p<b$ implies $p<a$, then $b \leqq a$).
6. If $a \leqq x \leqq b$, then there is a y with $a \leqq y \leqq b$, $x+y=b$, and $\dim a+\dim b=\dim x+\dim y$.

The condition of complementation gives the theory of linear dependence.

If L is a complemented σ-lattice, then $a+b \neq a, b$ implies that $1+\max\{\dim a, \dim b\} \leqq \dim (a+b) \leqq 1+\dim a+\dim b$.

With this in mind, we define, for the *general* σ-lattice: x and y are skew if $\dim (x+y)=1+\dim x+\dim y$; x and y are fully transversal if $\dim (x+y)=1+\max\{\dim x, \dim y\}$; the class $\{a_1, a_2, \cdots\}$ is called equi-transversal, written $E[a_1, a_2, \cdots]$ if any two are fully transversal and have the same dimension ($\dim (a_i+a_j)=1+\dim a_i$). This generalizes the relation of "coplanarity in pairs" used in §3. An equi-transversal class is called "maximal" if it contains an element on any given point. The ideal point in descriptive geometry was a maximal equi-transversal class; although planes, and so on, are added in the algebraic method, the maximal equi-transversal class is still the "backbone" of the ideal element.

In any σ-lattice, if $\dim a_1=\dim a_2$ and p is not on a_1+a_2, then $E[a, a_1]$, $E[a, a_2]$, $p \leqq a$ if and only if $a=(p+a_1)(p+a_2)$; and if $p_3<a_3<a_1+a_2$, $\dim a_3=\dim a_1$, then $E[a, a_3]$ if and only if $a_3=(p_3+a)(a_1+a_2)$. It consequently follows that if $E[a_1, a_2]$ and $a_1+a_2<1$, there cannot be more than one maximal equi-transversal class containing them, any two of its elements will determine it, and if a_1a_2 exists, the elements are given by $p+a_1a_2$ for all p not on a_1a_2.

To prove the existence of such maximal equi-transversal classes is a harder problem which will be considered in §2.

Consider now an Archimedean lattice for which

1. if p is not on a, then $p+a$ is prime over a;
2. if $a>ah>0$, then a is prime over ah;
3. if it is not true that $a \leqq b$, then there is a $p \leqq a$ with p not on b.

It is not difficult to show that such a lattice is complemented, so that Menger's axioms hold. We therefore call it a Menger lattice. It is also not very difficult to show that the modularity condition holds in the two cases (1) $bc>0$, (2) either $b\leqq c$ or $b\geqq c$. From this it follows that a complemented σ-lattice becomes a Menger lattice when the undefined products are defined to be 0, and, conversely, a Menger lattice in which those products are undefined for which $\dim a+\dim b >\dim (a+b)+\dim ab$ becomes a complemented σ-lattice.

A complemented σ-lattice is called irreducible if any line ($\dim=1$) contains at least three distinct points. The reason for this terminology, at least in the projective case, may be found in Menger, Birkhoff, and von Neumann. The primary use of this condition is in the proof that the projective extension is "minimal."

2. **Incidence geometry.** By an incidence geometry we mean an irreducible σ-lattice of dimension not less than 3 fulfilling the following condition:

Condition E. *If* $E[l_1, l_2, l_3]$, $E[l_1, l_2, l_4]$, *and* $l_1+l_2 \not> l_j$, *then* $E[l_3, l_4]$.

For dimension 3, the further condition T is required. See §3.

In an incidence geometry it is possible to prove the existence of a maximal equi-transversal class containing a_1 and a_2 if $E[a_1, a_2]$ and $a_1+a_2<1$; we have seen that the uniqueness holds in any σ-lattice. This result follows readily from three lemmas, the first two of which generalize Conditions E and E′ of §3.

Lemma B$_1$. *If* $E[a_1, a_2, a_3]$, $E[a_1, a_2, a_4]$, *and if it is not true that* $a_j\leqq a_1+a_2$, *then* $E[a_3, a_4]$.

For, taking $p_i<a_i$, p_i not on $\prod a_i$, $a_1=\sum_{j=1}^{r} l_{1j}$ where $p_1<l_{1j}$, $r=\dim a_i$, we get $a_i=\sum_{j=1}^{r} l_{ij}$ where $l_{ij}=a_i(p_i+l_{1j})$, and $E[l_{1j}, l_{2j}, l_{3j}]$, $E[l_{1j}, l_{2j}, l_{4j}]$ (by a dimensional count), so that Condition E gives $E[l_{3j}, l_{4j}]$, and we then have $a_3+a_4=\sum l_{3j}+\sum l_{4j}=\sum (l_{3j}+l_{4j})=\sum (l_{3j}+p_4)=\sum l_{3j}+p_4 =a_3+p_4$.

Lemma B$_2$. *If* $E[x, x_i, x_j]$, $i, j=1, 2, 3$, $x_i+x_j=x_j+x_k$, *and* $x\leqq x_i+x_j$ *does not hold, and if* p' *is not on* x, x_i+x_j, *or* $x'=\prod(x_i+b')$, *then* $E[x, x', x_i]$.

For p' is on at most one $x+x_i$; hence we consider the following cases:

Case 1. If $p'\leqq x+x_k$, let $x_i'=(x_i+p')(x+p')$; then $E[x, x_i, x_i']$ and $x_i'=(x+p')(x+x_k)$; thus $E[x_i', x, x_k, x_i]$, so that B$_1$ with $E[x, x_i, x_i']$, $E[x, x_i, x_j]$ gives $E[x_i, x_j, x_i']$; similarly $E[x_i, x_j, x_j']$ so that $x_i'=(x_i+p')(x_j+p')=x_j'$, therefore, $x_i'=x'=x_j'$, and so on.

CASE 2. If p' is not on $x+x_i$, $i=1, 2, 3$, let $x_i'=(x+p')(x_i+p')$ so that $E[x, x_i, x_i']$, and therefore B_1 and $E[x, x_i, x_j]$ give $E[x_i', x_j, x_i, x]$; therefore $x_i'=(x_i+p')(x_j+p')=x_j'$, therefore $x_i'=x'=x_j'$.

LEMMA B_3. *If* $E[x_1, x_2, x_3, x_4]$, $x_1+x_2 \not> x_i$, $x_3+x_4 \not> x_j$, *and* p *is not on* x_1+x_2, x_3+x_4, *then* $p \leqq x$ *has* $E[x_1, x_2, x]$ *if and only if* $E[x_3, x_4, x]$.

For $E[x_1, x_2, x_i]$ and B_1 give $E[x, x_i]$.

THEOREM B. *If* $E[x_1, x_2]$ *and* $x_1+x_2<1$, *then there is a (hence exactly one) maximal equi-transversal class containing* x_1 *and* x_2, *and* (1) *if* x, x', x'' *are in it*, $x'+x'' \not> x$, *then we have* $\dim(x+x'+x'')=2+\dim x$; (2) *it is determined by any two of its elements*; *and* (3) *if* xx' *exists, then any element is on* xx', *for example*, $x''=p''+xx'$.

It is only necessary to show that the constructions in §1: $x=(p+x_1)(p+x_2)$, and $x_3=(p_3+x)(x_1+x_2)$ have x_3 independent of x and x independent of x_1, x_2. This is the substance of Lemmas B_2 and B_3.

THEOREM. *If* $h \not> x$, $\dim x < \dim h$, *then there is one and only one maximal equi-transversal class containing* x *with elements on* h.

For if $p'<h$, $x'=(x+p')h$, $p''<h$, p'' not on x', $x''=(x+p'')h$, then $E[x, x', x'']$ (by $x'+x'' \leqq (x+p'+p'')h$ and a dimensional count), so that the class determined by x' and x'' is the one sought.

DEFINITION. *If* p_1, p_2 *are not on* h *and* $x_1 \geqq p_1$ *with* $\dim x_1 < \dim h$, *then* $\Gamma^h_{p_1p_2}(x_1)=x_2$ *is the element on* p_2 *of the maximal E.T. class of* x_1 *and* h.

THEOREM Γ. 1. *If* x_1h *exists and* $x_1h>0$, *then the same holds for* x_2h *and* x_1x_2, *and* $x_1h=x_2h=x_1x_2$, $x_2=p_2+x_1h$.

2. *If* $x_1 \leqq y_1$, *then* $x_2 \leqq y_2$.

3. $\Gamma^h_{p_1p_2}(p_1)=p_2$, *and, if* $x_1 \geqq p_1+p_2$, $x_2=x_1$.

The proof follows readily from $x_2=[p_2+(x_1+p)h][p_2+(x_1+p')h]$, $y_2=[p_2+(y_1+p)h][p_2+(y_1+p')h]$ where $p, p'<h$ with $\dim(y_1+p+p')=2+\dim y_1$.

LEMMA. *If* $E[x_1, y_1, z_1]$ *and* $z_1 \leqq x_1+y_1$, *then* $E[x_2, y_2, z_2]$ *and* $z_2 \leqq x_2+y_2$.

By Γ this needs proof only if x_1+y_1 is a hyperplane. We must show that $p_2' \leqq z_2$ implies $p_2' \leqq x_2+y_2$; that is, $l_2=p_2+p_2' \leqq z_2$ implies $l_2 \leqq x_2+y_2$; take $p_1' \leqq x_1+y_1$ and not on l_1; then the lines l_1, $(l_1+p_1')x_1$, $(l_1+p_1')y_1$ correspond to $l_2<z_2$ and lines on x_2, y_2; if we can show that these coplane, then $l_2 \leqq x_2+y_2$; hence the lemma requires proofs for lines only, and therefore follows from Γ if $\dim 1>3$. If $\dim 1=3$, Baker's proof applies (using results of §1, this is the only point in our development in which Desargues' theorem is needed).

THEOREM. $\Gamma^h_{p_1p_2}$ *may be extended to hyperplanes in one and only one way if the conditions of* Γ *are to hold.*

We merely take $h_1=x_1+y_1$, $E[x_1, y_1]$, and define $h_2=x_2+y_2$; the lemma then applies.

Theorem F of §3 is an obvious consequence of the lemma, using $\Gamma^{\pi}_{p_1p_2}$; it generalizes to the following theorem.

THEOREM. *If* $h=\sum_1^r l_{i1}=\sum_1^r l_{i2}$ *with* $E[l_{i1}, l_{i2}]$, $r=\dim 1$, *and if* p', p'' *are not on* h, $l_i' > p'$, $l_i'' > p''$ *belonging to the maximal E.T. class of* l_{i1}, l_{i2}, *then* $E[\sum_1^r l_i', \sum_1^r l_i'']$.

For $\Gamma^h_{p'p''}$ takes l_i' into l_i'', and hence $\sum l_i'$ into $\sum l_i''$.

Thus far, maximal E.T. classes are significant only if the elements are smaller than hyperplanes. Any class of hyperplanes is E.T. We can now distinguish the important ones.

DEFINITION. *A class of hyperplanes is called "regular" if there is one and only one on any* p, *and if, for every* h, h_1, h_2 *in it and* $p_i < h_i$, *we have* $h_2=\Gamma^h_{p_1p_2}(h_1)$.

THEOREM. *If* $h \neq h_1$, *there is one and only one R. class on them.*

The uniqueness is obvious. To prove existence, we must show that any h', h'' in the R. class determine it in the same way; that is, if $h_2=\Gamma^h_{p_1p_2}(h_1)$, $h'=\Gamma^h_{p_1p'}(h_1)$, $h''=\Gamma^h_{p_1p''}(h_1)$, then $h_2=\Gamma^{h'}_{p''p_2}(h'')$. We use the following lemmas to give $h_2=\Gamma^h_{p'p_2}h'$, $h''=\Gamma^h_{p'p''}h'$ (from $h_1=\Gamma^h_{p'p_1}h'$), hence $h_2=\Gamma^{h'}_{pp_2}h$, $h''=\Gamma^{h'}_{pp''}h$, therefore $h=\Gamma^{h'}_{p''p}h''$, and, finally, $h_2=\Gamma^{h'}_{p''p_2}h''$.

LEMMA 1. *If* $x_2=\Gamma^h_{p_1p_2}(x_1)$ *and* $x_3=\Gamma^h_{p_2p_3}(x_2)$, *then* $x_3=\Gamma^h_{p_1p_3}(x_1)$.

For x_1 and h give x_2, x_2 and h give x_3, hence x_1 and h give x_3; if $x_i=h_i$, split them up as usual.

LEMMA 2. *If* $p<h$, $p_i<h_i$, *and* $h=\Gamma^{h_1}_{p_2p}(h_2)$, *then* $h=\Gamma^{h_2}_{p_1p}(h_1)$.

For taking, as usual, $h_1=\sum_i^r l_{i1}$ with $p_1<l_{i1}$, $r=\dim h_1$, and $l_{i2}=h_2(p_2+l_{i1})$, we get $h_2=\sum_i^r l_{i1}$ and $l_{i1}=h_1(p_1+l_{i2})$; thus, on p, h_1 and l_{i2} give l_i; therefore l_{i1} and l_{i2} give l_i, hence l_{i1} and h_2 give l_i; but $\sum_i^r l_i=h$, which was to be proved.

The maximal E.T. classes and the R. classes give all the ideal elements except the ideal hyperplanes.

3. **Three dimensions.** This section will be devoted not only to filling in the gap mentioned above, but also to showing the relationship between our development and the classical work. In particular, in order to show that incidence geometry includes descriptive, projective, and affine geometry, we will start with the better known incidence axioms of geometry and show their relationship to our axioms.

We begin with the undefined terms "point," "line," and "plane" and the undefined relation "on," using the symbols p, l, π and $>$ or $<$ (depending upon the dimensionality). As axioms we take what are essentially the Hilbert incidence axioms:

(1) If $p<l$ and $l<\pi$, then $p<\pi$.

(2) If $p_1\neq p_2$, they are on one and only one line, called p_1+p_2.

(3) There is a p.

(4) For any p_1, there is a $p_2\neq p_1$.

(5) For any l, there is a p not on l.

(6) If p_3 is not on p_1+p_2, and $p_1\neq p_2$, the three points are on one and only one plane, called $p_1+p_2+p_3$.

From this the meaning of $p+l$ is clear.

(7) If $p<\pi_1$ and $p<\pi_2$ and $\pi_1\neq\pi_2$, then there is an $l>p$, with $l<\pi_1$, $l<\pi_2$; this l is called $\pi_1\pi_2$.

(8) Any line contains at least three distinct points.

(9) For any π, there is a p not on π.

The three space is defined similarly to $p+l$, and its uniqueness follows from Axiom (7); it will be designated by 1. We also use the null element 0. In general we define $a+b$ by using the maximum number of "independent" points which may be chosen from a and b. The product ab is not always defined, but we do take $ab=a$ if $a\leqq b$; $\pi_1\pi_2$ is defined in case the conditions of Axiom (7) hold; $l\pi=p$ if $\pi\not>l$, $p<l$, $p<\pi$; $l_1l_2=p$ if $l_1\neq l_2$, $p<l_i$, $pa=0$ if $a\not>p$; and $l_1l_2=0$ if $l_1+l_2=1$, that is, if l_i are "skew." It is easy to see that if l_1 and l_2 are skew and $p<l_1$, then $(p+l_2)l_1=p$.

The partial ordering axioms, Axioms 1–5′, are equivalent to Axiom 1 and the various definitions above. Modularity is easily verified by enumerating the six possibilities after the trivial cases (further $\leqq$ relations occurring) are eliminated, use being made of Axioms 2 and 6. The Archimedean axiom is obvious from the finite dimensionality. The axioms for a σ-lattice then follow from Axiom 7 above and the definitions following. The complementation condition follows from Axioms 2, 4, 5, 6, 9.

Beside the above axioms we also use the following one.

AXIOM T. *If* $p < l_1, \cdots, l_n$ *and* l *is skew to each of the* l_i, *then there is a* $\pi > l$, $\pi \not> p$ *which is transversal to the* l_i, *that is,* $\pi l_i = p_i$.

This axiom is obvious in projective geometry and follows readily in descriptive geometry.

DEFINITION. *The set of lines* $\{l_1, l_2, \cdots\}$ *is called equi-transversal if every two of them coplane. We write* $E[l_1, l_2, \cdots]$. *If there are more than two lines, we exclude the case where all lie in the same plane.*

In the development of ideal elements from descriptive geometry (see Pasch, Whitehead, and Baker), the following incidence theorems are proved.

THEOREM D. (Desargues' theorem on a point.) *If* $p < l_i, l_i',$ $= (i = 1, 2, 3)$, *and* $l_{ij} = (l_i + l_j)(l_i' + l_j') = l_{ji}$, $l^{ij} = (l_i + l_i')(l_j + l_j') = l^{ji}$, *then* l_{ij} *coplane if and only if* $l^{12} = l^{23} = l^{31}$.

THEOREM D′. (Desargues' theorem in a plane.) *If* $p_1 + p_2 + p_3 = p_1' + p_2' + p_3'$ *is a plane, and* $l_i'' = p_i + p_i'$, $l_{ij} = p_i + p_j = l_{ji}$, $l_{ij}' = p_i' + p_j' = l_{ji}'$, $l_{ij} l_{ij}' = p_{ij}$, $l_i'' l_j'' = p^{ij}$, *then* p_{ij} *colline if and only if* $p^{12} = p^{23} = p^{31}$.

THEOREM E. *If* $E[l_1, l_2, l_3]$, $E[l_1, l_2, l_4]$, *and* $l_1 + l_2 \not> l_j$, *then* $E[l_3, l_4]$.

THEOREM E′. *If* $\sum_1^3 l_i$ *is a plane,* p_4, p_5 *are not on* $\sum_1^3 l_i$, *and* $\prod_1^3 (l_i + p_4) = l_4$, *then* $\prod_1^3 (l_i + p_5) = l_5$ *and* $E[l_4, l_5]$.

THEOREM F. *If* $l_i, l_i' < \pi$, $(i = 1, 2, 3)$, *and* p^1, p^2 *are not on* π, *then* $l_i' = (l_i + p^1)(l_i' + p^1)$ *coplane if and only if* $l_i^2 = (l_i + p^2)(l_i' + p^2)$ *do.*

In this section we will establish those relationships among these theorems needed to show that all are consequences either of E or of E′, and will finally be led to the use of E (which is obviously a necessary condition for projectivization) as an axiom permitting the introduction of ideal elements. It is not our purpose here to consider whether it is independent of Axioms (1)–(9) and T. In projective geometry D and D′ are known to be true, E and E′ are obvious since all the lines concur, and F is obvious since the triangles l_i and l_i' are perspective from some line. Also E′ and F are the hypothesis and conclusion of a Desargues' theorem in the plane when the points p_{ij} and p^{ij} do not exist.

1. That D implies D′ is obvious upon projecting the configuration of D′ through a point not in the plane.
2. T and D′ imply D—obvious.

3. E′ implies E; for Axiom (8) may be used to find $p<l_1+l_2$ with p not on l_1, l_2, and Axiom (7) gives $(p+l_3)(l_1+l_2)=l$; taking $p_i<l_i$, we have $(p_3+l)(p_3+l_1)(p_3+l_2)=l_3$, hence, by E′, $(p_4+l)(p_4+l_1)(p_4+l_2)=l_4$, and $E[l_3, l_4]$.

4. E implies E′ is a special case of Lemma B_2, §2.

5. E implies D′; the proof parallels the classical construction with variations when certain intersections do not exist. Let $l=p_{23}+p_{31}$. We must show that $p_{12}<l$ if and only if $p^{31}<l_2''$. Take p_1 not on $\sum p_i$, $l_{31}''=p_1''+p_{31}$, $p_3''<l_{31}''$ with $p_3''\neq p_1''$, p_{31} (Axiom (8)). Thus $p_{31}=ll_{31}l_{31}'l_{31}''$. Let $l_i=p_i+p_i''$, $l_i'=p_i'+p_i''$ for $i=1$, 3, and take $l_{23}''=p_3''+p_{23}$; thus $p_{23}=ll_{23}l_{23}'l_{23}''$. Now take $l_2=(l_1+p_2)(l_3+p_2)>p_2$, so that $E[l_1, l_2, l_3]$ (for $l_i<p_i''+l_{31}$); take $l_{12}''=(l_1+l_2)(l_{31}''+l_{23}'')>p_1''$, $l_2'=(p_2'+l_{12}'')(p_2'+l_{23}'')>p_2'$, so that $E[l_2', l_{12}'', l_{23}'']$; and finally take $l'=(p^{31}+l_1)(p^{31}+l_3)>p^{31}$, so that $E[l', l_1, l_3]$.

Since $l_2<l_3+p_2=l_3+p_{23}$, $l_{23}''=p_3''+p_{23}<l_3+p_{23}$, the definition of l_{12}'' gives $E[l_2, l_{12}'', l_{23}'']$; since $l_i'<p_i''+l_{31}'$, $l'<p^{31}+l_i=p^{31}+p_i+p_i''=p^{31}+p_i'+p_i''=p^{31}+l_i'$, $l_2'<p_2'+l_{23}''=p_3'+l_{23}''>p_3'+p_3''=l_3$, we have $E[l', l_1', l_3']$ and $E[l_2', l_3']$. Consequently, applying E to $E[l', l_1, l_3]$, $E[l_2, l_1, l_3]$ and $E[l_2', l_{12}'', l_{23}'']$, $E[l_2, l_{12}'', l_{23}'']$, we get $E[l', l_1, l_2, l_3]$ and $E[l_2, l_2']$.

Now $p^{31}<l_2''$ if and only if $E[l', l_2']$; for since $E[l_2, l_2']$, $p^{31}<l_2''$ gives $l_2'<l_2+p_2'=l_2+l_2''=l_2+p^{31}=l_2+l'$, and, conversely, if $l_2''\not>p^{31}$ and $E[l', l_2']$, then $l_2+l_2'\not>l'$, so that $E[l', l_2', l_2]$; with $E[l_{23}'', l_2', l_2]$ and Theorem E, this gives $E[l', l_{23}'']$, so that we would have $p_{23}<l'+p_{23}=l'+l_{23}''=l'+p_3''=l'+l_3'=l'+p_3'=l'+l_3''$, and therefore $p_{23}<l_3''$, $l_{23}=l_3''=l_{23}'$, contradicting $l_{23}l_{23}'=p_{23}$. Thus we need only show that $E[l', l_2']$ if and only if $p_{12}<l$.

A. If $E[l_1, l_2']$, then $E[l', l_2', l_3']$ and $E[l', l_1', l_3']$ and Theorem E give $E[l_1', l_2', l_3']$; now if $l_1'+l_2'\not>l_{12}''$, we would have $E[l_{12}'', l_1', l_2']$ and therefore $p_1''=l_1'l_{12}''<l_2'$, hence $p_1''=l_1'l_2'<l_3'=p_3'+p_3''$, and therefore $p_3'<p_1''+p_3''=l_{31}''$ and $l_{31}''=p_{31}+p_3'=l_{31}'$, so that $p_1''<l_{31}'<\sum p_i$, a contradiction; hence $l_{12}''<l_1'+l_2'$, so that

$$
\begin{aligned}
p_{12}=l_{12}l_{12}' &= [(\textstyle\sum p_i)(l_1+l_2)][(\textstyle\sum p_i)(l_1'+l_2')]\\
&= (\textstyle\sum p_i)(l_1+l_2)(l_1'+l_2')\\
&< (l_1+l_2)(l_1'+l_2') = l_{12}''\\
&< l_{31}''+l_{23}''; \; p_{12}<(\textstyle\sum p_i)(l_{31}''+l_{23}'')=l.
\end{aligned}
$$

B. If $p_{12}<l$, then $p_{12}=ll_{12}=[(\sum p_i)(l_{31}''+l_{23}'')][(\sum p_i)(l_1+l_2)]=(\sum p_i)(l_{31}''+l_{23}'')(l_1+l_2)<l_{12}''$; hence $l_{12}''+l_2'=l_{12}''+p_2'=l_{12}''+l_{12}'$,

so that $E[l_1', l_2']$, therefore $E[l_1', l_2', l_3']$, and Theorem E gives $E[l', l_2']$, which was to be proved.

6. D, E, and E′ imply F. The proof is the same as that in descriptive geometry, where use is made of the perspectivity of the elements on p^1 and p^2 with respect to π. It has already been indicated in §2.

In the development, E and E′ are used as criteria for the identity of ideal points and F is used to determine the collinearity of ideal points, whereas in descriptive geometry D and D′ are used to prove E′; if E is accepted they are needed only to prove F, and even so only in three dimensions. E′ may be used to define $E[l_1, l_2, \cdots]$ in the case where the l_i coplane.

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WEIERSTRASS PREPARATION THEOREM

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H. Späth[1] has shown that the Weierstrass preparation theorem is a consequence of the following one:

THEOREM. *Let P and Q be power series in the variables $z_1, z_2, \cdots, z_m$ such that P is regular in z_1 of degree k, that is, $P(z_1, 0, \cdots, 0) = cz_1^k + dz_1^{k+1} + \cdots$, $(c \neq 0)$. Then there exist power series A and B such that*

$$Q - AP = B \tag{1}$$

where B does not contain powers of z_1 higher than the $(k-1)$st. The series A and B are uniquely determined.

It has not been observed that this general form permits of a simple proof by induction when we are concerned only with *formal* series. The solution of (1) is equivalent to the solution of the system

$$\begin{aligned}(q_n - a_0p_n - a_1p_{n-1} - \cdots - a_{n-1}p_1) - a_np_0 = b_n,\\ n = 0, 1, 2, \cdots,\end{aligned} \tag{2}$$

where $P = \sum p_n z_m^n$, $Q = \sum q_n z_m^n$, $A = \sum a_n z_m^n$, $B = \sum b_n z_m^n$. The theorem is evidently true for the case $m = 1$. Since p_0 is regular in z_1 of degree k, the existence of a formal solution of (1) and its uniqueness is an immediate consequence of an induction from $m-1$ to m variables.

PRINCETON, N.J.

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[1] H. Späth, Journal für die reine und angewandte Mathematik, vol. 161 (1929), pp. 95–100. This contains additional references.

SOME RESULTS CONCERNING THE BEHAVIOR AT INFINITY OF REAL CONTINUOUS SOLUTIONS OF ALGEBRAIC DIFFERENCE EQUATIONS[1]

OTIS E. LANCASTER[2]

The behavior at infinity of real continuous solutions of algebraic differential equations has been studied by Borel, Lindelöf, Hardy, Fowler, and Vijayaraghavan,[3] but, as far as the author is aware, the corresponding problem for difference equations has not been considered, except for the special case of solutions in the neighborhood of a double point.[4]

In this paper we propose to study the rate of increase, as the independent real variable x becomes infinite, of real continuous solutions of algebraic difference equations: that is, of equations of the form

$$P(y(x+m), y(x+m-1), \cdots, y(x), x) = 0, \tag{1}$$

where P is a polynomial with real coefficients in its arguments $y(x+m), y(x+m-1), \cdots, y(x)$, and x. Among the terms of the polynomial P, there is a term

$$T' = A'x^{\alpha'}y(x)^{\beta_0'}y(x+1)^{\beta_1'} \cdots y(x+m)^{\beta_m'}, \tag{2}$$

which has the property that if

$$T = Ax^{\alpha}y(x)^{\beta_0}y(x+1)^{\beta_1} \cdots y(x+m)^{\beta_m} \tag{3}$$

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[1] Presented to the Society, December 30, 1937.

[2] I am indebted to Professor G. D. Birkhoff for counsel and encouragement during the preparation of this paper.

[3] E. Borel, *Mémoire sur les séries divergentes*, Annales de l'École Normale Supérieure, Paris, (3), vol. 16 (1899), p. 26 ff.; E. Lindelöf, *Sur la croissance des intégrales des équations différentielles algébriques du premier ordre*, Bulletin de la Société Mathématique de France, vol. 27 (1899), pp. 205–215; G. H. Hardy, *Some results concerning the behavior at infinity of a real and continuous solution of algebraic differential equations of the first order*, Proceedings of the London Mathematical Society, (2), vol. 10 (1912), pp. 451–468; R. H. Fowler, *Some results on the form near infinity of real continuous solutions of a certain type of second order differential equations*, Proceedings of the London Mathematical Society, (2), vol. 13 (1914), pp. 341–371; T. Vijayaraghavan, *Sur la croissance des fonctions définies par les équations différentielles*, Comptes Rendus de l'Académie des Sciences, Paris, vol. 194 (1932), pp. 827–829.

[4] If $\lim_{x\to\infty} u(x) = U$, then U is a double point of the difference equation $u(x+p) = f[u(x+p-1), u(x+p-2), \cdots, u(x)]$, when it is a root of the algebraic equation $U = f(U, U, \cdots, U)$. See S. Lattès, *Sur les suites récurrentes non linéaires et sur les fonctions génératrices de ces suites*, Annales de la Faculté des Sciences de Toulouse, (3), vol. 3 (1911), pp. 75–124; J. Horn, *Zur Theorie der nicht linearen Differential- und Differenzengleichungen*, Journal für die reine und angewandte Mathematik, vol. 141 (1912), pp. 182–216.

is any other term of the polynomial P, then in the sequence of differences

$$(4) \qquad \beta_m' - \beta_m, \beta_{m-1}' - \beta_{m-1}, \cdots, \beta_1' - \beta_1, \beta_0' - \beta_0, \alpha' - \alpha$$

the first nonzero term is positive. We shall call T' the ***principal term*** of the equation. The proofs of the first four theorems are based upon the limits, as $x \to \infty$, of the ratios of the principal term to the other terms of the difference equation.

For the convenience of expressing the iterates of the logarithm and the exponential we shall use the notation that was employed by Hardy in his book *Orders of Infinity*, namely,

$$e_1(v) = \exp(v), \qquad e_n(v) = \exp[e_{n-1}(v)],$$
$$l_1(v) = \log(v), \qquad l_n(v) = \log[l_{n-1}(v)].$$

THEOREM 1. *A real continuous solution of an algebraic difference equation of the first order cannot equal or exceed the function* $Ce_2[xl_n(x)]$ *for all* $x > x_0(n)$, *where* C *is any positive constant and* n *is any integer.*

PROOF. First, assume that $Ce_2[xl_n(x)]$ is a solution of the first order difference equation (1). Divide the equation by T' and replace $y(x)$ by $Ce_2[xl_n(x)]$. Then, all terms except one, T'/T', are of the form T/T', where

$$(5) \quad T/T' = Bx^{\alpha-\alpha'}(e_2[xl_n(x)])^{\beta_0-\beta_0'}(e_2[(x+1)l_n(x+1)])^{\beta_1-\beta_1'}.$$

These terms T/T' are of three possible types:

$$\text{(a)} \qquad B\left\{\frac{x^{K_0}(e_2[xl_n(x)])^{K_1}}{e_2[(x+1)l_n(x+1)]}\right\}^{\beta_1'-\beta_1}, \qquad \beta_1' > \beta_1;$$

$$\text{(b)} \qquad B\left\{\frac{x^{K_2}}{e_2[xl_n(x)]}\right\}^{\beta_0'-\beta_0}, \qquad \beta_1' = \beta_1, \beta_0' > \beta_0;$$

$$\text{(c)} \qquad Bx^{\alpha-\alpha'}, \qquad \beta_1' = \beta_1, \beta_0' = \beta_0, \alpha' > \alpha,$$

where K_0, K_1, K_2 are rational numbers. The limit of each of these expressions as $x \to \infty$ is zero. Hence, we have a contradiction to the assumption that $y = Ce_2[xl_n(x)]$ is a solution of (1). For there exists an $x_0(n)$ such that for $x > x_0(n)$ the sum of all the terms T/T' is less than one in absolute value, whereas $T'/T' = 1$.

Second, a function $y(x)$ which is greater than $Ce_2[xl_n(x)]$ for $x > x_0(n)$ cannot[5] be a solution of the first order difference equation

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[5] The symbol $x_0(n)$ does not always denote the same value. In fact, it may represent several different values within one proof.

(1). For assume that $y(x)$ is such a solution of (1); then the ratios T/T' are again of the three types:

$$\text{(a}')\qquad B\left\{\frac{x^{K_0}y(x)^{K_1}}{y(x+1)}\right\}^{\beta_1'-\beta_1}, \qquad \beta_1' > \beta_1,$$

$$\text{(b}')\qquad B\left\{\frac{x^{K_2}}{y(x)}\right\}^{\beta_0'-\beta_0}, \qquad \beta_0' > \beta_0,$$

$$\text{(c}')\qquad Bx^{\alpha-\alpha'}, \qquad \alpha' > \alpha.$$

The ratios of types (b′) and (c′) approach zero, as x becomes infinite, for all rational values of K_2; hence, if $y(x)$ is a solution, not all of the ratios of type (a′) may approach zero, as $x\to\infty$, or we have a contradiction as above. Thus there is at least one ratio of type (a′) for which $K_1>0$. There may be several such ratios; however, the one which has the maximum value of K_1 is for all $x>x_0(n)$ greater in absolute value than any of the others. Hence if M_1 denotes this maximum value of K_1, then

$$\text{(6)}\qquad \frac{x^{K_0}y(x)^{M_1}}{y(x+1)} > \frac{1}{N}$$

for all x greater than or equal to $x_0(n)$. From this there follow in succession

$$\text{(7)}\qquad \begin{aligned} &y(x+1) \leqq Nx^{K_0}y(x)^{M_1} && x \geqq x_0,\\ &y(x_0+1) \leqq N[x_0y(x_0)]^{M}, && \text{where } M \geqq K_0,\ M_1,\ 2,\\ &\cdots\cdots\cdots\cdots\cdots\cdots, &&\\ &y(x_0+p) \leqq [Ny(x_0)]^{M^p}(x_0+p)^{M^{p+1}}, && p = 1, 2, 3, \cdots. \end{aligned}$$

Therefore,

$$\text{(8)}\qquad \lim_{p\to\infty}\frac{y(x_0+p)}{e_2[(x_0+p)l_n(x_0+p)]} \leqq \lim_{p\to\infty}\frac{[Ny(x_0)]^{M^p}(x_0+p)^{M^{p+1}}}{e_2[(x_0+p)l_n(x_0+p)]} = 0.$$

This contradicts the assumption that $y(x)$ is greater than $Ce_2[xl_n(x)]$. Thus, no continuous function $y(x)$, $y(x)\geqq Ce_2[xl_n(x)]$ for all $x>x_0(n)$, may be a solution of (1).

It is important to note that the above theorem does not state that a solution of a first order difference equation may not exceed $Ce_2[xl_n(x)]$ at an infinite number of points $x_1, x_2, x_3, \cdots, x_i\to\infty$. However, it does follow from the proof that if a solution $y(x)$ equals $e_2[xl_n(x)]$ at a point $x>x_0(n)$, then $y(x+1)<e_2[(x+1)l_n(x+1)]$.

Thus a solution $y(x)$ cannot exceed $e_2[xl_n(x)]$ over an interval of difference,[6] for $x > x_0(n)$.

Although we have not been able to generalize Theorem 1 so as to include all algebraic difference equations of the mth order, we have been able to establish the following facts:

THEOREM 2. *The function* $Ce_2[xl_n(x)]$, $(C \neq 0)$, *cannot be a solution of an algebraic difference equation for any value of* n.

The proof is similar to the first part of Theorem 1. Under the assumption that the theorem is false, $y(x) = Ce_2[xl_n(x)]$ is a solution of (1) and

$$\frac{T}{T'} = Bx^{\alpha-\alpha'} \prod_{j=0}^{m} \{e_2[(x+j)l_n(x+j)]\}^{\beta_j - \beta_j'}.$$

The considerations of the limits of these ratios divide into $m+2$ cases corresponding to the $m+2$ possible relations (4) between the exponents of the principal term and those of the other terms of (1). In the $(m+1-i)$th case, $\beta_m' = \beta_m$, $\beta_{m-1}' = \beta_{m-1}, \cdots, \beta_{i+1}' = \beta_{i+1}$, $\beta_i' > \beta_i$, the ratios are of the form

$$B\left\{\frac{x^{K_0}\prod_{j=0}^{i-1}\{e_2[(x+j)l_n(x+j)]\}^{K_{j+1}}}{e_2[(x+i)l_n(x+i)]}\right\}^{\beta_i' - \beta_i},$$

where the K_v, $(v = 0, 1, 2, \cdots, i)$, are rational numbers. The limit, as x becomes infinite, of such a quotient is zero for $i = 0, 1, 2, \cdots, m$ regardless of the magnitudes of the K_v. In the $(m+2)$nd case, the ratios also approach zero for they are of the form $Bx^{\alpha-\alpha'}$, $(\alpha' > \alpha)$. Thus, in all cases $\lim_{x\to\infty} T/T' = 0$. This contradicts the assumption that $Ce[xl_n(x)]$ is a solution of (1).

THEOREM 3. *No solution of an algebraic difference equation of the* m*th order can exceed a function* $Ce_2[xl_n(x)]$ *for all* $x > x_0(n)$, *if the terms of the equation are so related that when* $\beta_m' = \beta_m$ *then* $\beta_i' \geq \beta_i$, $(i = 0, 1, 2, \cdots, m-1)$.

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[6] It is interesting to observe that from this fact we can obtain a proof that a solution of a first order algebraic differential equation cannot equal or exceed $e_2[xl_n(x)]$ for all $x \geq x_0(n)$. For, when the interval of difference, say ω, is reduced, the intervals for which $y(x)$ may exceed $e_2[xl_n(x)]$ are also reduced. In the limit as $\omega \to 0$, the length of these intervals approaches zero and the difference equation approaches a differential equation. Hence a solution of an algebraic differential equation could be at most tangent to $e_2[xl_n(x)]$, $x \geq x_0(n)$. However, Hardy, Lindelöf and Borel have established lower bounds for the rate of increase of the solutions of an algebraic differential equation.

PROOF. Assume that $y(x)$, a solution of (1), is greater than or equal to $Ce_2[xl_n(x)]$ for $x>x_0(n)$. Under the hypothesis of the theorem the sequences (4) are of two general types: either the first term is positive and the other terms are arbitrary, or the first term is zero and all the other terms, except the last, are greater than or equal to zero. Corresponding to these two types of sequences the ratios T/T' take the forms

$$B\left\{\frac{x^{K_0}y(x)^{K_1}y(x+1)^{K_2}\cdots y(x+m-1)^{K_m}}{y(x+m)}\right\}^{\beta_m'-\beta_m} \tag{9}$$

and

$$B\left\{\frac{x^{k_0}}{y(x)^{k_1}y(x+1)^{k_2}\cdots y(x+i)^{k_{i-1}}}\right\}$$

respectively, where K_0, K_1, $K_2, \cdots, K_m$ are rational numbers, k_1, $k_2, \cdots, k_{i-1}$ are positive integers or zero, and k_0 is any positive or negative integer unless all $k_j=0$, $j=1, 2, \cdots, i-1$. The second set of ratios all approach zero as x increases indefinitely. Hence, not all of the ratios of the first type do approach zero or otherwise $y(x)$ is not a solution of (1). Of the finite number of terms of this type there is a maximum value for K_0, a maximum value for $K_1, \cdots$. Now consider the ratio

$$\left\{\frac{x^{M_0}y(x)^{M_1}y(x+1)^{M_2}\cdots y(x+m-1)^{M_m}}{y(x+m)}\right\},$$

where M_i is the maximum value of all the K_i for the ratios of type (9). Some positive constant power of this expression is greater, for $x>x_0(n)$, than the absolute value of any of the ratios T/T'. And since some of the ratios do not approach zero,

$$\frac{x^{M_0}y(x)^{M_1}y(x+1)^{M_2}\cdots y(x+m-1)^{M_m}}{y(x+m)} > \frac{1}{N}\cdot$$

That is,

$$y(x+m) \leqq Nx^{M_0}y(x)^{M_1}\cdots y(x+m-1)^{M_m}, \qquad x \geqq x_0(n),$$

or

$$y(x+m) \leqq N[xy(x)\cdots y(x+m-i)]^{M}, \quad \text{where} \quad M \geqq 2, M_j, \quad (j=0, 1, \cdots, m).$$

It follows from this that

$$y(x_0 + p + m) \leqq (x_0 + p)^{(M+1)^{p+2}}[Ny(x_0) \cdots y(x_0 + m - 1)]^{(M+1)^{p+1}},$$
$$p = 1, 2, 3, \cdots .$$

Thus,

$$\lim_{p\to\infty} \frac{y(x_0 + p + m)}{Ce_2[(x_0 + p + m)l_n(x_0 + p + m)]}$$
$$\leqq \lim_{p\to\infty} \frac{(x_0 + p)^{(M+1)^{p+2}}[Ny(x_0) \cdots y(x_0 + m - 1)]^{(M+1)^{p+1}}}{Ce_2[(x_0 + p + m)l_n(x_0 + m + p)]} = 0.$$

But this is contrary to the hypothesis that $y(x)$ is greater than $Ce_2[xl_n(x)]$ for $x > x_0(n)$. Therefore if the terms of equation (1) are so related that $\beta_i' \geqq \beta_i$, $(i = 0, 1, 2, \cdots, m-1)$, for every term T for which $\beta_m = \beta_m'$, then a solution cannot remain greater than $Ce_2[xl_n(x)]$ for all $x > x_0(n)$.

In Theorem 3, the difference equation was restricted but there were no restrictions as to the regularity of increase of the solution. Now we consider the general difference equation but make restrictions on the solutions.

THEOREM 4. *If $y(x)$ is a solution of an algebraic difference equation of the mth order and if there exists a constant such that the ratio $y(x)^K/y(x+1)$, where K is any rational number, gives a monotone sequence as x ranges over the values a, $a+1$, $a+2$, $\cdots$, then $y(x)$ cannot exceed $Ce_2[xl_n(x)]$, for any integer n, for all $x > x_0(n)$.*

The proof follows immediately from the above considerations. The ratios T/T', corresponding to the $(m+1-i)$th case for the sequence (4), are of the form

$$B\left\{\frac{x^{K_0}y(x)^{K_1} \cdots y(x + i - 1)^{K_i}}{y(x + i)}\right\}^{\beta_i' - \beta_i}.$$

When the conditions of the theorem are satisfied and $y(x)$ is *greater* than $Ce_2[xl_n(x)]$ for all x, then each of the ratios

$$\frac{[y(x + i - 1)]^K}{y(x + i)}, \qquad i = 1, 2, \cdots, m,$$

gives a monotone decreasing sequence as x takes on the values $x_1 = a + m - 1$, $x_2 = a + m$, $x_3 = a + m + 1$, $\cdots$. Moreover, $\lim_{x\to\infty} x^{K_0}/y(x) = 0$. Hence a ratio T/T' consists of a product of $x^{K_0}/y(x)$ times ratios which decrease monotonically as x ranges over the values x_2, x_3, x_4, $\cdots$. Thus, $\lim_{x_i\to\infty} T/T' = 0$, for all possible ratios, and this contradicts the assumption that $y(x)$ is a solution of (1).

DEFINITION. *A function $f(x)$ shall be said to be a regularly increasing function, if for every rational value of K, $[f(x)]^K/f(x+1)$ is a monotonic function $x>x_0$.*

COROLLARY. *If a solution $y(x)$ of an algebraic difference equation is a regularly increasing function, then*

$$|y(x)| < e_2[xl_n(x)], \qquad x > x_0(n).$$

The proof is evident.

We have shown that a regularly increasing solution of an algebraic difference equation cannot increase as rapidly as the function $e_2[xl_n(x)]$. The question naturally arises: Is there a function with a slower rate of increase which is also a bounding function for the regularly increasing solutions of algebraic difference equations? Since there are solutions of difference equations that increase as rapidly as the function a^{b^x}, where a and b are arbitrary numbers,[7] it is clear that a bounding function for all regularly increasing solutions of all algebraic difference equations must increase more rapidly than a function of the form $e_1(cx^n e^{Kx})$. Therefore, a bounding function must increase as rapidly as $e_2(v)$, where $v=xp(x)$ and $\lim_{x\to\infty} p(x)=\infty$. Hence, if there existed a better bounding function than $e_2[xl_n(x)]$ it would be of the form $e_2(v)$, where $p(x)$ increases slower than any $l_n(x)$. Hardy in *Orders of Infinity* makes the following statement: "It is possible to define functions whose increase . . . is slower than that of any $l_n(x)$; but this is not possible if we confine ourselves to functions defined by a finite and explicit formula involving the ordinary functional symbols of analysis." Therefore, if we confine ourselves to the ordinary functional symbols, we have obtained the best bounding function.

Now let us compare our results for the rate of increase of solutions of difference equations with those known for differential equations. The corollary above corresponds in some respects to the unpublished results of Vijayaraghavan relative to the maximum rate of increase of a solution of an algebraic differential equation. He states at the close of a note in the Comptes Rendus des Séances de l'Académie des Sciences:[3] "If $y(x)$, a function defined for all positive values of x, is a solution of an algebraic differential equation of the mth order, and if the increase of y satisfies certain very restrictive regularity conditions, then for $x>x_0$, $|y(x)| <e_m(kx^\Delta)$, where $e_m(x)$ designates the mth iterate of the exponential function and k and Δ are properly determined constants." If the regularity conditions of the two results

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[7] A solution of $y(x+1)=y(x)^b$, b a rational number, is $y(x)=a^{b^x}$, where $y(0)=a$.

are comparable, we see that the upper bound for the regularly increasing solutions of a differential equation for $m>1$ is much greater than that for the regularly increasing solutions of a difference equation. It is interesting to observe that there actually are differential equations of higher than first order which have regularly increasing solutions that increase more rapidly than any regularly increasing solution of any algebraic difference equation. In order to see this it is sufficient to verify that the differential equation

$$xyy'' - x(y')^2 - (2x^2 + 1)yy' = 0$$

is satisfied by $y(x) = e_2(x^2)$.

The story is quite different for equations of first order. There are regularly increasing solutions of some first order algebraic difference equations that increase more rapidly than any solution of any first order differential equation. For,[8] the solutions of a first order differential equation cannot increase as rapidly as $e_2(x)$ and, as was mentioned before, there are solutions of some difference equations of the first order that increase as rapidly as a^{b^x}, where a and b are arbitrary constants.

The above regularity conditions are very stringent. Are such regularity conditions necessary? As a partial answer to this question, we now show that unless some regularity conditions are imposed on the solutions, there is no bounding function for the solutions of all algebraic difference equations of higher than first order. The following theorem and its proof are modelled after the analogous theorem for differential equations that was given by Vijayaraghavan.[8]

THEOREM 5. *For a given function $\Phi(x)$ with an arbitrary rate of increase, there exists a second order algebraic difference equation*

$$P(y(x),\ y(x+1),\ y(x+2),\ x) = 0$$

that is satisfied for all real values of x by a real and continuous function $f(x)$, where $|f(x)| > \Phi(x)$ for an infinite number of values x_i, $\{x_i\} \to \infty$.

PROOF. We form the function

$$f(z) = (1/2)\left[\wp(\alpha z) + \overline{\wp(\alpha \bar{z})}\right], \tag{10}$$

where $\wp(z)$ is the Weierstrass $\wp$-function with the two fundamental periods ω and ω', $\bar{z}$ is the conjugate of z, and $\alpha = \omega + i\eta$, where η is yet to be determined.

We choose η real in such a manner that for an infinite number of integral values n_i and p_i

$$| p_j - n_j\eta/\omega' | < \frac{1}{2\omega'[\Phi(n_j)]^{1/2}}. \tag{11}$$

For this it is sufficient to take for η/ω' a continued fraction in which the denominators of the partial quotients increase in a sufficiently rapid manner.

For real values of z, $f(z)$ is the real part of $\wp(\alpha z)$; hence it is a real function. If $z = n_j\omega + p_j\omega' i + ti$, where t is real and very small, the real part of $\wp(z)$ is asymptotic to $-1/t^2$. Therefore, it follows from the inequality (11) that for $z = n_j$ and for j sufficiently large

$$-f(n_j) \sim \frac{1}{(n_j\eta - p_j\omega')^2} > \Phi(n_j). \tag{12}$$

Now by virtue of the addition formula for $\wp(z)$,

$$\overline{\wp(\alpha\bar{z})} = 2f(z) - \wp(\alpha z)$$

satisfies a first order algebraic difference equations. If $\wp(\alpha z)$ is eliminated between the difference equation for $2f(z) - \wp(\alpha z)$ and the one obtained from it by replacing z by $z+1$, we obtain a second order algebraic difference equation that is satisfied by the real function of $f(z)$ for all real values of z. This completes the proof.

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A FINITELY-CONTAINING CONNECTED SET[1]

P. M. SWINGLE

In a previous paper an example has been given of a set which, for every integer $n \geqq 2$, is the sum of n mutually exclusive connected subsets, but which is not the sum of *infinitely* many such subsets.[2] Here it is proposed to give an example of a connected set which, for every integer $n \geqq 2$, is the sum of n mutually exclusive *biconnected* subsets but which is not the sum of infinitely many mutually exclusive connected subsets. This example has the further property that, *for every such n, it contains n mutually exclusive connected subsets but it does not contain infinitely many such subsets*, being thus a *finitely-containing connected set.*[3] The method used will be a modification of that used by E. W. Miller to obtain a biconnected set without a dispersion point.[4] The *hypothesis of the continuum is assumed*, and use is made of the axiom of Zermelo.

The method used by Miller is dependent primarily upon showing the existence of a widely connected subset M of an indecomposable continuum K. It is only the part of this subset M which is contained within a square Q_0 which causes M to be biconnected and it is this fact which enables us to show the existence of the desired set of this paper. We will take a countable infinity of mutually exclusive such squares plus interiors, $Q, Q_1, Q_2, Q_3, \cdots$, each containing points of K and having the relation with K that Miller's square $ABCD$ has. We will use Q_i as Miller does to show that a subset B_{ni}, $(i=1, 2, \cdots, n+1;$ $n=1, 2, 3, \cdots)$, of a set M is biconnected. And Q will be used to show that there cannot be infinitely many mutually exclusive such subsets of M.

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[1] Presented to the Society, April 15, 1939.

[2] P. M. Swingle, *Generalizations of biconnected sets*, American Journal of Mathematics, vol. 53 (1931), pp. 387–388. I call such a set a *finitely-divisible connected set.* A connected set is defined here so as to contain at least two points. The example there given consists of a connected set which is the sum of infinitely many mutually exclusive biconnected subsets, each with a dispersion point, and a limit point of these subsets which none of them contains.

[3] Loc. cit., p. 395, Problem 7. This example also solves the questions raised in Problems 4, 5, and 6, pp. 394–395. Problem 2 was answered in part in American Journal of Mathematics, vol. 54 (1932), pp. 532–535. On p. 533 it is proved for $n=2$ that E_n is the sum of m mutually exclusive biconnected subsets where m is an integer greater than n. And it is said that the proof is similar for $n>2$. For E_2 the proof depends upon constructing 3 biconnected sets, having only the origin in common. That a similar construction holds for any E_n, $(n>1)$, is seen as follows. The half cones $x_1^2+x_2^2+\cdots+x_{n-1}^2=ax_n^2$, $(x_n \geqq 0,\ -\infty<a<\infty)$, of E_n are each $n-1$ dimensional surfaces. As each one is composed of concentric spheres $x_1^2+x_2^2+\cdots+x_{n-1}^2=r^2$ as is also E_{n-1}, each half cone and E_{n-1} are topologically equivalent. As for $n=3$, E_{n-1} is the sum of n biconnected sets, with only the origin in common, a mathematical induction proof will show that this is true for $n>3$. For let the a's be divided into $C_{n+1,n}$ ($C_{n+1,n}$ is a binomial coefficient) mutually exclusive sets $N_1, \cdots, N_c$, each dense in their sum. Let, for each a of N_i, $(i=1, \cdots, c)$, $x_1^2+x_2^2+\cdots+x_{n-1}^2=ax_n^2$ be the sum of parts of the same n biconnected sets, where there is a total of $n+1$ such sets B_j, mutually exclusive except that they have the origin in common. Those B_j's determined by N_i will be represented by the subscripts of that combination of $1, 2, \cdots, n+1$, taken n at a time, that i of N_i represents. Then the above is seen to be true.

[4] E. W. Miller, *Concerning biconnected sets*, Fundamenta Mathematicae, vol. 29, pp. 123–133.

Let V be a countable subset of K, which is dense in $K\cdot(Q_1+Q_2+Q_3+\cdots+Q)$. Let V_{ij}, $(i=1, 2, 3, \cdots; j=1, 2, \cdots, i+1)$, be a countable subset of V everywhere dense in V and such that (a) $V_{ij}\cdot V_{kl}$ is dense in V if $i\neq k$, (b) for any i the V_{ik}'s, $(k=1, 2, \cdots, i+1)$, are mutually exclusive, and (c) $V_{i1}+V_{i2}+\cdots+V_{i,i+1}=V$. For example V_{11} and V_{12} are mutually exclusive and $V_{11}+V_{12}=V$. Then V_{11} is divided into three mutually exclusive subsets, each dense in V, one for each of the sets V_{21}, V_{22}, V_{23} where V_{2j} is composed of such a set plus a similar subset of V_{12}. Each one of these three mutually exclusive subsets of V_{11} is then divided into four mutually exclusive sets, each dense in V, to obtain the parts of V_{31}, V_{32}, V_{33}, V_{34} contributed by V_{11}.

Let a division of V into infinitely many mutually exclusive subsets be $U_1, U_2, \cdots$, where each U_t, $(t=1, 2, \cdots)$, is everywhere dense in V. Either (1) there exists a region R of Q and a V_{ij} such that a U_t contains $R\cdot V_{ij}$, or (2) there does not exist such an R. If (2) is true, $V_{ij}-U_t\cdot V_{il}$ is dense in $V\cdot Q$ for each i, j, t. Consider case (1). Suppose for example that U_1 contains $R\cdot V_{32}$. Let R_1 be any region contained in R. Then U_1 contains a subset of V_{rj}, $(r>3)$, which is dense in $V_{rj}\cdot R_1$, since $V_{32}\cdot R_1$ contains such a subset because of (a) above. Hence U_t, $(t\neq 1)$, cannot contain a $V_{rj}\cdot R_1$, since U_1 and U_t are mutually exclusive. Suppose now that there exist a U_t, $(t\neq 1)$, U_2 say, which contains a $V_{3f}\cdot R_1$, $(f\neq 2$, but equals 1 say), for some R_1 of R. Hence as above U_t, $(t\neq 2)$, does not contain a $V_{rj}\cdot R_2$, where R_2 is any region of R_1. There may exist now a U_t, $(t\neq 1, 2)$, U_3 say, which contains a $V_{3f}\cdot R_2$ for $f\neq 1, 2$ but $f=3$ say. However since the U_t's are contained in $V_{31}+V_{32}+V_{33}+V_{34}$, there cannot exist a region R_3 of R_2 and a U_t, $(t\neq 1, 2, 3)$, such that U_t contains $R_2\cdot V_{3f}$, $(f\neq 1, 2, 3)$, for $R_2\cdot V_{34}$ must contain $R_3\cdot(U_4+U_5+\cdots)$. Thus in this case there exists an R_2 of R such that there are at most three U_t's which contain a $V_{ij}\cdot R_3$, where R_3 is any region of R_2. Hence there exists an R_3 of R

and a U_t, U'' say, such that for every V_{ij}, $V_{ij}-V_{ij}\cdot U''$ is dense in $V\cdot R_3$. Therefore in both cases (1) and (2) above there exists a region R'' of Q and a U_t, U'' say, such that for every V_{ij}, $V_{ij}-V_{ij}\cdot U''$ is dense in $V\cdot R''$.

The proof used by Miller to show that his widely connected set M is biconnected is dependent upon having a countable subset Δ of M and upon having a set of simple closed curves within the square $ABCD$ which have nothing in common with M except points[5] of Δ. One of these simple closed curves is taken for *each* subset of $\Delta=V$ which is dense in $V\cdot R$, where R is any region containing points of V. And the simple closed curve contains from the points of V only points from this subset of $V\cdot R$. The set of such possible subsets is c, the power of the linear continuum.

Following the method of Miller arrange in a well ordered sequence the continua C_a which separate K:

$$C_1, C_2, C_3, \cdots, C_a, \cdots, \qquad a<\Omega_c,$$

where Ω_c is the first transfinite ordinal number to correspond to the cardinal number c of the linear continuum. Let the regions of Q be well ordered as well as the possible divisions D_1, $D_2, \cdots, D_a, \cdots$ of V into infinitely many mutually exclusive subsets U_1, $U_2, \cdots$. As the power of this set of regions and the power of the set of D_a's are both c, let there be a one-to-one correspondence between each of these and the sequence C_1, $C_2, \cdots, C_a, \cdots$.

Choose for each C_a, having nothing in common with the interior of the square Q, a point set M_{ia} for each i and in each Q_i construct a simple closed curve J_{ia}, exactly as Miller does for his M, using, for each i, $Q_i\cdot V$ in place of his[6] $(ABCD)\cdot\Delta$. Thus in K, exterior to Q, we have infinitely many mutually exclusive sets, $N_1, N_2, \cdots, N_i, \cdots$ say, each exactly similar to Miller's biconnected set M, except for $K\cdot Q$. In each region R_a of Q let a simple closed curve J_a' be constructed, by a method similar to that used by Miller, so that each V_{ij} is dense in $K\cdot J_a'$. Each infinite division D_a above of V determines a U_a'' and an R_a'' of Q such that, for each i, j, $V_{ij}-V_{ij}\cdot U_a''$ is dense in $V\cdot R_a''$. In each R_a'' construct a simple closed curve J_a'' such that each V_{ij} is dense in $K\cdot J_a''$ but $J_a''\cdot U_a''=0$. For each C_a separating $Q\cdot K$ choose for each V_{ij} a point or vacuous set, according to whether or not $C_a\cdot V_{ij}$ is vacuous, obtaining for each such C_a an M_{ija} of Q with the properties of Miller's M_a's. No $J_a'+J_a''$ contains a point of an M_{ija} and no two M_{ija}'s consist of the same point.

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[5] E. W. Miller, loc. cit., p. 129.

[6] E. W. Miller, loc. cit., pp. 128–130.

The method used is dependent upon having chosen at any time during the process, under the hypothesis of the continuum, at most a countable infinity of points in $M\cdot(C_1+C_2+\cdots+C_a)$, where $M=N_1+N_2+\cdots+V+M_{111}+M_{121}+\cdots+M_{112}+M_{122}+\cdots$. This is true here just as it was for Miller's M_a's. As the set of composants of K is of the power of the linear continuum, new points can always be chosen for new C_a's, and each choice can be made so that no composant contains more than one point of M.

The set M is widely connected, for each C_a contains at least one point of M and no composant of K contains more than one point[7] of M. Let B_{1g}, $(g=1, 2)$, contain all of $N_g+[V_{1g}+\sum_{a=1}^{\Omega_0} M_{1ga}]\times Q$, and let in addition B_{11} contain all the rest of M, with the exception of the rest of M in Q_1, and let B_{12} contain this. Hence B_{11} and B_{12} are mutually exclusive sets whose sum is M. Each is connected, for every C_a contains a point of each. Just as Miller showed, each B_{1g} is biconnected, for suppose that B_{11}, say, is the sum of the two mutually exclusive subsets W_1 and W_2. As $W_1\cdot V$ must be dense in $Q_1\cdot V$, there exists a $J_{1a}\cdot M$ of Q_1 contained entirely in $W_1\cdot V$, according to the construction of the J_{1a}'s. As B_{11} is widely connected, this is impossible. Hence M is the sum of two mutually exclusive biconnected subsets B_{11} and B_{12}.

In a similar manner for $n>1$ it is seen that M is the sum of $n+1$ mutually exclusive biconnected subsets $B_{n1}, B_{n2}, \cdots, B_{n,n+1}$, where B_{nj} contains $N_j+[V_{nj}+\sum_a M_{nja}]\times Q$ of M and B_{n1} contains all the rest of M, except the rest of M contained in Q_1, and B_{n2} contains this.

It is seen however that M is not the sum of infinitely many mutually exclusive connected subsets $T_1, T_2, \cdots$, for every region of Q contains a J_a' and so each connected set T_i would contain a U_i dense in $V\cdot J_a'$ and so dense in $V\cdot Q$. This U_i is also dense in V because of the J_{ia}'s. Thus $T_1\cdot V, T_2\cdot V, \cdots$ is a division D_j of V into infinitely many mutually exclusive subsets $U_1, U_2, \cdots$ each dense in $V\cdot Q$. Hence one of these is a U'' which does not contain a point of some J_a''. Therefore the T_i, such that $U''=T_i\cdot V$, cannot be connected.

Thus it is seen that M is an example of a finitely-divisible connected set and similarly of a finitely-containing connected set, since each connected subset of M is widely connected.

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[7] E. W. Miller, loc. cit., p. 126, Theorem 7.

THE MINIMUM NUMBER OF GENERATORS FOR INSEPARABLE ALGEBRAIC EXTENSIONS[1]

M. F. BECKER AND S. MACLANE

1. Finite algebraic extensions of imperfect fields. A finite separable algebraic extension L of a given field K can always be generated by a single primitive element x, in the form $L=K(x)$. If K has characteristic p, while L/K is inseparable, there may be no such primitive element. The necessary and sufficient condition for the existence of such an element is to be found in Steinitz.[2] When there is no such primitive element, there is the question:[3] given K, what is the minimum integer m such that *every* finite extension L/K has a generation $L=K(x_1, x_2, \cdots, x_m)$ by not more than m elements?

The question can be answered by employing Teichmüller's[4] notion of the "degree of imperfection" of K. In invariant fashion, a field K of characteristic p determines a subfield K^p consisting of all pth powers of elements of K. If the extension K/K^p is finite, its degree $[K:K^p]$ is a power p^m of the characteristic, and the exponent m is called the *degree of imperfection* of K. For instance, let P be a perfect field of characteristic p and let x, y be elements algebraically independent with respect to P. Form the fields

$$S = P(x), \qquad T = P(x, y). \tag{1}$$

Then $S=S^p(x)$, $[S:S^p]=p$, while $[T:T^p]=p^2$, so that T is "more imperfect" than S.

THEOREM 1. *If the field K of characteristic p has a finite degree of imperfection m, then every finite algebraic extension $L\supset K$ can be obtained by adjoining not more than m elements to K. Furthermore, there exist finite extensions $L\supset K$ which cannot be obtained by adjoining fewer than m elements to K.*

PROOF. First consider the particular extension $K^{1/p}$ consisting of all pth roots of elements in K. Because of the isomorphism $a\longleftrightarrow a^{1/p}$,

$$[K^{1/p}:K] = [K:K^p] = p^m. \tag{2}$$

Each element y in $K^{1/p}$ satisfies over K an equation $y^p=a$ of degree p.

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[1] Presented to the Society, October 28, 1939.

[2] E. Steinitz, *Algebraische Theorie der Körper*, Berlin, de Gruyter, 1930, p. 72.

[3] This problem was suggested to one of us by O. Ore.

[4] O. Teichmüller, *p-Algebren*, Deutsche Mathematik, vol. 1 (1936), pp. 362–388.

If $K^{1/p}$ had generators $y_1, \cdots, y_n$ in number less than m, the degree $[K^{1/p}:K]$ could not exceed p^n, a contradiction to (2).

An explicit generation for K/K^p can be found by successively choosing elements x_i of K such that each x_i is not in $K^p(x_1, \cdots, x_{i-1})$. Then each x_i satisfies an irreducible equation[5] $x_i^p = a_i$ over $K^p(x_1, \cdots, x_{i-1})$. The adjunction of x_i is an extension of degree p; so

$$(3) \qquad K = K^p(x_1, x_2, \cdots, x_m), \qquad [K:K^p] = p^m,$$

where m is the degree of imperfection[6] of K.

Now let L be any finite extension of K. Because of the isomorphism $a \longleftrightarrow a^p$, one has $[L:K] = [L^p:K^p]$. Hence

$$(4) \qquad [L:L^p] = [L:K]\cdot[K:K^p]/[L^p:K^p] = [K:K^p] = p^m.$$

Therefore K and L have the same degree of imperfection. But L has an explicit generation $L = L^p(y_1, \cdots, y_m)$ like that of (3). If L^{p^n} denotes the field of all p^nth powers of elements of L, the isomorphism $a \longleftrightarrow a^{p^n}$ yields $L^{p^n} = L^{p^{n+1}}(y_1^{p^n}, \cdots, y_m^{p^n})$. By an induction on n,

$$(5) \qquad L = L^{p^n}(y_1, \cdots, y_m).$$

Since L/K is finite, there is an integer n so large that for each y in L the power y^{p^n} is separable over K. The separable extension[7] $K(L^{p^n})/K$ has a single generator $K(L^{p^n}) = K(y_0)$. Since y_0 is separable, the usual theorem[8] of the primitive element yields a single element y' such that $K(y_0, y_1) = K(y')$. Thus, by (5),

$$L = K(y_0, y_1, y_2, \cdots, y_m) = K(y', y_2, \cdots, y_m).$$

This is a generation by m elements, as required.

The degree of imperfection of a field K may be *infinite*, in the sense that the extension K/K^p used in the definition is infinite. Our arguments in this case give the following result.

THEOREM 2. *If the degree of imperfection of a field K is infinite, then for each integer $n > 0$ there exists a finite algebraic extension $L \supset K$ which cannot be obtained by adjoining fewer than n elements to K.*

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[5] For the usual properties of such equations, cf. A. A. Albert, *Modern Higher Algebra*, chap. 7.

[6] The set $\{x_1, \cdots, x_m\}$ of independent generators is called a *p-basis* for K. See O. Teichmüller, loc. cit., §3, or S. MacLane, *Modular fields*, I. *Separating transcedence bases*, Duke Mathematical Journal, vol. 5 (1939), pp. 372–393.

[7] Here $K(L^{p^n})$ denotes the field obtained from K by adjoining all elements of the field L^{p^n}.

[8] B. L. van der Waerden, *Moderne Algebra*, vol. 1, 1st edition, §34. Cf. also Steinitz, loc. cit., p. 72.

It might be thought that the minimum number of generators for an extension L/K is related to t, the transcendence degree of K over its maximum perfect subfield. However, this degree t may be larger than the degree of imperfection m. For a power series field K, Teichmüller observed that $m=1$, while t is infinite. Even when t and m are both finite, they can differ, as one of us showed by a more involved example[9] with $t=2$, $m=1$.

2. Infinite algebraic extensions of imperfect fields. In applying our criterion for the minimum number of generators one needs to compute the degree of imperfection of a given field. A perfect field contains pth roots of all of its elements, hence has degree of imperfection zero. A simple transcendental extension $K(t)$ has a degree of imperfection one greater than the degree of imperfection of K, as Teichmüller has proved (cf. also the examples (1)). On the other hand, the computation (4) proves the following theorem.

THEOREM 3. *The degree of imperfection of a field is not changed by a finite algebraic extension.*

There remains the case of an infinite algebraic extension L/K. Such an extension is purely inseparable (or, a "radical" extension) if for each element a of L some power a^{p^e} lies in K. In this case we have the following result.

THEOREM 4. *If K has a finite degree of imperfection m, then the degree of imperfection of a purely inseparable infinite extension of K is less than m, the degree of imperfection of K.*

Let L be a purely inseparable, infinite extension of K. We use a chain of intermediate fields

$$K \subset L_1 \subset L_2 \subset L_3 \subset \cdots \subset L_n \subset \cdots \subset L, \tag{6}$$

where L_n consists of all elements of L with p^nth power in K. The field L_{n+1} is obtained from L_n by adjoining pth roots of a sufficient number of elements of L_n. By (4), the degree of imperfection of each L_n is m. Hence, L_{n+1} is a field of degree at most p^m over L_n. Since $[L_n:K]$ is then finite, each L_{n+1} is larger than the preceding L_n.

By the definition of the tower (6), each $L_n \supset L_{n+1}^p$. Since $L^p \supset L_{n+1}^p$, any element α of L_n has over L_{n+1}^p a degree[10] $[\alpha:L_{n+1}^p] \geqq [\alpha:L^p]$. In other words,

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[9] S. MacLane, loc. cit., §10.

[10] In fact, $[\alpha:L_{n+1}^p]=[\alpha:L^p]$. Since α is an element of L_n, $[\alpha:L_{n+1}^p]=p$ or 1, hence $[\alpha:L^p]=p$ or 1. If $[\alpha:L^p]=1$, $\alpha^{p^{-1}}$ is in L and $(\alpha^{p^{-1}})^{p^{n+1}}=\alpha^{p^n}$ is in K. Thus, by definition of L_{n+1}, $\alpha^{p^{-1}}$ is in L_{n+1}, so $[\alpha:L_{n+1}^p]=1$.

$$[L^p(L_n):L^p] \leqq [L_{n+1}^p(L_n):L_{n+1}^p].$$

But $L_{n+1}^p(L_n)=L_n$, while $L_{n+1}>L_n$ and $[L_{n+1}^p:L_n^p]>1$. Thus

$$[L^p(L_n):L^p] \leqq [L_n:L_{n+1}^p] < [L_n:L_{n+1}^p]\cdot[L_{n+1}^p:L_n^p] = [L_n:L_n^p].$$

This degree $[L_n:L_n^p]$ is simply p^m, with m the degree of imperfection of L_n; so

$$[L^p(L_n):L^p] \leqq p^{m-1}. \tag{7}$$

The maximum value of these degrees in (7) thus determines an integer $\delta\leqq m-1$ with

$$p^\delta = \max\,[L^p(L_n):L^p], \qquad \delta < m. \tag{8}$$

We assert that δ is the degree of imperfection of L. In the first place, $L \supset L^p(L_n)$; so $[L:L^p]\geqq[L^p(L_n):L^p]=p^\delta$, where we have so chosen n as to give the maximum in (8). If, however, $[L:L^p]$ exceeds p^δ, there must be $\delta+1$ elements $a_0, a_1, \cdots, a_\delta$ in L such that

$$[L^p(a_0, a_1, \cdots, a_\delta):L^p] = p^{\delta+1},$$

contrary to the definition (8) of δ. Thus δ, the degree of imperfection of L, is less than the corresponding degree of imperfection for K.

The degrees used in the computation (8) of δ can be expressed explicitly by choosing a p-basis $x_1, \cdots, x_m$ for each L_n, for then

$$[L^p(L_n):L^p] = [L^p(L_n^p(x_1, \cdots, x_m)):L^p] = [L^p(x_1, \cdots, x_m):L^p].$$

Consider now an infinite extension L/K which is not purely inseparable, and let M denote the field of all elements of L separable over K. Even if M/K is infinite, M and K still have the same degree of imperfection, according to a result of Teichmüller.[11] If the *exponent* of L is taken to be the least integer e such that all powers a^{p^e} of elements a in L are separable over K, we then have the following theorem.

THEOREM 5. *If K has a finite degree of imperfection, then an algebraic extension L of K has the same or a smaller degree of imperfection according as L has a finite or an infinite exponent over K.*

3. **Generators for given extensions.** In §1 we determined the minimum number of generators for all algebraic extensions of a fixed base field. Suppose, however, L is a specific extension of K. We wish to

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[11] Any p-basis for K is also a p-basis for an arbitrary separable algebraic extension M of K; cf. Teichmüller, loc. cit., p. 170.

get the minimum number of generators for this particular extension. It clearly suffices to consider L a purely inseparable finite extension of K.

Whether the degree of imperfection of L is finite or infinite there is a subset U in L such that $L^p(U)=L$. By the same argument as in §1

$$L = L^{p^n}(U) \tag{9}$$

for integral n.

Consider, now, the field $L^p(K)$ between L and K. For L/K finite, $L/L^p(K)$ is finite and $[L:L^p(K)]=p^r$. Since the pth power of every element in L is contained in $L^p(K)$, r elements $X_1, X_2, \cdots, X_r$ in L can be chosen such that

$$L = L^p(K)(X_1, \cdots, X_r) = L^p(K, X_1, X_2, \cdots, X_r).$$

If e is the exponent of L/K, using (9) we obtain

$$L = L^{p^e}(K, X_1, \cdots, X_r) = K(X_1, X_2, \cdots, X_r),$$

Hence L/K can be generated by r elements.

Moreover, r is the minimum number of generators. For if $L=K(Y_1, \cdots, Y_s)$ where $s<r$,

$$L^p = K^p(Y_1^p, \cdots, Y_s^p), \qquad L^p(K) = K(Y_1^p, \cdots, Y_s^p),$$
$$p^r = [L:L^p(K)] = [K(Y_1, \cdots, Y_s):K(Y_1^p, \cdots, Y_s^p)] \leqq p^s,$$

and $r \leqq s$, against assumption.

THEOREM 6. *If L is a purely inseparable finite extension of K, the minimum number of generators of L/K is r, the exponent determined by the degree $[L:L^p(K)]=p^r$.*

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THE ANNUAL MEETING OF THE SOCIETY

The forty-sixth Annual Meeting of the American Mathematical Society was held at the Ohio State University, Columbus, Ohio, from Tuesday to Friday, December 26–29, 1939, in conjunction with the meetings of the American Association for the Advancement of Science, the Mathematical Association of America, and the National Council of Teachers of Mathematics. This was the fifth time the Society has met at the Ohio State University, the other meetings being in the years 1899, 1915, 1926, and 1931.

There were four general and nine sectional sessions of the Society, at which eight addresses and one hundred two research papers (sixty-two in person) were given. The Gibbs Lecture on Wednesday afternoon was given in Campbell Hall; the joint session on Friday morning with Section E was held in Mendenhall Hall; all other sessions were held in the Horticulture and Forestry Building of the Ohio State University.

Arrangements for the meetings were made by a committee consisting of Professors S. E. Rasor (Chairman), Lincoln LaPaz, J. H. Weaver, and F. B. Wiley. The dormitory and social rooms of Neil Hall were made available for the use of the visiting mathematicians.

Over four hundred persons registered including the following two hundred eighty-six members of the Society, the largest attendance at an annual meeting in the history of the Society:

V. W. Adkisson, R. P. Agnew, Leonidas Alaoglu, A. A. Albert, G. E. Albert, H. W. Alexander, C. B. Allendoerfer, W. E. Anderson, Emil Artin, Max Astrachan, J. V. Atanasoff, C. S. Atchison, H. T. R. Aude, Frank Ayres, W. L. Ayres, R. W. Babcock, F. R. Bamforth, G. M. Bareis, I. A. Barnett, R. C. F. Bartels, E. F. Beckenbach, E. R. Beckwith, M. M. Beenken, A. A. Bennett, S. F. Bibb, G. D. Birkhoff, H. L. Black, Archie Blake, Henry Blumberg, L. M. Blumenthal, R. P. Boas, Salomon Bochner, Paul Boeder, D. G. Bourgin, J. W. Bower, M. G. Boyce, J. W. Bradshaw, J. B. Brandeberry, A. T. Brauer, W. C. Brenke, F. L. Brooks, O. E. Brown, G. S. Bruton, C. T. Bumer, R. S. Burington, Jewell H. Bushey, W. E. Byrne, W. D. Cairns, J. W. Calkin, H. H. Campaigne, H. C. Carter, W. B. Carver, E. W. Chittenden, R. V. Churchill, E. H. Clarke, W. W. S. Claytor, R. F. Clippinger, L. M. Coffin, L. W. Cohen, J. B. Coleman, A. H. Copeland, W. A. Cordrey, A. P. Cowgill, L. C. Cox, M. M. Culver, H. B. Curtis, J. H. Curtiss, A. H. Diamond, R. P. Dilworth, L. L. Dines, H. A. DoBell, J. L. Doob, H. H. Downing, W. C. Doyle, Arnold Dresden, R. J. Duffin, P. S. Dwyer, J. J. Eachus, E. D. Eaves, Samuel Eilenberg, W. H. Erskine, G. C. Evans, G. W. Evans, H. P. Evans, F. A. Ficken, B. F. Finkel, F. A. Foraker, L. R. Ford, W. B. Ford, Orrin Frink, T. C. Fry, D. G. Fulton, A. S. Galbraith, M. C. Garvin, B. E. Gatewood, H. M. Gehman, Abe Gelbart, F. J. Gerst, Lachlan Gilchrist, Wallace Givens, L. M. Graves, J. W. Green, V. G. Grove.

D. W. Hall, N. A. Hall, P. R. Halmos, O. G. Harrold, M. C. Hartley, M. L. Hartung, G. G. Harvey, O. C. Hazlett, E. R. Hedrick, A. E. Heins, M. H. Heins, E. D. Hellinger, Fritz Herzog, M. R. Hestenes, H. C. Hicks, J. J. L. Hinrichsen, D. L. Holl, T. R. Hollcroft, P. M. Hummel, E. V. Huntington, W. A. Hurwitz, D. H. Hyers, Dunham Jackson, R. D. James, E. D. Jenkins, Walter Jennings, Fritz John, R. P. Johnson, F. E. Johnston, B. W. Jones, H. A. Jordan, Mark Kac, Samuel Kaplan, Wilfred Kaplan, L. C. Karpinski, Chosaburo Kato, D. K. Kazarinoff, J. L. Kelley, R. B. Kershner, J. R. Kline, L. A. Knowler, Alfred Korzybski, H. W. Kuhn, A. C. Ladner, W. D. Lambert, K. W. Lamson, O. E. Lancaster, H. G. Landau, R. E. Langer, H. L. Langhaar, Lincoln LaPaz, C. G. Latimer, C. E. Leach, Solomon Lefschetz, D. H. Lehmer, R. A. Leibler, J. H. Levin, Norman Levinson, G. H. Ling, Marie Litzinger, M. I. Logsdon, A. T. Lonseth, E. R. Lorch, C. I. Lubin, P. H. McGrath, C. C. MacDuffee, Saunders MacLane, H. M. MacNeille, M. M. Maloney, J. D. Mancill, Dorothy Manning, R. G. Mason, Karl Menger, A. N. Milgram, D. D. Miller, E. W. Miller, Virginia Modesitt, M. G. Moore, Max Morris, Richard Morris, R. C. Morrow, Marston Morse, E. J. Moulton, J. R. Musselman, F. C. Ogg, Rufus Oldenburger, E. G. Olds, E. J. Olson, E. R. Ott, J. R. Overman, W. V. Parker, E. W. Paxson, D. T. Perkins, O. J. Peterson, Everett Pitcher, V. C. Poor, G. B. Price, Tibor Radó, J. F. Randolph, S. E. Rasor, F. W. Reed, Eric Reissner, R. G. D. Richardson, C. E. Rickart, P. R. Rider, R. F. Rinehart, H. P. Robertson, W. H. Roever, J. B. Rosenbach, J. E. Rosenthal, Barkley Rosser, E. H. Rothe, N. E. Rutt, O. F. G. Schilling, I. J. Schoenberg, G. E. Schweigert, C. E. Sealander, Seymour Sherman, C. G. Shover, D. R. Shreve, D. T. Sigley, M. E. Sinclair, E. R. Sleight, M. M. Slotnick, M. F. Smiley, E. R. Smith, F. C. Smith, J. P. Smith, T. L. Smith, W. F. Smith, Virgil Snyder, D. C. Spencer, G. W. Starcher, N. E. Steenrod, H. E. Stelson, C. N. Stokes, R. W. Stokes, E. C. Stopher, W. T. Stratton, E. C. Strayhorn, Otto Szász, J. S. Taylor, M. E. Taylor, W. C. Taylor, H. P. Thielman, J. M. Thomas, R. M. Thrall, E. W. Titt, C. C. Torrance, H. C. Trimble, W. J. Trjitzinsky, P. L. Trump, A. W. Tucker, F. W. Urban, J. L. Vanderslice, J. H. Van Vleck, J. I. Vass, R. W. Wagner, G. W. Walker, A. D. Wallace, J. L. Walsh, S. E. Warschawski, J. H. Weaver, Warren Weaver, M. S. Webster, M. J. Weiss, C. P. Wells, M E. Wescott, G. T. Whyburn, Norbert Wiener, L. R. Wilcox, R. L. Wilder, F. B. Wiley, C. W. Williams, K. P. Williams, E. W. Wilson, Aurel Wintner, F. L. Wren, C. R. Wylie, M. M. Young, J. W. T. Youngs, R. T. Zoch, Max Zorn.

The meeting opened at 2:00 Tuesday afternoon with sections for Analysis and for Topology. Professors L. L. Dines and G. T. Whyburn presided at these two sessions.

Provost E. R. Hedrick presided at the general session on Wednesday morning. On Wednesday afternoon there were three sessions, for Analysis, for Algebra, and for Logic and the Theory of Sets. Professors M. R. Hestenes, C. C. MacDuffee, and L. M. Blumenthal presided. On Thursday afternoon there were sessions for Applications and for Topology, at which Professors Norbert Wiener and Solomon Lefschetz presided. On Friday morning Professor Arnold Dresden presided at the section for Analysis.

On Wednesday afternoon Professor Theodore von Kármán, Director of the Guggenheim Aeronautical Laboratory of the California Institute of Technology, delivered the fifteenth Josiah Willard Gibbs Lecture before a joint session of the Society and the American Association for the Advancement of Science. His subject was *The engineer grappling with non-linear problems.* Professor G. C. Evans, President of the Society, presided at this lecture.

The opening session of the American Association for the Advancement of Science was held on Wednesday evening at which Dr. W. C. Mitchell delivered the retiring presidential address on *The public relations of science.* This was followed by a reception for all members of the A.A.A.S. and affiliated societies at the Deshler-Wallick Hotel.

The general session on Wednesday morning opened with the business meeting and annual election. At this time the Frank Nelson Cole Prize in Algebra was awarded to Professor A. A. Albert for his papers *On the construction of Riemann matrices*, published in the Annals of Mathematics, vol. 35 (1934), pp. 1–28; vol. 36 (1935), pp. 376–394. This was followed by a joint session of the Society, the Association and Section A of the A.A.A.S. at which Professor J. R. Kline, as retiring Vice President of the A.A.A.S. and Chairman of Section A, gave an address entitled *The Jordan curve theorem.* The President of the Society, Professor G. C. Evans, presided.

At the general session on Thursday afternoon, by invitation of the Committee on Invited Speakers, Professor D. H. Lehmer delivered an address on *The application of Bernoulli polynomials to some problems in Diophantine analysis.* Professor H. W. Kuhn presided at this lecture.

On Friday morning the section for Applied Mathematics met jointly with the Association and Sections A and E of the A.A.A.S. At this session the following five addresses were given: *The beginnings of mathematical geophysics in Great Britain* by W. D. Lambert, *Use of mathematics in the delineation of magnetic and electric anomalies* by Professor Lachlan Gilchrist, *Gravimetric and seismic methods in exploratory geophysics* by Dr. M. M. Slotnick, *Mathematical problems in seismology* by Dr. Archie Blake, *Some seismological problems* by Professor Perry Byerly. In the absence of Professor Byerly, his prepared address was read by Professor W. D. Cairns. Professor Marston Morse, Chairman of Section A, presided at this joint session.

Sessions of the Mathematical Association were held on Friday afternoon and Saturday morning, and sessions of the National Council of Teachers on Friday morning and afternoon.

A joint dinner for the mathematical organizations and their guests was held at the Faculty Club on Thursday evening. The attendance was two hundred eighty-three. Professor Tibor Radó was toastmaster and introduced Dean B. L. Stradley who welcomed the visitors to Columbus and the Ohio State University. Professor H. C. Christofferson called attention to some interesting trends in enrollments of grade school, secondary and college students in the past fifty years. Professor R. E. Langer discussed some proposed policies of the Association. Professor G. D. Birkhoff spoke of the origin and workings of the Putnam Prize Competition. Professor J. M. Thomas pointed to the vigorous but alarming growth in published research in this country and proposed that authors "Write less better." At the close of the dinner Dean G. H. Ling presented resolutions, which were adopted unanimously, expressing the appreciation of the mathematical groups to the Ohio State University and the local committee for the excellent arrangements and generous hospitality.

The ladies of the local Department of Mathematics served tea in Neil Hall on Tuesday and Friday afternoons for visiting mathematicians and guests. There was also a tea at the Governor's Mansion on Thursday afternoon.

At the meeting of the Board of Trustees at 7:30 P.M., Tuesday, December 26, in a social room of Neil Hall, there was not a quorum present. An adjourned meeting, therefore, was held on January 1, at 12:30 P.M., in the Men's Faculty Club of Columbia University, New York City.

The Council met on Wednesday, December 27, at 7:30 P.M. in a social room of Neil Hall. An adjourned meeting was held on Thursday at 11:15 P.M.

The Secretary announced the election of the following twenty-eight persons to membership in the Society:

Sister Anastasia Maria, Immaculata College, Immaculata, Pa.;
Miss Lois E. Bell, Independence Junior College, Independence, Kan.;
Mr. Kenneth Arthur Bush, Bard College, Annandale-on-Hudson, N.Y.;
Miss Mary Louise Constable, Philadelphia High School for Girls;
Mr. Commodore Columbus Dearman, Jr., East Central Junior College, Decatur, Miss.;
Mr. Bernard Dimsdale, University of Idaho;
Professor Solomon Dobrin, Jr., Essex Junior College, Newark, N.J.;
Mr. William Holt Glenn, Jr., Pasadena Junior College;
Dr. Michael Golomb, School of Electrical Engineering, Cornell University;
Professor Alberto Gonzalez Dominguez, Brown University;
Dr. Albert Edward Heins, Purdue University;
Dr. Stephen Arthur Jennings, Yale University;
Professor Pete Singleton Leach, Middle Georgia College, Cochran, Ga.;
Mr. Joseph Harmon Levin, Wayne University;

Mr. Hunter John McConnell, Salt Lake City, Utah;
Reverend Philip H. McGrath, St. Peter's College, Jersey City, N.J.;
Mr. Thomas Henry Murtaugh, Fordham University;
Dr. Andrewa R. Noble, Berkeley, Calif.;
Dr. Eric Reissner, Massachusetts Institute of Technology;
Mr. W. Murray Robertson, Pennsylvania Railroad, Narberth, Pa.;
Dr. Louis D. Rodabaugh, University of Alabama;
Dr. Peter Scherk, Taft School, Watertown, Conn.;
Mr. Morris Herbert Shamos, New York, N.Y.;
Professor Herman Walton Smith, Oklahoma Agricultural and Mechanical College;
Dr. Chester Snow, National Bureau of Standards, Washington, D.C.;
Dr. Donald Clayton Spencer, Massachusetts Institute of Technology;
Dr. Charles P. Wells, Michigan State College;
Mr. Max Wyman, California Institute of Technology.

It was reported that the following had been elected as nominees on the Institutional Memberships of the institutions indicated:

Bowdoin College: Mr. Dan Edwin Christie (at Princeton University).
Brooklyn College: Mrs. Mary I. Draper Boeker.
Brown University: Dr. Willy K. Feller, Messrs. George Elmer Forsythe, John George Herriot, Joseph Albert Joseph, and Olaf Schmidt.
Bryn Mawr College: Dr. Hilda Geiringer and Miss Dorothy Maharam.
University of California: Mr. Douglas George Chapman and Miss Orla Virginia Wood.
University of California at Los Angeles: Messrs. Jule Gregory Charney and Harry Lass.
University of Chicago: Miss Katherine Elizabeth Hazard, Messrs. William Karush and Herman Lewis Meyer, Jr.
College of the City of New York: Mr. Kenneth J. Arrow.
Columbia University: Messrs. Seymour Jablon, Robert Alex McKean, Don Mittleman, and Fred Supnick.
Cornell University: Professor Walter Rue Murray (Franklin and Marshall College) and Mr. Edward Oscar Stephany.
Harvard University: Messrs. Maurice Haskell Heins and Edwin Norman Nilson, and Hon. John Lord O'Brien (of Buffalo, N.Y.).
University of Illinois: Messrs. David Harold Blackwell, Abraham Charnes, Robert W. Gibson, Nathan Goldman, and Yudell Leo Luke.
Indiana University: Mr. David Gilbarg and Miss Margaret Ellen Stump.
Institute for Advanced Study: Drs. Alfred Theodor Brauer and Clifford Hugh Dowker, Professor Guido Fubini, Mr. Albert Edward Ingham (King's College, Cambridge, England), and Professor Charles Eugene Springer (University of Oklahoma).
The State University of Iowa: Mr. Carl Elroy Noble.
Iowa State College: Mr. Roy Herbert Cook.
The Johns Hopkins University: Mr. Samuel G. Bourne, Miss Florence Jessie Collinson, Mr. Alfred Levine, Miss Yael Naim, and Mr. Harold Kenneth Sohl.
University of Kentucky: Mr. William Glenn Clark.
Lehigh University: Mr. Eugene Park.
Massachusetts Institute of Technology: Messrs. Francis Begnaud Hildebrand, Carl Gustaf Allan Nordling, and Claude E. Shannon.

University of Michigan: Dr. Samuel Eilenberg and Mr. Sheng Chin Fan.
University of Minnesota: Miss Margaret Pearl Martin, Messrs. William Robert McEwen and Frank Joseph Polansky.
Northwestern University: Messrs. Theodore Wilbur Anderson, Jr., Harold Elra Burns (Indiana University Extension, East Chicago, Ind.), Thomas Elias Caywood, Edwin Leonidas Godfrey (Indiana University Extension, East Chicago, Ind.), and Ralph Eldon Lane (Flitcraft, Inc., Oak Park, Ill.).
University of Pennsylvania: Mr. Hyman N. Laden.
University of Rochester: Mr. Henry Pearce Atkins.
The College of St. Thomas: Mr. George Joseph Haltiner (of St. Paul).
Stanford University: Messrs. Charles Lester Clark and Stanley Eugene Rauch.
Syracuse University: Mr. Barnard Hinkle Bissinger.
University of Virginia: Mr. Franklin Guy Myers.
University of Washington: Misses Kathryn Eleanor Benson and Mary Elizabeth Layne.
Wellesley College: Professor Harriet Whitney Allen (Hollins College, Hollins, Va.).
University of Wisconsin: Messrs. Burton Houston Colvin, Wilhelm S. Erickson, William Grenfell Leavitt, and Clarence Bernhart Lindquist.

The University of Minnesota was elected to Institutional Contributing Membership in the Society.

Professor Francesco Tricomi of the University of Turin was admitted to the Society in accordance with the reciprocity agreement with the Unione Matematica Italiana.

The following appointments by President G. C. Evans were reported: as representative of the Society at the inauguration of Frank Jay Prout as President of Bowling Green State University on October 21, Professor J. B. Brandeberry; as representative of the Society at the inauguration of John Benjamin Magee as President of Cornell College on October 27, Professor E. W. Chittenden; as representatives of the Society at the Centenary Celebration of the American Statistical Association in Boston on November 27, Professor G. D. Birkhoff, and in Philadelphia on December 29, Professor S. S. Wilks; as representative of the Society at the inauguration of Homer Price Rainey as President of the University of Texas on December 9, Professor R. L. Moore; as a Committee on Arrangements for the 1940 Spring Meeting in Washington, D.C., Professor F. E. Johnston (chairman), Dr. Archie Blake, Professors T. R. Hollcroft, A. E. Landry, Florence M. Mears, E. W. Titt, and F. M. Weida; as a Committee on Arrangements for the Summer Meeting of 1940, Professors F. W. Perkins (chairman), W. D. Cairns, T. R. Hollcroft, Robin Robinson, L. L. Silverman, and H. L. Slobin; as a Committee on Arrangements for the Annual Meeting of 1940, Professors S. T. Sanders (chairman), W. L. Ayres, H. E. Buchanan, W. D. Cairns, W. V. Parker, and N. E. Rutt; as additional members of the Committee on Publicity, Mr. R. M. Foster and Dr. J. M. Thompson; as tellers for the election at the 1939 Annual Meeting, Professor F. R. Bamforth and Dr. E. R. Lorch; as auditors of the Society's books for the year 1939, Professors R. G. Archibald and A. E. Meder, Jr.; as the Committee on Nomination of Officers and Members of the Council for 1941, Professors Arnold Dresden (chairman), L. M. Graves, E. R. Hedrick, R. L. Moore, and J. L. Walsh.

The following representatives of the Society were appointed: on the Editorial Board of the Annals of Mathematics for a period of three years beginning with 1940, Professors T. H. Hildebrandt, Saunders MacLane, and G. T. Whyburn; on the Division of Physical Sciences of the National Research Council for a period of three years beginning July 1, 1940, Professor G. C. Evans; on the Council of the American Association for the Advancement of Science for the year 1940, Professors Arnold Dresden and J. R. Kline; on the advisory board of The Kosciuszko Foundation to assist Polish scholars, Professor J. R. Kline.

The Secretary reported that the ordinary membership in the Society is now 2,283, including 200 nominees of institutional members and 79 life members. There are also 87 institutional members. The total attendance of members at all meetings in 1939 was 1,354; the number of papers read was 489; the number of invited addresses was 10; the number of members attending at least one meeting was 841.

At the annual election which closed on December 27, and at which 406 votes were cast (no person not on the ballot receiving more than one vote for any office) (375 for the list nominated by the Council), the following officers were elected:

Vice-Presidents, Dr. T. C. Fry and Professor F. D. Murnaghan.

Secretary, Dean R. G. D. Richardson.

Associate Secretaries, Professors W. L. Ayres and M. H. Ingraham, and Dean T. M. Putnam.

Member of the Editorial Committee of the Bulletin, Professor P. A. Smith.

Member of the Editorial Committee of the Transactions, Professor C. C. MacDuffee.

Member of the Editorial Committee of the Colloquium Publications, Professor J. D. Tamarkin.

Members of the Council, Professors L. E. Dickson, C. G. Latimer, N. H. McCoy, Saunders MacLane, and W. M. Whyburn.

The Secretary requested that the Society accept his resignation to take effect not later than the end of 1940.

The reports of the Treasurer and auditors (Professors R. G. Archibald and A. E. Meder, Jr.) showed a balance of $9,405.61 exclusive of the balances in the Colloquium, Sinking Fund, and special funds. The Society's investments including Endowment Fund, Sinking Fund, and other special funds had a market value on November 30, 1939, of approximately $150,000.00. The net interest income for the year was $5,377.25; dues from Institutional Memberships amounted to $6,530.00; dues from Contributing Memberships, $1,163.92; and dues from Ordinary Memberships were $13,541.71. The Colloquium Fund showed a balance of $8,075.38. The total received from the sale of the Society's publications was $11,534.24. The Board of Trustees adopted a budget for 1940 showing estimated expenditures and receipts of the Society (including Mathematical Reviews) as $53,800.00 and $40,800.00 respectively.

The Librarian reported that the Library of the Society now contains about 9,475 volumes.

The American Journal of Mathematics, which is published jointly by the Johns Hopkins University and the Society and to which the Society gives an annual subvention of $2,500, printed 1,008 pages during 1939.

An invitation from Dartmouth College to hold the 1940 Summer Meeting in Hanover, New Hampshire, was accepted, and the meeting was set for September 10–12. On invitation, the 1941 Summer Meeting is to be held at the University of Chicago in connection with its Semicentennial Celebration. There will be a meeting of the Society at Seattle, Washington, in June 1940, in affiliation with the American Association for the Advancement of Science. Times and places of additional meetings during 1940 were set as follows: February 24 and October 26 in New York, April 6 in Berkeley, December 30, 1940–January 1, 1941 in Baton Rouge.

On recommendation of the Committee on Special Membership Rates, the following addition to Article VI of the by-laws was recommended to the Society:

> Section 9. Any member who has been retired from active service on account of age and has been a member of the Society for twenty years or more may, upon notification to the Secretary of such retirement, have his dues remitted, on the understanding that he will thereafter receive the programs of the meetings, but not the Bulletin.

This change in the by-laws will be submitted to the Society for vote at the meeting on February 24, 1940.

A proposal from the Committee on the Disposition of Interest on Special Funds is to lie over for consideration until the next meeting of the Council. This proposal involves the inauguration by the Society of a new series of books, somewhat smaller in size than books of the Colloquium Series and devoted chiefly to exposition of advanced topics.

Copies of the first number of Mathematical Reviews were on exhibition at the meeting. Eight hundred subscribers are already on the books. Three hundred fifty mathematicians, in all parts of the world, have accepted invitations to be reviewers. Arrangements are being made with various organizations for joint sponsorship of the journal with the Society.

A grant of one thousand dollars was announced from the Rockefeller Foundation for an experimental period of two years for the filming of all mathematical papers reviewed in the Mathematical Reviews.

The Emergency Executive Committee of the International Congress of Mathematicians which was to have been held in Cambridge, Massachusetts, in September 1940, announced that on December 20, 1939, notice was sent to all persons interested that the Congress has been postponed.

It was decided that no Josiah Willard Gibbs Lecture would be given during the year 1940.

Professor R. L. Wilder was invited to give the Colloquium Lectures in 1942. It was announced that Professor G. T. Whyburn had been asked to give his Colloquium Lectures in 1940. A book entitled *Lattice Theory* by Professor Garrett Birkhoff was accepted for publication in the Colloquium Series.

The Transactions editors announced that space is now available promptly in that journal. Professor E. T. Bell's term having expired, Professor Saunders MacLane has been appointed associate editor in his place.

Professor M. H. Stone proposed that a letter be sent to sister mathematical organizations in various parts of the world, suggesting that mathematicians use their influence to conserve the scientific resources of the world against the day when peace shall reign once more and to mitigate in every way possible the horrors of war. This proposal is to be presented to the Society for its action at the February meeting.

Titles and cross references to the abstracts of papers read at the sessions follow below. The papers were read as follows: papers numbered 1 to 6 in the section for Analysis on Tuesday afternoon; papers 7 to 12 in the section for Topology on Tuesday afternoon; papers 13 to 20 in the general session on Wednesday morning; papers 21 to 28 in the section for Analysis on Wednesday afternoon; papers 29 to 35 in the section for Algebra on Wednesday afternoon; papers 36 to 40 in the section for Logic and the Theory of Sets on Wednesday afternoon; papers 41 to 49 in the section for Applications on Thursday afternoon; papers 50 to 55 in the section for Topology on Thursday afternoon; papers 56 to 62 in the section for Analysis on Friday morning; and papers 63 to 102, whose abstract numbers are followed by the letter *t*, were read by title. Professor H. W. Smith was introduced by Professor O. H. Hamilton, Dr. Stefan Bergman by Professor W. T. Martin, Dr. M. H. Heins by Professor J. L. Walsh, Dr. D. C. Spencer and Dr. Samuel Eilenberg by Professor W. L. Ayres, Dr. A. T. Brauer by Professor Hermann Weyl, Dr. Alexander Wundheiler by Professor D. J. Struik, Professor André Gleyzal and Dr. M. S. Hendrickson by Professor Henry Blumberg, Dr. A. E. Heins by Professor A. H. Smith, and Dr. W. A. Patterson by Professor Lincoln LaPaz. Paper 4 was presented by Dr. J. J. Eachus, paper 8 by Dr. P. V. Reichelderfer, paper 12 by Professor E. W. Miller, paper 28 by Professor Marston Morse, paper 42 by Dr. Eric Reissner, paper 48 by Professor Norbert Wiener, and paper 55 by Dr. D. W. Hall.

1. Mark Kac: *On the partial sums of the exponential series.* (Abstract 46-1-90.)

2. Norman Levinson: *A proof of Hardy's theorem on the zeros of the zeta function.* (Abstract 46-1-96.)

3. D. G. Bourgin: *The Dirichlet problem for the damped wave equation.* (Abstract 46-1-49.)

4. R. J. Duffin and J. J. Eachus: *Sets invariant under a polynomial operator.* (Abstract 46-1-63.)

5. Otto Szász: *On strong summability of Fourier series.* (Abstract 46-1-127.)

6. Alfred Korzybski: *General semantics: extensionalization in mathematics, mathematical physics, and general education.* III. *Over/under defined terms.* (Abstract 45-11-424.)

7. A. D. Wallace: *Relatively non-alternating transformations.* (Abstract 46-1-131.)

8. Tibor Radó and P. V. Reichelderfer: *On cyclic transitivity.* (Abstract 46-1-30.)

9. J. W. T. Youngs: *A remark on cyclic transitivity.* (Abstract 46-1-136.)

10. O. G. Harrold (National Research Fellow): *Exactly* (k, 1) *transformations on linear graphs.* (Abstract 46-1-76.)

11. G. E. Schweigert: *A local property arising from certain interior transformations*. (Abstract 46-1-120.)

12. Samuel Eilenberg and E. W. Miller: *On 0-dimensional upper-semi-continuous collections*. (Abstract 46-1-67.)

13. R. D. James: *A problem analogous to the problem of prime pairs*. (Abstract 46-1-85.)

14. A. A. Albert: *On p-adic fields and cyclic algebras*. (Abstract 46-1-44.)

15. W. J. Trjitzinsky: *Some problems in the theory of singular integral equations*. (Abstract 45-11-438.)

16. G. B. Price: *On the theory of integration*. (Abstract 46-1-114.)

17. I. J. Schoenberg: *On metric arcs of vanishing Menger curvature*. (Abstract 46-1-119.)

18. R. P. Boas: *Some uniqueness theorems for entire functions*. (Abstract 46-1-48.)

19. Karl Menger: *On Cauchy's theorem in the real plane*. (Abstract 46-1-104.)

20. Norbert Wiener: *A canonical series for symmetric functions in statistical mechanics*. (Abstract 46-1-133.)

21. H. W. Smith: *The oscillation of solutions of the differential boundary value problems of the fourth order*. (Abstract 46-1-125.)

22. C. E. Sealander: *Some third order irregular boundary value problems*. (Abstract 46-1-122.)

23. Stefan Bergman: *On the approximation of functions satisfying a linear partial differential equation*. (Abstract 46-1-46.)

24. J. W. Calkin: *Semi-bounded forms and self-adjoint boundary value problems*. (Abstract 46-1-52.)

25. M. H. Heins: *On the conformal mapping of a multiply-connected region into itself*. (Abstract 46-1-78.)

26. D. C. Spencer: *On mean one-valent functions*. (Abstract 46-1-126.)

27. Abe Gelbart: *On the growth of a function of two complex variables given by its power series expansion on certain hypersurfaces*. (Abstract 46-1-71.)

28. Marston Morse and C. B. Tompkins: *Minimal surfaces of unstable type*. (Abstract 46-3-149.)

29. D. T. Sigley: *k-set groups*. (Abstract 45-11-436.)

30. B. W. Jones: *Related genera of quadratic forms*. Preliminary report. (Abstract 46-1-87.)

31. C. G. Latimer: *On a certain equation*. (Abstract 46-1-95.)

32. R. P. Dilworth: *Ideals in Birkhoff lattices*. (Abstract 46-1-61.)

33. Max Zorn: *Alternative rings with nilpotent elements.* (Abstract 46-1-137.)

34. A. T. Brauer: *On the density of the sum of sets of positive integers.* II. (Abstract 46-1-50.)

35. R. F. Clippinger: *On matrix products of positive powers of given matrices.* Preliminary report. (Abstract 46-1-55.)

36. A. H. Diamond: *Postulates for the theory of strict implication.* (Abstract 46-1-60.)

37. Walter Jennings: *Some implications of the continuum hypothesis.* (Abstract 46-1-86.)

38. L. W. Cohen: *On topological completeness.* (Abstract 46-1-56.)

39. Seymour Sherman: *Some new properties of transfinite ordinals.* (Abstract 46-1-123.)

40. André Gleyzal: *A general theorem on the structure of linear orders.* (Abstract 46-1-73.)

41. A. E. Heins: *The solution of the discrete wave equation.* (Abstract 46-1-16.)

42. Eric Reissner and H. A. Wood: *On boundary value problems of bi-potential theory for an infinite sector.* (Abstract 46-1-115.)

43. Rufus Oldenburger: *Symbolic elements in dynamics.* (Abstract 46-1-111.)

44. A. H. Copeland: *Transformations on probability sequences.* (Abstract 46-1-57.)

45. J. L. Doob: *On a certain type of family of chance variables.* (Abstract 46-1-62.)

46. M. E. Wescott: *Sets of Newton polynomials analogous to Laguerre's polynomials.* (Abstract 46-1-132.)

47. Barkley Rosser: *On the computation of logarithms to a large number of decimal places.* (Abstract 46-1-117.)

48. Alexander Wundheiler: *Are complex numbers vectors?* (Abstract 46-1-135.)

49. H. W. Alexander: *On the edge of regression of a general surface.* (Abstract 46-1-45.)

50. J. L. Kelley: *On the hyperspaces of a given space.* Preliminary report. (Abstract 46-1-94.)

51. Samuel Eilenberg: *An invariance theorem for subsets of* S^n. (Abstract 46-1-64.)

52. R. L. Wilder: *Local connectedness and generalized manifolds.* (Abstract 46-1-134.)

53. A. N. Milgram: *Iterations of mappings of a set on its square.* (Abstract 46-1-105.)

54. A. W. Tucker: *The algebraic structure of complexes.* II. Preliminary report. (Abstract 46-1-128.)

55. D. W. Hall and G. T. Whyburn: *On arc-preserving and tree preserving transformations.* (Abstract 46-3-143.)

56. E. R. Lorch: *Spectral analysis of weakly almost periodic transformations in reflexive vector spaces.* (Abstract 45-11-401.)

57. M. S. Hendrickson: *On certain properties of arbitrary functions.* (Abstract 46-1-79.)

58. Dunham Jackson: *Note on certain orthogonal polynomials.* (Abstract 46-1-84.)

59. M. F. Smiley: *A note on the Jacobi condition for extremaloids.* (Abstract 46-1-124.)

60. W. A. Patterson: *Inverse problems of the calculus of variations for multiple integrals.* (Abstract 46-1-112.)

61. J. D. Mancill: *On the Carathéodory condition for unilateral variations.* (Abstract 46-1-102.)

62. M. R. Hestenes: *A theorem on quadratic forms and its application in the calculus of variations.* (Abstract 46-1-83.)

63. H. L. Garabedian: *Theorems associated with the Riesz and the Dirichlet's series methods of summation.* (Abstract 45-11-417-*t*.)

64. A. D. Hestenes: *A solution of the nonhomogeneous ϕ-difference equation of the first order.* Preliminary report. (Abstract 46-1-81-*t*.)

65. R. G. Lubben: *Separabilities of higher orders and related properties.* (Abstract 46-1-99-*t*.)

66. F. B. Jones: *Almost cyclic elements and simple links of a continuous curve.* (Abstract 46-1-88-*t*.)

67. H. A. Arnold: *Postulates for the defective group.* (Abstract 45-11-407-*t*.)

68. Mark Kac: *On a problem concerning probability and its connection with the theory of diffusion.* (Abstract 46-1-19-*t*.)

69. Garrett Birkhoff: *An ergodic theorem for arbitrary semi-groups.* (Abstract 46-1-9-*t*.)

70. W. T. Martin: *Analytic functions and multiple Fourier integrals.* (Abstract 46-1-23-*t*.)

71. F. J. Murray: *Nullifying functions.* (Abstract 46-1-26-*t*.)

72. Salomon Bochner: *Finitely additive integral.* (Abstract 46-1-10-*t*.)

73. R. P. Boas: *A completeness theorem.* (Abstract 46-1-47-*t*.)

74. R. S. Phillips: *On linear transformations.* (Abstract 46-1-113-*t*.)

75. O. G. Harrold (National Research Fellow): *The dimensionality of continuous transforms of an n-cell.* (Abstract 46-1-77-*t*.)

76. O. G. Harrold (National Research Fellow): *Continua of finite degree and certain product sets.* (Abstract 46-1-75-*t*.)

77. H. F. MacNeish: *A set of postulates for a finite geometry represented by the Pappus configuration.* (Abstract 46-1-100-*t*.)

78. H. F. MacNeish: *A sufficient condition for integrability.* (Abstract 46-1-101-*t*.)

79. H. L. Garabedian: *A new formula for the Bernoulli numbers.* (Abstract 46-1-69-*t*.)

80. G. N. Garrison: *Quasi-groups.* (Abstract 46-1-70-*t*.)

81. Richard Brauer and H. S. M. Coxeter: *A generalization of theorems of Schönhardt and Mehmke on polytopes.* (Abstract 46-1-51-*t*.)

82. R. L. Moore: *Concerning accessibility.* (Abstract 46-1-108-*t*.)

83. R. L. Moore: *Concerning domains whose boundaries are compact.* (Abstract 46-1-109-*t*.)

84. F. B. Jones: *Certain consequences of the Jordan curve theorem.* (Abstract 46-1-89-*t*.)

85. W. T. Scott: *Interpolation by continued fractions.* (Abstract 46-1-121-*t*.)

86. F. A. Lewis: *Generators of permutation groups simply isomorphic with $LF(2, p)$.* (Abstract 46-1-97-*t*.)

87. I. M. Niven: *Integers of quadratic fields as sums of squares.* (Abstract 46-1-110-*t*.)

88. E. R. van Kampen: *On uniformly almost periodic multiplicative and additive functions.* (Abstract 46-1-91-*t*.)

89. J. D. Mancill: *On the Jacobi condition for unilateral variations.* (Abstract 46-1-103-*t*.)

90. Samuel Eilenberg: *Fixed points for periodic transformations.* (Abstract 46-1-65-*t*.)

91. Samuel Eilenberg: *On the homotopy type of S^n.* (Abstract 46-1-66-*t*.)

92. Benjamin Epstein: *Growth properties of analytic functions of two complex variables.* II. Preliminary report. (Abstract 46-1-68-*t*.)

93. Harriet M. Griffin: *The abelian quasi-group.* (Abstract 46-1-74-*t*.)

94. Leonard Carlitz: *A set of polynomials.* (Abstract 46-1-53-*t*.)

95. Leonard Carlitz: *On certain sums involving polynomials in a Galois field.* (Abstract 46-1-54-*t*.)

96. L. I. Wade: *Certain quantities transcendental over the field $\Phi(x)$, where $\Phi = GF(p^n)$.* (Abstract 46-1-129-*t*.)

97. P. W. Gilbert: *Two-to-one transformations on linear graphs.* Preliminary report. (Abstract 46-1-72-*t*.)

98. A. D. Hestenes: *On the solutions of a certain functional equation* $\phi_c(x) - c \log \phi_c(x) = x - c \log x + 1$. Preliminary report. (Abstract 46-1-80-*t*.)

99. A. D. Hestenes: *On certain matrix* ϕ*-difference equations*. Preliminary report. (Abstract 46-1-82-*t*.)

100. Deane Montgomery and Leo Zippin: *A Hilbert axiom for topological transformation groups of space*. (Abstract 46-1-107-*t*.)

101. W. C. Graustein: *Harmonic minimal surfaces*. (Abstract 46-3-142-*t*.)

102. E. D. Hellinger: *Classes of monotone functions*. Preliminary report. (Abstract 46-3-145-*t*.)

W. L. AYRES,
Associate Secretary

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BOOK REVIEWS

Elementary Number Theory. By J. V. Uspensky and M. A. Heaslet. New York and London, McGraw-Hill, 1939. 10+484 pp.

Numerous historical references and applications of the theory presented, as well as detailed proofs, make this book especially suited to novices in the field. As noted in the introduction "owing to self-imposed limitations in the size of the book, many topics of interest had to be omitted." Analytical and geometrical methods have been avoided, while topics such as continued fractions and integral transformations of forms have been omitted entirely. On the other hand the book covers with thoroughness various fundamental problems which have inspired much of the research in this field. The results are illustrated by numerous examples, some solved in the text, others left to the reader.

The first five chapters are devoted to various problems which can be readily treated without the use of congruences. In fact, the notion of congruence is deferred until the sixth chapter when the reader has become familiar with a number of the fundamental concepts of number theory. The first chapter contains a discussion of elementary properties of integers, a method of computing polygonal numbers, and a discussion of scales of notation. It also contains an analysis, in terms of binary representations of numbers, of the generalized Chinese game of Nim. In the second chapter there is a treatment of common divisors and multiples, and a solution of the Diophantine equation $x^2+y^2=z^2$ in integers. The third chapter contains Lame's theorem, Euclid's least remainder algorithm, and application of this algorithm to the solution of linear Diophantine equations. The fourth chapter is devoted to a discussion of prime numbers. The topics considered are the sieve of Eratosthenes, the unique factorization theorem, the number and sum of divisors of an integer, perfect numbers, Mersennes's numbers, and the distribution of primes. In the fifth chapter there is a discussion of relative primeness, and in particular of Euler's function $\phi(n)$ and Moebius's function $\mu(n)$, with results based on a well known combinatorial theorem. This theorem is applied to the problem of determining the number of primes less than a given integer, and the chapter closes with Meissel's formula for this number.

After the introduction of the concept of "congruence," Chapter 6 is devoted to Fermat's theorem concerning the congruence $a^{p-1}\equiv 1 \pmod{p}$ and the Euler generalization on $a^{\phi(m)}\equiv 1 \pmod{m}$. Several proofs are given for Fermat's theorem, followed by Gauss's proof of Wilson's theorem. Chapter 6 is followed by an appendix on the problem of constructing magic squares with the solution based on the notion of congruence. In Chapter 7 the authors discuss the number of roots of polynomial congruences of arbitrary degree with application to the theory of residues of lth powers of a prime p. First degree Diophantine equations treated in the third chapter are solved here by a little known method due to Voronoi. The chapter closes with the Chinese method for solving sets of linear congruences in one unknown. The theory of linear congruences is applied in an appendix on calendars. Chapter 8 is devoted to residues of powers with special emphasis on primitive roots and indices, and there is an application to the solution of congruences. The concept of "exponent to which a number a belongs modulo m" is applied in an appendix on card shuffling. In Chapter 10 the authors take up various aspects of the theory of quadratic residues. The chapter is concerned primarily with two fundamental problems: Is a number a a quadratic residue or non-residue of a given prime p, and for what prime moduli is a given number a a quadratic residue or non-residue? The first problem is solved by means of a criterion of Euler. For $a=-1$, 2 and -2 the second is solved by elementary direct methods. Legendre and Jacobi's symbols are introduced, and the corresponding quadratic reciprocity laws are proved. The chapter ends with the solution of general quadratic congruences.

In Chapter 11 the authors develop some special aspects of the theory of quadratic forms. The chapter begins with a discussion of Fermat's equation $t^2-au^2=1$, wrongly attributed by Euler to Pell, and the generalization $x^2-ay^2=m$ of this equation. The results are applied to determine conditions that a given number m be prime. The chapter contains Kummer's proof of the reciprocity law based on Fermat's equation, and Dickson's proof that each integer is a sum of four squares. In Chapter 12 results of earlier chapters are applied to special Diophantine equations, such as $x^2+ay^2=z^m$, $x^2+c=y^3$. This is followed by a proof that the equation $x^4+y^4=z^2$ has no solution in integers none of which is zero, a proof accomplished by the standard technique of showing that if $x^4+y^4=z^2$ has a solution (x_1, y_1, z_1), this equation has a solution (x_2, y_2, z_2) with $z_2<z_1$. The final chapter (Chapter 13) is devoted to the general arithmetical identities proved by Liouville by elementary methods. These identities are used to prove that the primes of the form $8k+1$ or $8k+3$ are sums of a square and the double of a square, and are also employed to give an elementary proof of Jacobi's result on the number of repre-

sentations of an integer as a sum of four squares. Finally, a proof, based on these identities, is given for the theorem of Gauss that all integers except those of the form $4^k(8N+7)$, $k \geqq 0$, are representable as a sum of three squares. The idea of this proof goes back to Kronecker.

An exception to the other chapters of the book which form a closely knit and well-integrated unit is Chapter 9 on Bernoulli numbers. This chapter is independent of the rest of the book except for a minor reference to it later. The chapter treats some fundamental properties of Bernoulli numbers. In particular, the authors give Staudt's theorem on the fractional parts of Bernoulli numbers, and use a theorem of Voronoi to yield information on the types of factors which occur in the numerators and denominators of Bernoulli numbers. The chapter ends with Kummer's congruence on Bernoulli's numbers. There are no applications.

Although the style is clear and detailed, more clarity would have been obtained if there had been uniformity in the statement of the theorems and corollaries. Sometimes the theorems and corollaries are incorporated in the text, with a part of the proof, while in other places the theorems and corollaries stand out in paragraphs by themselves. In the treatment of roots of polynomial congruences one misses the important notion of a "field of numbers," since some of the results, such as that a congruence of degree n with prime modulus has no more than n incongruent roots, are merely theorems valid for any field of numbers and in particular for modular fields. The use of the concept of field, so lacking in books on number theory, would emphasize the importance of many of the problems considered. Some number theorists might take exception to the statement, pages 19–20, "The theory of numbers, unlike some other branches of mathematics, is a purely theoretical science without practical applications." Although certainly deficient in this sense compared with other fields of mathematics, such problems as the calendar problems considered in this text seem to fall in the class of practical applications. Historians might take exception to the statement, page 20, "Pierre de Fermat (1601–1665) was the first man to discover really deep properties of numbers."

We mention a few errors that came to our attention. In line 2 from the bottom of page 18 the following phrase "$r_3+s_3=4 \leqq k$, so that we adjoin the number 3, and change the units in column l_3 to 0's" should read "$r_3+s_3=4>k$, whence we change one zero in column l_3 to 1." On page 19 the columns (which we write as rows) *, 0, $1 \rightarrow 0$, $1 \rightarrow 0$, $1 \rightarrow 0$, 0, 0 and $0110<1110$, $0110<1101$, $0100<1001$, $10<11$, 100, 10, /440, should read *, $0\rightarrow 1$, 1, 1, 1, 0, 0 and $0111<1110$, $0111<1101$, $0101<1001$, 11, 100, 10, /444, respectively. The expression "$n\geqq 4$," line 7, page 89, should read "$10>n\geqq 4$." In line 15, page 100, the symbol $[n/]$ should be $[n/p]$. On page 194, line 2, the phrase "modulo p^{a-1}" should follow "is congruent." On page 299 the symbol $(p_1\,q)$ in the 9th line from the bottom should read (p_1/q), and $[(p_k-1)/2]$ in line 3 from the bottom should read $[(p_i-1)/2]$.

On the whole the authors have succeeded in presenting number theory in a fascinating and highly instructive manner. Their work will fill an important place in the literature.

RUFUS OLDENBURGER

Differential Geometrien in den Kugelräumen. Vol. 1. *Konforme Differentialgeometrie von Liouville und Möbius.* By T. Takasu. Tokyo, Maruzen, 1938. 18+457 pp.

This handsome textbook gives a comprehensive account of the differential geometries in spaces in which spheres are taken as spatial element. It is the outgrowth of a series of fundamental investigations in which the author has been engaged since 1924, and which have, so far, been laid down in 47 papers. He has also used, in the widest and most generous sense of the word, the work of other authors, which he lists in an extensive and probably complete bibliography of 202 titles. This book is, therefore, the most authoritative source on the subject, and the reader will at the same time be pleased with the variety and beauty of the results and amazed at the thoroughness of the work.

The main body of research already has been published in various "Science Reports of the Tôhoku Imperial University," beginning in March, 1928. The present volume often follows the previous publications, but also departs from it, giving more material and more references. The author, in the preface, gives his own classification of the different geometries as derived from Lie's geometry of the spheres, embracing several types of conformal geometry, including that of Laguerre, of affine, and of non-euclidean geometry. We repeat his fundamental principle as follows:

"1. The conformal space of Möbius is a non-euclidean space with movable absolute spheres, hence also with variable curvature.

"2. The space of Laguerre is a euclidean space with movable absolute circles, hence also with variable unit of angular measurement.

"3. The space of Lie is a conformal space with a movable absolute complex of spheres and at the same time a dual-conformal space with movable dual-absolute complex of spheres, so that the unity of measure of length and angle are functions of position."

These three principles, according to the author, allow a perfectly clear interpretation of the analytical construction of these geometries, and one which has no gaps. He believes that this elucidates completely a field which has had too little attention.

The geometry of spheres has, indeed, been little investigated and mostly in an unsystematic way. Pentaspherical and hexaspherical coordinates, which are the appropriate types of coordinates for such geometries, were originally only used by Darboux (1889) and Coolidge (1916), until after 1922 when a new series of investigations began, with which the names of Vessiot, Thomsen, Blaschke, Liebmann and Kubota are connected, of whom some also use cartesian coordinates. There is a close relation in method between this book of Professor Takasu and the third volume of Blaschke's *Differentialgeometrie*, written by Thomsen, and which appeared in 1928. There is also a considerable difference between the books, Thomsen's book being only partly devoted to differential geometry, Takasu investigating in a very systematic way only differential geometry.

This book of 457 pages is only the first volume of a planned set of three. It deals with the geometries of Liouville and Möbius. The second volume will give the differential geometry of Laguerre, the third volume that of Lie.

The book consists of two "Abschnitte," a short first one of twenty-five pages giving introductory theories, explaining the relation of the three geometries of the sphere, the second one bringing the theory of systems of ∞^1 circles, of nets of curves, of systems of ∞^1 and ∞^2 spheres, of systems of ∞^1 circles in space, and of triply orthogonal systems. It follows closely the classical exposition of ordinary differential geometry. The systems of ∞^1 circles correspond to the curves in the plane (taken as envelopes of straight lines), the systems of ∞^1 spheres to curves in space, the systems of ∞^2 spheres to surfaces, the systems of ∞^1 circles in space to ruled surfaces, the correspondence is with non-euclidean geometry rather than with euclidean because of the nondegenerated quadratic relations between the coordinates.

It is impossible to mention even a small part of the rich material. We have generalizations to conformal space of Bertrand curves, of the four vertex theorem, of Meusnier's and Euler's theorems. Many classical results receive for the first time their proper setting. Dualities are often shown by comparison of theorems and proofs on the right and left side of the page. The systematic exposition can often reframe an old proof with the new and unified methods in the new conformal language and notation. There is a certain finality about the content which makes it the foundation of every further study on the subject.

It is not always easy to follow the author and we would like to ask him to be kind with his readers when he gives the final touches to the other two volumes. As an example, let us take the beginning of the book. On page 1 we hurry immediately into the midst of things. An oriented sphere is defined by 5 pentaspherical coordinates connected by a quadratic relation. The meaning of these coordinates is not explained, we must take this from other books, and the quadratic relation is only given as $(yy)_5=1$, which leaves it to the reader to discover what it means. We are not informed either, whether the coordinates are ordinary or general pentaspherical coordinates, and have to discover this later, from the context. On page 2 we read that the different projective orientation processes must be understood in the sense of the author's non-euclidean geometry. This is an essential point, and we must therefore first go to volume 26 of the Tôhoku Mathematical Journal (1926) to find what it means. The same thing happens on page 3, where we are referred to another paper to find the meaning of certain equations expressing a doubly oriented sphere. Such difficulties could easily be avoided if the author, at the beginning, would not presuppose more than an average college knowledge of pentaspherical coordinates, projective and non-euclidean geometry, as, for example, Blaschke, Thomsen, or Fubini-Cech have done in their related expositions. We are sure that Professor Takasu will only do justice to his beautiful investigations if he can agree to such modifications in his presentation.

D. J. STRUIK

Superficie Razionali. By Fabio Conforto. Bologna, Zanichelli, 1939. 16+549 pp.

Although there is an extensive literature on rational surfaces, it is scattered through the periodicals in various languages, and the methods of proof differ widely as the theory gradually develops through more than a century. On the other hand, a knowledge of this field is indispensable to the study of algebraic geometry of more than two dimensions, and to some phases of analysis.

The purpose of the present book is to supply this need of a systematic development of the subject from the present point of view, starting at the beginning and providing all the necessary details of the general theory, but referring to original papers for further perusal of particular features. A knowledge of plane geometry, including plane Cremona transformations and algebraic plane curves is presupposed. Throughout the book the guiding principle is that of continuity of the parameters in a linear system, the methods of proof following closely those of the four volume treatise of Enriques and Chisisni[1] to which frequent reference is made.

The volume is divided into two parts, the first being concerned with rational surfaces of order less than 5, developed largely along traditional lines, and the other to the systematic theory. But this first part includes one feature that deserves special mention. After discussing those quartic surfaces that have a double curve, the book provides an exhaustive analysis of all the quartic surfaces that are rational and have just one isolated singular point. Here, in particular, the principle of continuity of the parameter in a linear system is employed in such a way that a limiting value of the parameter defines a surface with a singularity of the desired form. It is clearly shown that the threee types of Noether quartics correspond to the three types of plane involutions of order two, the images, on the surface, of a pair of conjugates on the plane, being a pair of residual points on a line through the double point. Other features in this first part are the exhaustive discussion of all possible cases, including the Steiner surface and Cayley's cubic scroll.

The second part begins with surfaces having a pencil of rational curves, then those having only rational plane sections, followed by elliptic, hyperelliptic ($p=2$) and those of genus 3, not hyperelliptic. In each case, the characteristic properties of the representative net of plane curves are established. Apparent digressions are the classifications of plane involutions of order two and the conditions for rationality of a double plane, but both results are shown to be necessary to insure that the classification of rational surfaces is complete.

A long chapter is devoted to other rational surfaces; it includes the quintics having a multiple curve, but does not claim to be exhaustive. The last chapter contains a complete treatment of the theorem of Castelnuovo that all plane involutions, of any order n, are rational. In the main, the proof in the original paper is reproduced, but important simplifications are achieved by systematic use of continuity in linear systems.

Frequent and extensive references to the literature are made, and a convenient index is provided, but much of the recent literature is not included, especially that of non-European origin; moreover, many of the earlier theorems are not referred to their sources, but to later proofs of them given in works supposed to be more accessible.

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[1] *Lezioni sulla Teoria Geometrica delle Equazioni e delle Funzioni Algebriche*, Bologna, Zanichelli, 1915, 1918, 1923, 1934.

The writing and proofreading have been done with great care; apart from a few trivial errors in orthography, the only error is the reference to G. Kantor, footnote on page 131, instead of S. Kantor.

This book will be welcomed by workers in algebraic geometry; it competently fills a gap in the preparatory literature.

VIRGIL SNYDER

Modern Science, A Study of Physical Science in the World Today. By Hyman Levy. New York, Knopf, 1939. 736 pp. 160 illustrations.

Professor Levy's ability to present general relationships and abstractions in an interesting simple way has been commended by reviewers of his previous recent book "A Philosophy for a Modern Man." This faculty is again evident in his analysis and evaluation of social and intellectual forces in the development of physical science.

Part I of the book deals with the background of social life within which science has developed as one among many "channels" of human energy. In an interesting discussion Professor Levy contends that the work of men like Newton was largely a consequence of commercial and other social factors rather than a spontaneous intellectual activity.

Part II is concerned with the nature, methods and unity of science. Considerable stress is laid on the common occurrence of sequences of phases separated by discontinuities in physical processes. Many usually overlooked examples of these phases are given.

Parts III, IV and V deal with mathematical symbols and physical (theoretical) models. Some general algebraic and geometrical concepts are discussed. The idea of limits is presented in a remarkably concrete way. The rigor is surprisingly good for a popular discussion, but there are some undesirable implications, for example, termwise differentiability of infinite series is accepted without question. An instructive treatment of mass, momentum, impulse and energy is given. Aeronautical science is used to illustrate the unity of theory and experiment. The development of non-euclidean geometries is used to illustrate how science can "shake off the past."

Part VI is rather discursive in its discussion of astronomy, geology, atomic theory and relativity. The treatment of mountain building does not include the more modern theories of Joly and others.

Part VII on the "Age of Light" uses the history of methods of illumination and their social consequences to illustrate the dependence of various branches of science and human activity on one another.

While the reviewer can recommend this book as interesting, clear and stimulating for the moderately well informed reader, he feels that it is unnecessarily long due to redundance of examples, and that its lack of references is an unfortunate weakness.

J. K. L. MacDonald

Punktreihengeometrie. By E. A. Weiss. Leipzig and Berlin, Teubner, 1939. 8+232 pp.

The purpose of this book is to serve as an introduction to the projective geometry of higher dimensional spaces. The author adopts the point of view of Reye in regarding such space as the map space of various three-dimensional configurations and he thereby succeeds in bringing together a large variety of topics.

After a brief recapitulation of the elementary geometry of the line, the author introduces the concept of a (linear) point range (punktreihe) on a line obtained by equating to zero a bilinear form in variables (ξ_1, ξ_2) and (τ_1, τ_2), which he writes using the Clebsch-Arnold symbolic notation as $(\gamma\xi)(\mu\tau)=0$. He interprets (τ_1, τ_2) as a parametrization of the points (ξ_1, ξ_2) of the line, and he subsequently defines a point range more generally as a "one-dimensional rational manifold provided with a definite parametrization." By means of the coefficients $\gamma_i\mu_j$ linear point ranges on the line can be put in 1-1 correspondence with the points of projective 3-space; singular ranges map into a ruled quadric, pencils and bundles of ranges into lines and planes. From the properties of singular ranges the author deduces the elementary properties of the quadric.

The two main chapters of the book deal with point ranges (and their duals) in two and three dimensions. By using symbolic notation, the Clebsch correspondence principle, and similar devices, the author derives easily such fundamental results as the harmonic properties of a quadrilateral, the projective generation of a conic, and the polar theory in the plane. Further properties of a conic follow by considering it as a point range of the second order defined analytically by $(um)(\gamma\tau)^2=0$. Pascal's configurations are obtained by mapping binary quadratic forms on the points of a plane.

Linear point ranges in the plane can be mapped on R_5, projective space of five dimensions, and singular ranges correspond to a Segre manifold of three dimensions and of order 3. The singular ranges of a pencil or bundle will map into rational cubic curves and surfaces.

The third chapter deals with geometry in R_3 and the development parallels that of the second. Now the map space of linear point ranges is an R_7, the Segre manifold is quartic and four-dimensional, and properties of quartic curves and surfaces follow by considering the singular ranges in pencils and bundles. Here, as in the preceding chapters, the author studies the collineations of the map space which leave fixed the corresponding Segre manifold. By noting that quadric hypersurfaces in R_7 are the maps of linear complexes, the author leads naturally to the principle of triality of Study and Cartan, null systems, the line sphere transformation, and properties of oriented spheres.

The final chapter deals with trilinear forms and some applications to the preceding situations as well as to non-euclidean geometry.

The author has succeeded in bringing together a large variety of geometrical configurations and studying them in a way which will stimulate the student's interest and arouse his admiration for geometrical methods.

HARRY LEVY

Mathematical Recreations and Essays. 11th edition. By W. W. Rouse Ball. Revised by H. S. M. Coxeter. New York, Macmillan, 1939. 16+418 pp.

For almost half a century the earlier editions of this book[1] have provided a rich supply of mathematical topics commonly known as recreations, which although they often involve fundamental mathematical methods and notions, yet make their appeal in the spirit of a game or a puzzle, rather than with an eye to the usefulness of their conclusions. Since no knowledge of the calculus or of analytic geometry is presupposed, many of these recreations make excellent subjects for student talks in undergraduate mathematics clubs.

The eleventh edition, revised by H. M. S. Coxeter after Ball's death, not only "aims to preserve the spirit of Ball's delightful book," but does. It is, nonetheless, a thorough-going revision of the tenth edition. The chapters on Mechanical Recreations, Bees and their Cells, and String Figures have been omitted. In their stead we find the following material.

(1) A large new section of arithmetical recreations (chap. II), in-

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[1] Editions of this book have also appeared in French and Italian. A three-volume edition in French with considerable additional material, especially in the history of numbers, was edited by J. Fitz-Patrick in Paris (1907–1909), and a new edition appeared in 1926–1927 with new material by A. Margossian, Reinhart, J. Fitz-Patrick and A. Aubry. Two Italian editions were printed in Bologna, the first in 1911 by D. Gambioli, 398 pp., and the second in 1927 by D. Gabioli and G. Loria.

cluding discussions of derangements of n quantities, repeating decimals, rational right-angled triangles, finite arithmetics, and the distribution of primes; (2) a new chapter (chap. V) on polyhedra—a subject in which the author is expert—written in a manner which is intelligible to a beginner equipped with a modicum of ability in space perception, and supplemented by two excellent plates showing models of the regular and semiregular solids; (3) a more extended treatment of magic squares and their generalizations; (4) a discussion, in the chapter on map colouring problems, not only of the famous four-colour problem in the plane, but of the seven-colour problem on the torus, and of other topics connected with the stereographic projections of the regular polyhedra.

The first edition of 1892 was entitled "Mathematical Recreations and Problems" and consisted of two parts, the one on Recreations and the other on Problems and Speculations. Of the five chapters in the latter part only the one on Three Classical Problems has been retained in all editions. The chapters on Astrology, Hyper-space, Time and its Measurement, and the Constitution of Matter, became the basis for a much enlarged second half of the fifth edition (1911) which bore the title of Mathematical Essays. The space devoted to essays was considerably reduced in the tenth edition (1922). It now includes the Three Classical Geometrical Problems (chap. XII), an essay on calculating prodigies (chap. XIII), and an essay on cryptography (chap. XIV), the latter being completely revised for the present edition by Abraham Sinkov, a cryptanalyst in the U. S. War Department.

A partial summary of the contents of the book is as follows. *Chapter I* includes arithmetical recreations whose interest is mainly historical rather than arithmetical. Some of these are of the "think of a number" type, others involve digit notations, and still others are tricks with cards or games with counters. *Chapter II* opens with a series of arithmetical fallacies, continues with problems of probability derangements and arrangements, decimal expansions, rational triangles, finite arithmetics, D. H. Lehmer's number sieve for prime factors, and concludes with a discussion of perfect numbers, Mersenne's numbers, and Fermat's theorem. *Chapter III* is composed mainly of geometrical fallacies and paradoxes, problems in dissection, cyclotomy and area-covering. The deltoid solution to Kakeya's minimal problem, erroneously attributed to Kakeya (p. 100) was really suggested by Professors Osgood and Kabota according to Question 39, American Mathematical Monthly (1921), p. 125. *Chapter IV* is concerned with statical and dynamical games of position.

Among topics discussed are some extensions of the game of three in a row, tessellations of the plane, problems with moving counters, and the effect of cutting a Möbius strip in various ways. *Chapter V* gives a comprehensive elementary discussion of the relations between the faces, edges, and vertices and the associated angles of the regular solids and the Archimedean solids, which is well illustrated by good figures. Stellated polyhedra, solid tessellations, and the kaleidoscope each receive some attention. The use of the term Platonic for the regular solids might be questioned since they were known before Plato. *Chapters VI* and *VII* contain familiar recreations associated with the chessboard and with magic squares. Similar problems with dominoes and with magic cubes are also discussed. *Chapter VIII* treats the general theory of the four-colour problem more elaborately than the earlier editions of this book, mentions briefly such matters as orientable surfaces and dual maps, and more fully the seven-colour mapping problem on the torus, and finally considers various colouring problems on the regular polyhedra. *Chapter IX* discusses mazes and other similar problems whose solutions depend on the unicursal tracing of a route through prescribed points (nodes) over various given paths. *Chapter X* features certain combinatorial problems known under the title of Kirkman's school-girl problems, and ends with a similar problem about arranging members of a bridge club at tables so that different members shall play together in successive rubbers. *Chapter XI*, on Miscellaneous Problems, contains an account of the Fifteen Puzzle, the Tower of Hanoï, Chinese Rings, and various mathematical card tricks. *Chapter XII* contains the famous classical problems concerning the duplication of the cube, trisection of an angle, and quadrature of the circle. *Chapter XIII* is an essay on calculating prodigies which introduces over a dozen famous mental calculators beginning with Jedediah Buxton and Thomas Fuller in the eighteenth century, and including two American calculators Zerah Colburn and Trueman Henry Safford, and gives something of their histories and the type of problems they could solve. *Chapter XIV* is a chapter on cryptography and cryptanalysis written by Dr. Abraham Sinkov. It presents in easily understandable form the chief elements in a cryptographic system, and gives various possible ways for attempting to solve such a cipher.

J. S. Frame

Einführung in die Zahlentheorie. By Arnold Scholz. (Sammlung Göschen, no. 1131.) Berlin, de Gruyter, 1939. 136 pp.

The topics which should be taken up in an introduction to the theory of numbers do not seem to have been well standardized. Each author discusses a certain minimum set of topics and then proceeds to whatever subject interests him. In this book the choice of further topics is well taken and quite varied.

After an introduction to the series of natural numbers the author takes up the subjects common to most such books. Included here is a descriptive section on the distribution of prime numbers in which a number of interesting numerical examples are given. The later part of the book takes up the representation of integers by binary quadratic forms in considerable detail and it includes a chapter on the numerical solution of various problems that arise in number theory.

The principle followed in the preparation of this book seems to have been conciseness. This is apparent, not only in the typography, but even in the author's manner of writing. The earlier chapters have been written with care but this brief manner of writing makes the later chapters more difficult to read.

Because of its conciseness this book seems more suitable as supplementary reading rather than as a first introduction to the theory of numbers.

H. S. ZUCKERMAN

Anwendung der Eulerschen Reihentransformation zur Summierung der Dirichletschen Reihen, der Fakultätenreihen und der Newtonschen Reihe. By N. Obreschkoff. Berlin, Verlag der Akademie der Wissenschaften, 1938. 36 pp.

This pamphlet is a reprint of an article published in the Sitzungsberichte der Preussischen Akademie der Wissenschaften, Physikalish-mathematische Klasse, 1938. The author applies the Euler-Knopp method of summability to Dirichlet series, to factorial series and to Newton's series, obtaining results which show that this summability method is very useful in such cases and that it is ordinarily more powerful than the methods of summation of Cesàro and of Riesz.

The ordinary Dirichlet series $\sum a_n/(n+1)^s$ is first investigated. The author shows that if this series is E_k-summable for $s=s_0$, then it is E_k-summable for every s for which $R(s)>R(s_0)$, and he gives a formula for the generalized sum there. Use is made, in the proof, of the fundamental Silverman-Toeplitz conditions for summability. A formula is derived for the abscissa of E_k-summability in terms of the E_k-transform of the series $\sum a_n$. Corresponding results are obtained for absolute E_k-summability.

Similarly, it is shown that if the factorial series $a_0/s+\sum n!a_n/s(s$

$+1)(s+2)\cdots(s+n)$ is E_k-summable for $s=s_0$, then it is E_k-summable for every s for which $R(s)>R(s_0)$, and a formula is given for the sum. Also, a formula is derived for the abscissa of E_k-summability. Extensions are made to absolute E_k-summability. Similar results are obtained for the Newton series $a_0+\sum(-1)^n a_n(s-1)(s-2)\cdots(s-n)/n!$.

LLOYD L. SMAIL

NOTES

A Conference in Topology will be held at the University of Michigan June 24–July 6, 1940.

Mr. L. L. Locke has placed his collection of old calculating machines in the Smithsonian Institution. This collection contains many unique and interesting machines, among them the first two machines made in the United States and the first direct multiplication machine.

A mathematics teaching seminar on a post-doctoral level is being conducted this year at Reed College, with Professor F. L. Griffin in charge. The four fellows who are participating are: Mr. H. E. Goheen, Dr. L. Louise Johnson, Mr. R. A. Rosenbaum, and Dr. Henry Scheffé.

Professor P. A. M. Dirac of the University of Cambridge has been awarded a medal by the Royal Society, London, for his work in the new quantum mechanics.

The University of London has conferred the degree of doctor of science on Dr. Niels Bohr, director of the Institute of Theoretical Physics at the University of Copenhagen.

Professor V. V. Narlikar of Benares Hindu University has been elected a fellow of the National Institute of Sciences of India.

The Institute of Mathematical Statistics has elected Associate Professor S. S. Wilks of Princeton University as president, Associate Professor A. T. Craig of the University of Iowa and Associate Professor C. C. Craig of the University of Michigan as vice presidents, and Professor P. R. Rider of Washington University as secretary-treasurer.

Professor A. B. Coble of the University of Illinois has been elected vice president of the American Association for the Advancement of Science. Professor J. L. Walsh of Harvard University has been elected a member of the Sectional Committee for Mathematics.

Professor L. E. Dickson of the University of Chicago has been named Eliakim Hastings Moore Distinguished Service Professor Emeritus of Mathematics.

Professor William Gillespie of Princeton University has been given the title of professor emeritus.

Professor Gabriele Mammana of the University of Naples and Professor Luigi Sobrero of the University of Rome have been appointed to professorships at the University of Rio de Janeiro.

Assistant Professor R. V. Blair of Vanderbilt University has been promoted to an associate professorship.

Associate Professor E. R. Breslich of the University of Chicago has retired.

Assistant Professor R. C. Bullock of North Carolina State College has been promoted to an associate professorship.

Dr. A. S. Gale of the University of Rochester has resigned as dean of the College for Men. He will continue as chairman of the department of mathematics.

Associate Professor A. E. Gault of Bradley Polytechnic Institute has been promoted to a professorship.

Mr. Coleman Herpel of Pennsylvania State College at Hazleton has been promoted to an assistant professorship.

Dr. W. A. Patterson of Fenn College has been promoted to an assistant professorship.

Professor G. Y. Rainich of the University of Michigan will be on leave for the second semester.

Professor T. M. Simpson of the University of Florida has been appointed acting dean of the graduate school.

Reverend J. P. Smith of St. Peters College, Jersey City, has been appointed to an associate professorship at Georgetown University.

The following appointments to instructorships are announced: University of California: Dr. J. L. Brenner; University of Detroit: Dr. Alvin Sugar; Georgia School of Technology: Dr. Nelson Robinson; University of Illinois: Dr. I. M. Niven; Johns Hopkins University: Dr. Herbert Busemann; University of Maine: Mr. A. W. Jones; University of Maryland: Dr. F. J. Weyl; Scranton-Keystone Junior College: Mr. B. J. Trapani; Wayne University: Mr. Morris Friedman, Mr. Joseph Levin.

Professor Basilio Manià of the University of Milan died September 26, 1939.

The death of Gaetano Bernardino Scorza, an editor of Annali di Matematica Pura ed Applicata, has been announced.

The death of Dr. H. T. Burgess has been reported. He had been a member of the Society since 1909.

Dr. S. C. Harry of Baltimore, Maryland, died September 19, 1939.

The death of Reverend Albert O'Brien has been reported. He had been a member of the Society since 1922.

Professor Emeritus F. N. Willson of Princeton University died on November 15, 1939, at the age of eighty-three years. He joined the Society in 1891.

The death of Professor Emeritus Ruth G. Wood of Smith College has been reported. She had been a member of the Society since 1899.

ABSTRACTS OF PAPERS

SUBMITTED FOR PRESENTATION TO THE SOCIETY

The following papers have been submitted to the Secretary and the Associate Secretaries of the Society for presentation at meetings of the Society. They are numbered serially throughout this volume. Cross references to them in the reports of the meetings will give the number of this volume, the number of this issue, and the serial number of the abstract.

138. G. E. Albert: *On contiguous point spaces and their applications.*

In topologizing the class P^* of cyclic elements of a Peano space P, R. L. Moore (Rice Institute Pamphlet, vol. 23, no. 1) was led to the concept of contiguity spaces. The presence of pairs of contiguous points seems to make the theory of Moore's spaces quite complicated. In a simpler axiomatic treatment of contiguity, Theodore Hailperin (this Bulletin, vol. 45 (1939), p. 172) modified Hausdorff's neighborhood space by denying the separation axiom and defining two points as contiguous if every neighborhood of each contains the other. The symmetry of this definition implies that the relation is transitive; this should render his theory inapplicable to the study of P^*. In the present paper a neighborhood space is introduced which differs from ordinary topological spaces in essentially one respect, namely, the Hausdorff separation axiom is replaced by the unsymmetrical Kolmogoroff axiom: of every two points, at least one has a neighborhood not containing the other. If every neighborhood of the point x contains the point $y\neq x$, then x is said to be contiguous to y. It is the purpose of the paper to construct, on this basis of contiguity, a simple and comprehensive theory of the class P^*. (Received December 11, 1939.)

139. Garrett Birkhoff: *On a class of positive matrices.*

The paper deals with matrices of non-negative elements, the sum of the terms in every row and column being the same. It is first shown combinatorially that every such matrix is a positive linear combination of permutation matrices. Applications are then made to the theory of dependent probabilities, where such matrices occur; in particular, an ergodic theorem over arbitrary semigroups is proved—valid also for an analogous class of linear (stochastic) operators on the space (L). (Received December 21, 1939.)

140. Nathaniel Coburn: *A characterization of Schouten's and Hayden's deformation methods.*

Another approach to deformation problems, in an n-dimensional space with a connection L_n, is given. The principal idea is that all deformations take place in the local tangent space E_n at each point of L_n. First, the laws of deformation of differential and ordinary vector fields are examined. These laws are distinct in non-Finsler space; they coincide in Finsler space. It is shown that: (1) Schouten's deformation formulas are obtained when the coordinate differential vectors and the unit affinor (A_μ^λ) of L_n are deformed by E_n parallelism; (2) Hayden's deformation formulas are obtained when the measure vectors of L_n are deformed by L_n parallelism, and the vectors of a subspace as well as the unit affinor of L_n are deformed by E_n parallelism. Finally, a particular type of deformation condition is examined. The following theorems are proved: (1) if this deformation is compatible with a motion in L_n, then the Christoffel symbols of L_n are not deformed; (2) if this deformation is possible, then L_n is generally a space of absolute parallelism; (3) subcases of (2) are discussed. (Received December 4, 1939.)

141. Nathaniel Coburn: *Conformal unitary spaces.*

In this paper the properties of unitary spaces of n dimensions K_n and $'K_n$ whose fundamental tensors are related by the conformal transformations are studied. First, the relation between conformal spaces and the geodesics of these spaces is discussed. The following theorems are proved: (1) two unitary spaces, both with symmetric connection (without torsion), cannot be conformal; (2) if two unitary spaces are conformal, then their geodesics correspond; (3) conversely, if two unitary spaces, one of which is without torsion, are in a *restricted* geodesic correspondence and if a certain curvature affinor ${}'C_{\alpha\bar{\mu}\nu}{}^{\cdot\cdot\cdot\beta}$ vanishes, then the two spaces are conformal. The paper concludes with a study of some special conformal unitary spaces. After defining the terms: (1) unitary k-spaces; (2) k-spaces which are k-conformal, the following theorems are proved: (1) the affinor ${}'C_{\alpha\bar{\mu}\nu}{}^{\cdot\cdot\cdot\beta}$ is a conformal invariant of all k-spaces which are k-conformal; (2) conversely, if two unitary spaces are conformal and the affinor ${}'C_{\alpha\bar{\mu}\nu}{}^{\cdot\cdot\cdot\beta}$ is a conformal invariant, then the spaces are k-spaces. A special case of these k-spaces has been studied by Bergmann, Mitrochin, Fuchs and others. (Received December 4, 1939.)

142. W. C. Graustein: *Harmonic minimal surfaces.*

A minimal surface in a euclidean space of three dimensions is harmonic if it is representable in terms of Cartesian coordinates (x_1, x_2, x_3) by an equation of the form $U(x_1, x_2, x_3) = \text{const.}$, where U is a harmonic function. It is the purpose of this paper to determine all the families of harmonic minimal surfaces. The families found are a pencil of planes, a family of right helicoids, a family of imaginary transcendental surfaces, a family of imaginary quartic surfaces, a family of imaginary sextic surfaces (for which U depends on an elliptic integral), and families of imaginary cylinders with isotropic rulings. The method employed introduces three mutually orthogonal congruences of curves, with unit tangent vector fields α, β, γ, which are closely associated with the required family of surfaces, and expresses the prescribed properties of these surfaces by a suitable choice of the coefficients in the equations of variation of α, β, γ, with respect to the arcs of the curves of the three congruences. These equations of variation constitute the differential system finally to be integrated. Their conditions of integrability yield a second differential system of ten partial differential equations of the first order in five dependent and three independent variables. (Received December 7, 1939.)

143. D. W. Hall and G. T. Whyburn: *On arc-preserving and tree preserving transformations.*

In an earlier abstract (45-11-399) the authors have obtained results concerning arc-preserving transformations. In the present paper the following additions to the theory are obtained: (a) Condition (i) (a) of the earlier abstract characterizes tree preserving transformations. (b) If $T(A) = B$ be continuous, then in order that T be tree preserving it is necessary and sufficient that the image of every simple arc in A be a tree in B. (c) If B is cyclic, then the following types of transformations are equivalent: arc-preserving, tree preserving, A-set reversing, monotone retracting. The set A is, throughout the paper, a compact locally connected continuum. (Received December 27, 1939.)

144. P. R. Halmos: *Statistics, set functions, and spectra.*

The purpose of this paper is to exploit the analogy existing between the distribution functions of statistics on the one hand and spectral families of projection operators on the other hand. It is shown that the class of all bounded measurable functions in the unit interval satisfies the axiomatic system developed by von Neumann (Matematicheskii Sbornik, vol. 43 (1936)) for the class of all bounded Hermitian operators on Hilbert space. Certain well known theorems, such as the equality of the expectation of a chance variable with the first moment of its distribution function, and the Radon-Nikodym theorem concerning absolutely continuous set functions are special cases of the general theory considered. These theorems are suggested by the application of Hilbert space theorems to real function theory. Working in the converse direction, certain concepts (particularly independence) of real function theory are extended to operators and the behavior of sets of independent operators is investigated, yielding results parallel to the usual results of probability theory. (Received January 3, 1940.)

145. E. D. Hellinger: *Classes of monotone functions.* Preliminary report.

The problem is to investigate the manner of increase of monotone functions over an interval or over a perfect set of points and to form classes of functions with common behavior in this respect. A function $g(x)$ is in the class defined by $h(x)$ if it can be represented by an integral of the form $\int_a^x (df(x))^2/dh(x)$ with a suitable $f(x)$; the integral is of the kind introduced by the author as a generalization of the Stieltjes integral (T. H. Hildebrandt, this Bulletin, vol. 24 (1918), p. 194). By these and other related integrals conditions are found under which two functions shall belong to the same class; further representatives of the different classes are given. It is shown that the theory is closely connected with the theory of absolute continuity. The theory can be applied, for instance, to the problem of the orthogonal equivalence of infinite quadratic forms or symmetric operators, bounded as well as non-bounded, which is just the problem which suggested these investigations. (Received December 27, 1939.)

146. Henry Hurwitz: *Total regularity of infinite matrix transformations.*

Total regularity of a matrix transformation was defined by W. A. Hurwitz who found a necessary and sufficient condition for it in the case of a real infinite triangular matrix (this Bulletin, vol. 28 (1922), p. 30). In the present paper a necessary and sufficient condition is deduced for a real regular infinite square matrix transformation given by $X = \lim \sum_{k=0}^{\infty} a_k(t) x_k$. There must eventually be only a finite number of negative elements $a_k(t)$ for each t, and there must be no sequence t_i such that a suitably defined set of negative elements appearing in the sequence is not properly "guarded" by large positive elements which also appear. Several modifications of the condition are possible. (Received December 15, 1939.)

147. A. N. Lowan and Gertrude Blanch: *Analysis of computing errors in the process of analytic continuation.*

When the values of a function $f(x)$ are to be tabulated over a fairly wide range of the argument x, it is frequently convenient to compute the function and its derivatives for certain equally spaced "key arguments" and to sub-tabulate by means of Taylor expansions around the key arguments. (It is assumed that all required derivatives are continuous over the entire range.) The scheme adopted by the "Project for the Computation of Mathematical Tables," W.P.A., New York City, involves computing in succession derivatives of all orders needed for every point in the range, beginning with those of the highest order, and ending with the function itself, keeping the interval constant. No estimate of the manner in which the error in the method is propagated from stage to stage seems to be available in the literature, and because of the importance of the method as a means of computation, an upper bound of the error is derived in this paper. (Received December 20, 1939.)

148. A. V. Martin and J. H. Roberts: *Two-to-one transformations of 2-manifolds.* Preliminary report.

If A is a compact 2-manifold, then a continuous two-to-one transformation can be defined on A if and only if the Euler characteristic $\lambda(A)$ is even. If B is the image space, then $\lambda(B)=\lambda(A)/2$. The possible image spaces include all the manifolds B (in number 0, 1, or 2) with $\lambda(B)=\lambda(A)/2$, as well as certain 2-complexes obtained by identifying pairs of points of a manifold. (Received December 29, 1939.)

149. Marston Morse and C. B. Tompkins: *Minimal surfaces of unstable type.*

The authors continue the theory initiated in the April issue of the Annals of Mathematics extending it to surfaces bounded by several contours. A typical result extending a classical result well known for surfaces of revolution bounded by two circles and generated by a catenary is as follows. Let g_1 and g_2 be two simple rectifiable closed curves whose convex envelopes do not intersect. Suppose that the ratio of chord to arc on g_i, $(i=1, 2)$, is bounded from zero. Term a surface S a ring-surface or disc-surface if S is the continuous image of a circular ring or a circular disc, respectively. Referring the term "minimizing" to the Douglas-Dirichlet integral, our theorem states the following. If g_1 and g_2 bound a ring mimimal surface of minimizing type (relative or absolute), then g_1 and g_2 also bound a ring minimal surface not of minimizing type, or else at least one of the two contours g_i bounds a disc minimal surface of unstable (non-minimizing) type. This result is easily verified for the classical case of surfaces of revolution, where the ring surface of unstable type always exists under the hypotheses. (Received December 1, 1939.)

150. Tibor Radó: *On a lemma of McShane.*

In his work on the semi-continuity of double integrals in the calculus of variations, McShane (Annals of Mathematics, (2), vol. 34 (1933), pp. 829–830, Lemma 7) established an important lemma which plays a fundamental part in his proofs. Using the notations of that paper, the main result of the present paper can be described by the following statement. The conclusion of the lemma of McShane remains valid if the functions x_0^i are subjected to the (obviously necessary) condition that the Jacobians X_0^i be summable. The proof is based on methods developed by the author in his work on the area of surfaces. Applications will be considered on another occasion. (Received December 13, 1939.)

151. A. L. Whiteman: *Additive prime number theory in real quadratic fields.*

In a series of three papers (Abhandlungen aus dem Mathematischen Seminar der

Hamburgischen Universität, vol. 3 (1924), pp. 109–163 and pp. 331–378; Mathematische Zeitschrift, vol. 27 (1926), pp. 321–426) Rademacher considered the problem of representing numbers in an algebraic field as the sum of primes. By applying the Hardy-Littlewood method he showed that if a certain hypothesis concerning the distribution of the zeros of Hecke's $\zeta(s, \lambda)$-functions is true, then every sufficiently "large" "odd" number in an algebraic field is the sum of three primes. In this paper it is proved that the same hypothesis implies that "almost" every "even" integer in a real quadratic field is the sum of two primes. The proof makes use of the Hardy-Littlewood-Rademacher method and also employs certain simplifications due to Landau. (Received December 22, 1939.)

152. R. P. Agnew: *Tauberian conditions.*

Let $K>0$ and $0<\theta<\pi/2$. Corresponding to each angle ψ, let $S(K, \theta, \psi)$ denote the "sector with vertical angle less than π" consisting of all points z of the complex plane representable in the form $z=-Ke^{i\psi}+\rho e^{i(\psi+\phi)}$ where $\rho\geqq 0$, $-\theta\leqq\phi\leqq\theta$. A series $\sum u_n$ of complex terms is said to satisfy the Tauberian condition T if K, θ, λ, and $\psi_1, \psi_2, \psi_3, \cdots$ exist such that $K>0$, $0<\theta<\pi/2$, $\lambda>1$, and, for each sufficiently great index k, $nu_n \in S(K, \theta, \psi_k)$ when $k\leqq n<\lambda k$. The condition T is sufficiently general to include classic Tauberian conditions of "order" and "gap" types, and hence provides the means of both generalizing and unifying classic Tauberian theorems. Characterizations and properties of the class of series satisfying T are developed. A condition T^* which is more general than T (and is related to T in much the same way that the condition that $\sum u_n$ be "langsam oszillierende" is related to the condition $n|u_n|<K$) is given, characterized, and discussed. Tauberian theorems involving T and T^* are proved for C_1 and other methods of summability. (Received January 10, 1940.)

153. R. P. Agnew and Mark Kac: *Translated functions and statistical independence.*

In connection with some problems of Kampé de Feriet concerning the theory of turbulence, Steinhaus proposed the following question: Does there exist a continuous function $f(t)$, $(-\infty<t<\infty)$, such that, for each sequence of real numbers $\lambda_1, \lambda_2, \cdots$, the functions $f(t+\lambda_1), f(t+\lambda_2), \cdots$ are statistically independent? (For the definition of statistical independence see M. Kac and H. Steinhaus: *Sur les fonctions indépendantes* IV, Studia Mathematica, vol. 7, pp. 1–15, and P. Hartman, E. R. van Kampen and A. Wintner: *Asymptotic distributions and statistical independence*, American Journal of Mathematics, vol. 61 (1939), pp. 477–487.) A positive answer to this question is given by the example $f(t)=\sin e^{t^2}$. A more general class of functions having the above property can be obtained. It may be mentioned that once the statistical independence is established, one can get different results concerning superposition of translations of the function. For instance, the relative measure of the set of those t's for which $f(t+\lambda_1)+\cdots+f(t+\lambda_n)<\alpha(n)^{1/2}$, $(f(t)=\sin e^{t^2})$, tends to $\pi^{-1/2}\int_{-\infty}^{\alpha}\exp(-u^2)du$ as $n\to\infty$. (Received January 13, 1940.)

154. G. E. Albert: *On quasi-metric spaces.* Preliminary report.

The present note studies a modified form of the quasi-metric spaces introduced by W. A. Wilson (American Journal of Mathematics, vol. 53 (1931), p. 675). Here, a space S will be called quasi-metric if for every pair of points x, y in S two non-negative numbers xy and yx can be defined such that (1) $xy=0=yx$ if and only if $x=y$, and (2) $xz\leqq xy+yz$ for every triple of points x, y, z in S. (Wilson used the axiom: $xy=0$ if and only if $x=y$.) If $xy=0$ but $yx\neq 0$, then x is called contiguous to y. Let $\{x_n\}$ be a sequence of points; one says that $\{x_n\}$ converges to x as an l-limit, r-limit, or c-limit accordingly as $xx_n \to 0$, $x_n x \to 0$, or both, respectively. One defines the corresponding classes of l-closed, r-closed, and c-closed sets as well as l-, r-, and c-open sets. The relationships between quasi-metric spaces and topological spaces are studied in detail. In particular, it is shown that the former include as special cases the contiguity spaces recently defined by the author (abstract 46-3-138) with neighborhood as the primitive concept. (Received January 30, 1940.)

155. Reinhold Baer: *Abelian groups that are direct summands of every containing abelian group.*

A characterization of those abelian groups, admitting a ring of operators, which are direct summands of every containing abelian group is given. It is shown that they are essentially a generalization of the groups "with division" and that they have similar properties. (Received January 5, 1940.)

156. A. C. Berry: *The accuracy of the Gaussian approximation to a sum of independent variates.*

The sum of finitely many variates frequently possesses an almost Gaussian distribution. Let M denote the least upper bound of the modulus of the difference between the distribution function of the variate sum and that of a related normal variate. M measures the "abnormality" of the variate sum. For uniformly bounded, totally independent variates, the following inequality is established: $M \leqq 2L/\sigma$. Here L is the least common bound of the *individual* variates, none of the given variates differs from its mean value by more than L except at most in cases of zero probability, and σ is the standard deviation of the variate *sum*. This result is an arithmetical refinement of the theorem of Liapounoff that $M \to 0$ when $L/\sigma \to 0$. The constant 2 is not the best possible. The existence of a best constant is proved, also that it cannot be less than $(2\pi)^{-1/2}$. The result rests on an equality connecting the difference of distribution functions with the difference of *characteristic* functions, an instance of the Parseval theorem in the theory of Fourier transforms. With the aid of a method of Paul Lévy, an extension is obtained for the case of unbounded variates. (Received January 25, 1940.)

157. W. Z. Birnbaum and H. S. Zuckerman: *On the properties of a collective.*

This paper is concerned with the concepts of a "collective" and a "selection" ("Stellenauswahl"), used in von Mises' foundations of the calculus of probabilities. The central theorem is: The set of all infinite selections can be interpreted as a space in which a Lebesgue measure is defined; then, if a sequence of 0's and 1's fulfills the first postulate of von Mises, it fulfills the second postulate in the sense that the limit of the relative frequencies is invariant for almost all selections in the space. (Received January 22, 1940.)

158. A. B. Brown: *On transformation of multiple integrals.*

The formula for transformation of a multiple Riemann integral under a change of variables $x_i = x_i(u_1, \cdots, u_n)$, $(i = 1, \cdots, n)$, depends on the formula for the volume V in (x)-space of the image, under the transformation, of a solid R in (u)-space, namely $V = \int \cdots \int_R |J_{xu}| du_1 du_2 \cdots du_n$, where J_{xu} is the Jacobian. The standard proof of the latter formula is sufficiently forbidding, for $n \geqq 3$, to discourage one's presenting it to a class in advanced calculus. In the case $n=2$, the proof does not make it clear what formula will be obtained when n is greater than 2. A different method of proof gives the result quickly and naturally, but has not been given completely in the literature, to the writer's knowledge. In this note a simple but complete derivation of the formula is given following the second method. (Received January 17, 1940.)

159. L. E. Bush: *An asymptotic formula for the average sum of the digits of integers.*

Let $S(r, N)$ be the sum of the digits of all non-negative integers less than N, when these numbers are written in the scale of notation of radix r, so that $S(r, N)/N$ is the average sum of the digits of all numbers less than N (including zero) when these numbers are written in the r-scale. An asymptotic formula is found for $S(r, N)/N$ by which it is shown that for a sufficiently large N the average sum of the digits of all non-negative integers less than N is least when the numbers are written in the binary scale, and in general is smaller for a smaller radix than for a larger. (Received January 19, 1940.)

160. J. Hobart Bushey: *Expansions of products of certain symmetric orthogonal polynomials.*

In 1878 Adams gave (Proceedings of the Royal Society of London, vol. 27, pp. 63–71) a formula for the expansion of the product $P_m(x)P_n(x)$ of two Legendre polynomials in a series of Legendre polynomials, and made use of this expansion to evaluate $\int_{-1}^{1}P_m(x)P_n(x)P_r(x)dx$, $(m, n, r=0, 1, 2, \cdots)$. In the present paper, analogous results are obtained for symmetric Jacobi polynomials and for Hermite polynomials. (Received January 29, 1940.)

161. Sister M. Patricia Callaghan: *Generalized Frégier curves.*

The author finds equations of envelopes of generalized Frégier theorems. If in the Frégier theorem a constant angle is used instead of the right angle, the envelope is a conic. The equation of the envelope of this envelope is found as the point moves around the given conic. If through any fixed point on a conic, pairs of lines are drawn making supplementary angles with a fixed line in the plane, the envelope of the chords which join the intersections of these pairs of lines with the conic is, in general, a hyperbola. The equation of the envelope of this envelope is found as the fixed point moves around the given conic. (Received January 27, 1940.)

162. R. H. Cameron and W. T. Martin: *An unsymmetric Fubini theorem.*

Under fairly general conditions the authors prove that $\int s(u)d\int p(x, u)dk(x) = \int dk(x)\int s(u)dp(x, u)$, where the integrals are Lebesgue-Stieltjes (Radon) integrals and all the limits are from $-\infty$ to ∞. (Received January 12, 1940.)

163. Richard Courant and N. Davids: *Minimal surfaces spanning closed manifolds.*

The methods for the solution of the Plateau-Douglas problem must be considerably modified if "free boundaries" on prescribed boundary manifolds are considered. The present paper, extending previous results (Proceedings of the National Academy of Sciences, vol. 24 (1938), p. 97 ff., and Acta Mathematica, vol. 72 (1940)) solves the problem of constructing an extremal minimal surface whose boundary is free on a given closed manifold of genus $p>0$. The boundary is required to be linked with a prescribed cycle which itself is linked with the manifold. This novel type of topological conditions in the calculus of variations is essential for the formulation and solution of a wide class of problems concerning minimal surfaces. (Received January 17, 1940.)

164. M. M. Day: *Linear methods of summability.*

Let Y be a directed set; that is, a partially ordered set with the transitivity and composition properties of Moore and Smith (American Journal of Mathematics, vol. 44 (1922), pp. 102–121). Let V be a space of real-valued functions on Y such that $\lim_y f(y)$ exists in the Moore-Smith sense if $f \epsilon V$. Let X be another directed set and for each $x \epsilon X$ let U_x be a functional on V; the transformation thus defined is regular on V if $\lim_x U_x(f) = \lim_y f(y)$ for all $f \epsilon V$. Extensions of the theorem of Toeplitz for simple sequences are shown to hold when V is a Banach space and the set X is essentially sequential in the sense that there exists a sequence $\{x_n\} \subset X$ such that each $x \epsilon X$ is followed by some x_n. The weak topologies in the conjugate space of V and the descriptive theory of directed sets, due to Tukey (Thesis, Princeton University, 1939), are used in showing that some such restriction is necessary. The results are applied to finding regularity conditions for various function classes defined on particular directed sets, such as multiple sequences, functions of n real variables, continuous functions, and unconditionally convergent series. (Received January 26, 1940.)

165. M. M. Day: *The spaces* L^p *with* $0<p<1$.

The class $L^p_\mu(Y)$ is the set of all real-valued functions f defined on Y, measurable μ and such that $\int_Y |f|^p d\mu < \infty$, where μ is a measure (Saks, *Theory of the Integral*, chap. 1) and $0<p<1$. It is shown that a non-identically-zero, additive, and continuous functional on $L^p_\mu(Y)$ exists if and only if there is a measurable set E with $0<\mu(E)<\infty$, such that E cannot be divided into two disjoint sets of positive measure. The proof proceeds stepwise, first assuming $\mu(Y)<\infty$, next that Y can be split into a countable sum of sets of finite measure, and finally removing all such restrictions. As a corollary, no such nonzero functionals exist when Y is any measurable subset of a euclidean n-space and μ is Lebesgue measure. In the case when they exist, a general form of the linear functionals is given, and the class of them is shown to be equivalent to the class of bounded functions on a properly chosen set. (Received January 22, 1940.)

166. R. P. Dilworth: *A characterization of lattices of ideals.*

An element a of a continuous lattice is said to be finite dependent if $a \supset \Pi(S)$ implies $a \supset \Pi(S')$ where S' is a finite subset of S. The following theorem is proved: A continuous lattice $\mathfrak{S}$ is isomorphic to the lattice of ideals of a sublattice if and only if the finite dependent elements are closed with respect to union and generate $\mathfrak{S}$ under infinite crosscut. It follows that every exchange lattice is a lattice of dual ideals. (Received January 29, 1940.)

167. F. G. Dressel: *A Stieltjes integral equation.*

The paper shows how to solve a particular type of Young-Stieltjes integral equation. Also it is shown that only one of the three conditions imposed by Fischer (Annals of Mathematics, (2), vol. 25 (1923–1924), pp. 142–158) on the kernel of a Stieltjes integral equation is needed to insure the solution of this equation. (Received January 24, 1940.)

168. F. G. Dressel: *The fundamental solution of the parabolic equation.*

The existence of the fundamental solution for the parabolic equation with variable coefficients is proved in this paper. (Received January 24, 1940.)

169. W. K. Feller: *On the integrodifferential equations of the purely discontinuous Markoff processes.*

Consider a random variable $X(t)$ varying in some space E so that if $X(t)$ coincides with the point x the probability that no change will occur during $(t, t+\Delta t)$ is $p(t, x)\cdot\Delta t+o(\Delta t)$, and the probability of $X(t+\Delta t)$ being contained in the set $\Lambda \not\ni x$ is $p(t, x)\Pi(t, x, \Lambda)\cdot\Delta t+o(\Delta t)$, where $\Pi(t, x, \Lambda)$ is a probability distribution. It is examined to what extent these conditions determine the transition probabilities $P(\tau, x; t, \Lambda)$ of some stochastic process. Generalizing previous results (*Zur Theorie der stochastischen Prozesse*, Mathematische Annalen, vol. 113 (1936), and W. Dubrovski, *Eine Verallgemeinerung der Theorie der rein unstetigen stochastischen Prozesse von W. Feller*, Comptes Rendus de l'Académie des Sciences de l'URSS, vol. 19 (1938)) it is shown that, if all functions depend continuously on t, a derivative $\partial P/\partial t$ exists for a suitable class of sets and almost all t. The problem reduces to two integrodifferential equations which determine uniquely an additive function of sets $P(\tau, x; t, \Lambda)$ with $0 \leqq P \leqq 1$. In the case of an unbounded $p(t, x)$, however, $P(\tau, x; t, E)$ may fall short of unity, although all other requirements of the theory are always fulfilled. In the case of temporally homogeneous processes, necessary and sufficient conditions for $P(\tau, x; t, E)=1$ are given, which are related to the ergodic properties of the system. (Received January 28, 1940.)

170. Tomlinson Fort: *Summability and the definition of a limit.*

Let $S_0^x f(t)\Delta_\omega t$ denote a principal solution of the difference equation $\Delta_\omega y=f(x)$ as defined by Nörlund. In the present paper a study is made of the transformation $S_0^x K(x, t)f(t)\Delta_\omega t$ analogous to studies that have been made of the integral transformation $\int_0^\infty K(x, t)f(t)dt$. A variety of sufficient conditions are obtained that the transformation be regular, limit-producing, and so on. (Received January 18, 1940.)

171. H. L. Garabedian and H. S. Wall: *Hausdorff matrices and continued fractions.*

In a recent paper (abstract 46-1-130) Wall characterized totally monotone sequences in terms of continued fractions. In this paper the subclass of regular totally monotone sequences is so characterized. If $c_0, c_1, c_2, \cdots$ is totally monotone, then the row, column, and diagonal sequences of the difference matrix $(\Delta^m c_n)$ are totally monotone. Necessary and sufficient conditions are obtained in terms of the continued fraction in order that these be regular sequences. This is accomplished by means of the curious result that when $c_0-c_1x+c_2x^2-\cdots \sim c_0/1+g_1x/1+g_2(1-g_1)x/1+g_3(1-g_2)x/1+\cdots$, then $c_0-\Delta c_0x+\Delta^2c_0x^2-\Delta^3c_0x^3\cdots \sim c_0/1+(1-g_1)x/1+g_1g_2x/1+\cdots$, where the second continued fraction is obtained from the first by replacing g_{2n-1} by $1-g_{2n-1}$, $(n=1, 2, 3, \cdots)$. A hypergeometric summability obtained from the continued fraction of Gauss is investigated as to inclusion relations, and the methods of summability arising from the difference matrix for the base sequence are considered. Finally a general theorem is given on the effectiveness of the methods considered relative to analytic continuation of power series outside the circle of convergence. (Received January 13, 1940.)

172. G. A. Hedlund: *A new proof for a metrically transitive system.*

Two distinct methods have been used to prove that the flows defined by the geodesics on suitably restricted surfaces of constant negative curvature are metrically transitive. The first of these (cf. Annals of Mathematics, (2), vol. 35 (1934), p. 787) involves the use of symbolism to characterize the geodesics and is restricted to those surfaces for which a suitable symbolism has been devised. The second of these methods (cf. E. Hopf, *Ergodentheorie*) makes use of the theory of harmonic functions and is valid for all complete surfaces of constant negative curvature and of finite area. Both of the methods seem to involve excessive machinery and it would seem desirable to have a simpler and more straightforward proof of this result. The present paper gives a new and elementary proof of the metric transitivity of the flow defined by the geodesics on any closed orientable surface of constant negative curvature. The method can be readily extended to the case of complete surfaces of constant negative curvature and finite area. (Received January 25, 1940.)

173. G.A. Hedlund: *Surfaces of negative curvature and metric transitivity.*

Let U be the unit circle $z\bar{z}=1$, $z=x+iy$, and let Ψ be its interior. The author considers the quadratic form (A) $ds^2=\lambda^2(x, y)(dx^2+dy^2)/(1-x^2-y^2)^2$, $(x^2+y^2<1)$; where $\lambda(x, y)$ is a function defined in Ψ and satisfying the following conditions: (I) $\lambda(x, y)$ is of class C^7 in Ψ; (II) there exist positive constants a and b such that $a\leqq\lambda(x, y)\leqq b$ in Ψ; (III) there exist positive constants c and d such that if $K(x, y)$ denotes the Gaussian curvature of (A), $-d^2\leqq K(x, y)\leqq -c^2$ in Ψ; (IV) there exists a positive constant e such that if $H(P, Q)$ denotes the hyperbolic distance between arbitrary points P and Q of Ψ, then $|K(P)-K(Q)|\leqq eH(P, Q)$; (V) $\lambda(x, y)$ is invariant under a Fuchsian group which has U as principal circle, which is of the first kind, and which has a finite number of generators. If points which are congruent under F are considered identical, there is defined a manifold M of negative curvature. It is shown that the geodesic system on M is metrically transitive. The result has been attained previously only in the case when $K(x, y)$ is a negative constant. (Received January 25, 1940.)

174. Fritz Herzog: *Uniqueness theorems for rational functions.*

The uniqueness theorems, dealt with in this paper, are of the type of Nevanlinna's theorem which says that two meromorphic functions must be identical if they have *five* identical distributions, that is, if they assume five complex values (finite or infinite) at the same points. No assumption whatever is made about the multiplicity with which the two functions assume these five values at the various points. While five is the smallest number for which this uniqueness theorem holds true for meromorphic functions, the paper shows that for rational functions, *four* identical distributions are sufficient for the identity of the two functions. *Three* identical distributions, however, are not sufficient to insure the identity of the two functions. Several examples of pairs of rational functions with three identical distributions are given, also a triplet of such functions. Under certain restrictive conditions, concerning for instance the degrees of the two functions, uniqueness theorems for rational functions with three identical distributions are derived. It is also shown that two polynomials with identical distributions with respect to two finite values are identical. Finally, there can be no more than a finite number of rational functions with given distributions with respect to three given values. (Received January 6, 1940.)

175. F. B. Hildebrand and P. D. Crout: *A least square procedure for improving a method for solving integral equations by polynomial approximation.*

This paper continues the development of an approximate method for solving integral equations begun in a recent paper (Crout, Journal of Mathematics and Physics, Massachusetts Institute of Technology, vol. 19 (1940), pp. 34–92). This method has given very good results when applied to a number of physical problems, and consists essentially in replacing the unknown function by an algebraic polynomial determined by $2n+1$ unknown ordinates, using for convenience the Lagrangean form, tables for which are given. By interchanging the order of integration and summation and requiring that the integral equation be satisfied at a number of points, there is obtained a set of linear equations in the unknown ordinates which fix the polynomial. The least square process described is written into a single matrix multiplication, and gives a great increase in accuracy compared with the original technique; furthermore it permits a simple consideration of the error. In setting up physical problems and in applying the method described for solving the resulting integral equations, the work is greatly simplified by the continual appearance of matrices; also the work involved in any one problem automatically includes most of that required by a variety of other problems. (Received January 30, 1940.)

176. Einar Hille and I. E. Segal: *Blaschke products as Laplace-Stieltjes integrals.*

The possibility of representing a Blaschke product $B(z)$ of the half plane type as an absolutely convergent Laplace-Stieltjes integral in some half plane is investigated. It is shown that such a representation is impossible if the zeros of $B(z)$ are all real and tend to infinity, or if the zeros satisfy certain other conditions. A necessary and sufficient condition for $(z+1)^{-\alpha}B(z)$, $(0<\alpha\leqq 1/2)$, to be representable as an absolutely convergent Laplace-Stieltjes integral is given for the case where the zeros are all real and tend to infinity. The result is qualitatively that the more slowly the zeros tend to infinity, the larger α must be in order for $(z+1)^{-\alpha}B(z)$ to be representable as an absolutely convergent Laplace-Stieltjes integral in some half plane. (Received January 26, 1940.)

177. S. A. Jennings: *Nilpotent groups and nilpotent rings.*

Consider a nilpotent ring N which is free of operators and which satisfies an ascending chain condition for two sided ideals. Such a ring may be generated by a finite number of elements, and possesses a finite basis $n_1, \cdots, n_r$ so that any $n \varepsilon N$ may be written $n=\sum \alpha_i n_i$, where $\alpha_i n_i$ denotes $n_i+n_i+\cdots+n_i$ to α_i terms. Adjoin a unit element 1 to N and consider the set of elements of the form $1+n$. These elements form under multiplication a nilpotent group G, which is generated by a finite number of elements and whose class is not greater than the exponent of N. The order of G will be finite or infinite as the order of the additive group of N is finite or infinite. Conversely, let G be any nilpotent group with a finite number of generators. By using a result of W. Magnus (Journal für die reine und angewandte Mathematik, vol. 177 (1937), pp. 105–115) on the representation of the free group in a certain ring, it is shown that G determines a nilpotent ring of the above type, whose exponent is not greater than the class of G. There is, therefore, a 1:1 correspondence between nilpotent rings with a finite basis and nilpotent groups generated by a finite number of elements. (Received January 30, 1940.)

178. I. N. Kagno: *On a certain non-separating graph on an orientable surface.*

Let G be a connected linear graph, containing no vertices of degree less than 3, which is mapped on an orientable surface Σ_p of genus $p>0$ in such a manner that Σ_p-G is a single 2-cell. The structure of G is determined and it is shown that G has at most $6p-3$ arcs, and Σ_p-G has at most $12p-6$ sides. This result is applied to the universal covering surface of Σ_p and it is shown that a fundamental domain of the group of *decktransformations* of the covering surface has at most $12p-6$ sides. In particular, if Σ_p is an algebraic Riemann surface of genus $p>1$, the results of this paper can be applied to determine an upper bound for the number of sides of a fundamental domain of the group of linear transformations to which an automorphic function belongs. (Received January 19, 1940.)

179. E. R. Kolchin: *On the exponents of differential ideals.*

After defining the concept, "exponent of a differential ideal," theorems are given connecting the exponent of a differential ideal with the exponents of the factor ideals in the representation given by J. F. Ritt (Proceedings of the National Academy of Sciences, vol. 25 (1939), pp. 90–91) and showing the invariance of the exponent under differential field adjunction. These results are applied to the study of the exponent of the differential ideal generated by a single form in one unknown of order unity. It is seen that much depends on the nature of the singular solutions of the form. Complete results are not obtained, but the exponent is determined for a wide class of such ideals. Finally, there is a brief discussion of chains of differential ideals. A theorem is given relating the length to the exponent of a principal chain. (Received February 3, 1940.)

180. D. P. Ling and Leon Recht: *Geodesics on a paraboloid of revolution.* Preliminary report.

This paper is concerned with the study of the system of geodesics on a paraboloid of revolution. It is shown that the surface is divided into an infinite number of "zones" bounded by circles whose planes are perpendicular to the axis. The zone containing the vertex is characterized by the fact that no geodesic starting from a point in this zone can return to its starting point. Through every point in the next zone there is exactly one geodesic which returns to this point after one turn around the paraboloid, but none which returns after more than one turn. Through a point in the kth zone there are k geodesics which return after $1, 2, \cdots, k$ turns respectively, but none which returns after more than k turns. Finally, a method for approximating the positions of the circles between these zones is given. (Received January 19, 1940.)

181. J. K. L. MacDonald: *Existence theorems for node and amplitude properties near singularities of linear differential equations.*

Necessary and sufficient conditions for forms of solutions of linear, ordinary, second order differential equations near singular end points are established. Comparison, transformation and iteration theorems are generalized on the basis of previous work of the writer (this Bulletin, vol. 45 (1939), pp. 164–171, and abstract 44-9-356). Comparison criteria (analogous to the sequence of logarithmic tests for series convergence and divergence) are developed for determining whether or not nodes accumulate near the singular end points. Transformations leading in all cases to coefficients of the same sign are discussed. The phase angle and the transformed linear equations are expressed in forms which show their interrelationship in a clear way. (Received January 27, 1940.)

182. Saunders MacLane: *Note on the relative structure of p-adic fields.*

A problem of relative structure for p-adic fields, which arose in a previous paper (Annals of Mathematics, (2), vol. 40 (1939), pp. 423–442), can now be solved. Let k be a p-adic field with a residue class field $\mathfrak{k}$ of characteristic p. Let $\mathfrak{K}$ be any extension of this residue class field. A necessary and sufficient condition that any two p-adic extensions K and K' of k, both having the same residue class field $\mathfrak{K}$, be analytically equivalent over k is the requirement that the given residue class extension $\mathfrak{K}/\mathfrak{k}$ preserve p-independence (see definition in Duke Mathematical Journal, vol. 5 (1939), pp. 372–393). Here an *analytical equivalence* of K to K' is to be an isomorphism of K to K' which leaves values and residue classes of all elements in K fixed. The necessity proof is an extension of previous examples, and is based on a suitable analysis of those extensions $\mathfrak{K}/\mathfrak{k}$ which do not preserve p-independence. (Received January 29, 1940.)

183. R. S. Phillips: *On the space of completely additive set functions.*

This paper is devoted to an investigation of a class of Banach spaces typified by the space of completely additive set functions on a given Borel field of sets. The space of functions of bounded variation is a particular instance of such a space. The central result is that a space of this type can be represented as the direct product of spaces of completely additive set functions, each absolutely continuous with respect to a unique measure function of a certain orthogonal set. The author shows that to each set function $x(\tau)$ of the space corresponds a denumerable number of disjoint sets τ_i such that $x(\tau)$ is uniquely representable in the form $x(\tau)=\sum x(\tau\cdot\tau_i)$ where $x(\tau\cdot\tau_i)$ belongs to one of the component spaces. By means of this decomposition it is easy to demonstrate that the space is weakly complete and that it possesses a generalized base. Finally the author is able to define the general linear functional on the space and characterize the conjugate space. (Received January 9, 1940.)

184. Tibor Radó and P. V. Reichelderfer: *Note on an inequality of Steiner.*

Let $L(f)$ denote the Lebesgue area of a continuous surface $z=f(x, y)$ defined over the unit square Q. If $z=f_n(x, y)$ is a sequence of continuous surfaces defined over Q and converging uniformly to $z=f(x, y)$ on Q for which $L(f_n)$ and $L(f)$ are finite, then McShane (Annals of Mathematics, (2), vol. 33 (1932), pp. 125–138) has shown that smallness of $\epsilon_n=[L(f_n)+L(f)]/2-L([f_n+f]/2)$ implies smallness with respect to measure of $\delta_n(x, y)=[(f_{nx}-f_x)^2+(f_{ny}-f_y)^2]^{1/2}$, whence he concludes that if $L(f_n)$ converges to $L(f)$, then $\delta_n(x, y)$ converges to zero in measure. This result is improved in the present paper by replacing smallness with respect to measure by smallness with respect to exponent. It is shown that smallness of ϵ_n implies smallness of $I_n(\lambda)=\iint_Q\delta_n^\lambda(x, y)dxdy$ for every exponent λ satisfying $0<\lambda<1$, and thus if $L(f_n)$ converges to $L(f)$ then $I_n(\lambda)$ converges to zero for every λ satisfying $0<\lambda<1$. Since McShane has shown that $I_n(1)$ does not generally converge to zero when $L(f_n)$ converges to $L(f)$, the present result is the best obtainable under the stated hypotheses. (Received January 19, 1940.)

185. Eric Reissner: *On Green's function of $\nabla^2\nabla^2 w=0$ for the half plane.*

The following result, of interest mainly for the theory of bending of thin elastic plates, is proved. The explicit form of Green's function of $\nabla^2\nabla^2 w=0$ for the half plane $x \geqq 0$ satisfying the boundary conditions $w=0$ and $(\partial w/\partial x)=\lambda\nabla^2 w$ along $x=0$ is: $w=(1/8\pi)\Re n\left[|z-a|^2\log(z-a)/(z+a)+2az+8ax\exp((z+a)/2\lambda)\operatorname{Ei}((z+a)/2\lambda)\right]$. In the formula for w, Ei (x) denotes the "exponential integral." The solution for the special case $\lambda=0$ has been given by J. H. Michell (Proceedings of the London Mathematical Society, vol. 34 (1902), pp. 223–228). (Received January 25, 1940.)

186. H. J. Riblet: *Certain theorems for algebraic differential fields.*

In this paper several theorems are proved for finite algebraic differential fields. One states that in a finite algebraic differential field $\overline{K}(\theta_1)$ there exists an element which is linearly independent of its $n-1$ conjugates. This is helpful in proving that if every element of $\overline{K}(\rho)$ satisfies a linear homogeneous differential equation of order not greater than n over $\overline{K}$, then ρ is algebraic over $\overline{K}$ of degree not greater than n. (Received January 16, 1940.)

187. Jenny E. Rosenthal: *Generating functions and properties of certain orthogonal polynomials.* Preliminary report.

The differential equation satisfied by the generating functions $U(x, t) = \sum_{m=0}^{\infty} R_m(x)t^{m+\alpha}$ of a set of orthogonal polynomials is found to be $\sum_{j=0}^{n} t^j[a_j+(b_jx+c_j)t+d_jt^2]\cdot\partial^j U/\partial t^j=0$, where the various a's, b's, c's, and d's are constants. Depending on the values of these constants, the solution of the equation for $n-1$ gives three new functions in addition to the generating functions of the Legendre, Laguerre, and Hermite polynomials. The properties of the orthogonal polynomials generated by the new functions are investigated. The range of orthogonality and the weight factors are derived by a simple method in two of the cases. The weight factors are of the form of gamma- and beta-functions. Possible generalizations of the results are discussed. (Received January 27, 1940.)

188. Peter Scherk: *Estimations with integers in the demesne of the α-β-hypothesis.*

The "number function" $D(x)$ of a set D of positive integers d denotes the number of the d not greater than x. Let A, B be sets of positive integers respectively a, b; and let C be the set of all the numbers of the form a, b or $a+b$. The author gives two new estimates of the number function of the set C through those of A and B. One of them is the simplest possible symmetric in A and B, the other being related to it by Khintchine's "Umkehrformel." Both are exact. The proofs are based on a classical special case of this formula. The methods are similar to those used in two earlier papers by Besicovitch and by the author. See Landau's Cambridge Tract and the author's *Bemerkungen zu einer Note von Besicovitch*, Journal of the London Mathematical Society, vol. 14 (1939), pp. 185–192. (Received January 19, 1940.)

189. F. K. Schmidt and Saunders MacLane: *On inseparable fields.*

The original proof of the structure theorem for p-adic fields depended on an analysis of the residue class field by means of certain Steinitz field towers. Saunders Mac Lane has given examples to show that the original analysis of these towers could not be correct (Transactions of this Society, vol. 46 (1939), pp. 23–45). The authors now develop an alternative treatment of modified Steinitz towers, in terms of which the proof of the structure theorem can be carried out. The first modification is the device of replacing a denumerable transcendence basis $t_0, t_1, t_2, \cdots$ by another basis $t_0, t_1^p, t_2^{p^2}, \cdots$, which is almost a p^∞th power. The second modification consists in generating an arbitrary extension K preserving p-independence over a modular field

k by a transfinite sequence of extensions, each of which has a denumerable transcendence basis and preserves p-independence. (Received January 29, 1940.)

190. A. R. Schweitzer: *Concerning general abstract relational spaces.*

The author constructs a set S of postulates for relational spaces in terms of an undefined transitive and symmetrical relation K_H effective between $(n+1)$-ads of elements $(n=1, 2, 3, \cdots)$ and invariant under any arbitrarily selected subgroup H of the symmetric group G on $n+1$ elements, including G. When H is the alternating group and $n=3$ the set S is equivalent to Axioms 1–6, 11–13, 16 of the author's descriptive system $^3K_3$ (American Journal of Mathematics, vol. 31 (1909), pp. 395–396). A feature of the general theory is the generalized axiom of dimensionality (generalizing Axiom 16, above) which expresses disjunctively relative classification of $(n+1)$-ads corresponding to the classification of the substitutions of the symmetric group G on $n+1$ elements into the subgroup H and cosets with respect to H. Particular instances of the preceding relational spaces are rotation spaces of the regular polygons and polyhedra and other metric spaces. When H coincides with G, instances include the author's system of postulates $^nK^*$ for projective geometry (this Bulletin, abstract 42-11-443). (Received January 28, 1940.)

191. W. T. Scott: *Approximation to real irrationals by certain classes of rational fractions.*

The set of all irreducible rational fractions is divided into three classes, according as the numerator and denominator are even or odd integers. For each of these classes it is found that there are infinitely many fractions p/q satisfying $|\omega - p/q| < k/q^2$ when $k \geqq 1$, regardless of the value of the real irrational number ω. If $k < 1$, there exist irrationals everywhere dense on the real axis for which the inequality holds for at most a finite number of fractions of a given one of the three classes. The method of proof depends on the geometric properties of elliptic modular transformations. (Received January 6, 1940.)

192. Seymour Sherman: *A comparison of linear measures of point sets in the plane.* Preliminary report.

The purpose of this investigation is to compare the linear measures defined by Carathéodory and Deltheil. H. Steinhaus (Comptes Rendus du Premier Congrès des Mathématiciens des Pays Slaves, 1929, pp. 348–354) has proved that the Deltheil measure of a rectifiable plane Jordan curve is twice its length and so twice its Carathéodory linear measure. Here it is shown that, contrary to Steinhaus's belief, there exist sets (in particular every set irregular in the sense of Besicovitch) with Deltheil measure zero and Carathéodory measure nonzero. In general if a set has finite Carathéodory linear measure, then it is Deltheil measurable with Deltheil measure equal to twice the Carathéodory linear measure of its regular part. Deltheil measurability does not imply linear measurability in the sense of Carathéodory. Nothing is proved concerning sets of noncountably infinite Carathéodory linear measure and no extensions are made to sets in more general product spaces or even to sets in higher dimensional euclidean spaces. (Received January 11, 1940.)

193. M. F. Smiley: *Measurability and modularity in the theory of lattices.*

In a previous note (*A note on measure functions in a lattice*, to appear in this Bulletin), a notion of measurability (with respect to a function μ) of elements of an arbitrary lattice was introduced. Results of V. Glivenko (*Contributions à l'étude des systèmes de choses normées*, American Journal of Mathematics, vol. 59 (1937), pp. 933–934) and of L. R. Wilcox and the author (*Metric lattices*, Annals of Mathematics, (2), vol. 40 (1939), p. 313) indicate that the idea of measurability and that of modularity are intimately related. The purpose of this note is to exhibit a further relationship which does not depend on metric properties of the function μ. Under the assumption that $a \leqq b$ with $\mu(a) = \mu(b)$ implies that $a = b$, it is found that the measurability of each subelement of a fixed element c implies the complete modularity of each subelement of c. A further consequence is the proof of a theorem in the reverse direction in case the lattice is of finite dimension and satisfies the conclusion of Jordan's theorem. (Received February 1, 1940.)

194. Gabor Szegö: *Remarks on a note of R. Wilson and related subjects.*

Let $w(x)$ be a weight function in $[-1, +1]$ and $\{p_n(x) = k_n x^n + \cdots\}$ the associated set of orthonormal polynomials. (1) Assuming the existence of $\int_0^\pi \log w(\cos\theta)d\theta$, a simple proof is given for the relations $\lim_{n\to\infty} \max |p_n(x)|^{1/n} = (1/2)\cdot\lim_{n\to\infty} k_n^{1/n} = 1$, $-1 \leqq x \leqq +1$, recently discussed by R. Wilson (this Bulletin, vol. 45 (1939), p. 190). (2) An older theorem of Shohat stating that the equivalency of the relations $2^{-n}k_n = O(1)$, $\lim_{n\to\infty} 2^{-n}k_n$ exists is proved in an arrangement slightly different from that of Shohat. (3) It is shown that the relation $2^{-n}k_n = O(1)$ (or the existence of $\lim_{n\to\infty} 2^{-n}k_n$) is equivalent to the existence of the integral mentioned in (1). (Received January 22, 1940.)

195. G. B. Thomas: *Regular positive ternary quadratic forms.* Preliminary report.

Let $f = f(x, y, z) = ax^2 + by^2 + cz^2 + 2ryz + 2sxz + 2txy$ be a positive, reduced, properly or improperly primitive form with integral coefficients and Hessian $H = \Omega^2\Delta$, where Ω is the g.c.d. of the 2-rowed minors of H. Then f is said to be *regular* if the integers not represented by f coincide with the positive integers in certain sets of arithmetic progressions. (L. E. Dickson, Annals of Mathematics, (2), vol. 28 (1927), pp. 333–341.) The problem of determining all such regular forms with $r = s = t = 0$ has been solved. (Dickson, loc. cit.; B. W. Jones and Gordon Pall, Acta Mathematica, vol. 70 (1939), pp. 165–191.) Combining some results due to B. W. Jones (Transactions of this Society, vol. 33 (1931), pp. 92–124) with certain theorems regarding primes in arithmetic progressions, the present paper shows that there are not more than 99 nor fewer than 94 regular forms with $\Omega = 1$ and r, s, t not all zero. The regular forms are listed, together with the progressions of integers they do not represent. (Received January 16, 1940.)

196. Leonard Tornheim: *Linear forms in function fields.*

The following analogue for function fields of a well known theorem of Minkowski on linear forms is proved algebraically. Let L_i be n linear expressions in n variables x_j with coefficients a_{ij} in $F(z)$ where z is an indeterminate over a field F and with determinant $|a_{ij}|$ of degree d. If the sum of n integers c_i is greater than $d - n$, there exists a set of values for the x_j in $F[z]$ such that each L_i has degree at most c_i. (Received January 15, 1940.)

197. H. S. Vandiver: *Elements of a theory of abstract discrete semi-groups.*

A set of axioms is given for a semi-group, the latter being a system elsewhere roughly described by the author as a set closed under an associative operation. A feature of the treatment employed is the fact that the parentheses are not used, there being an axiom of substitution which leads to a result which is equivalent to the associated law, but which does not contain parentheses in its statement. The symbol of relation ($\backcong$) (called equivalence) does not necessarily stand for equality in the usual sense; for example, the integers 1, 2, 3, 4 are said to form a semi-group under multiplication, modulo 5. In this case equivalence means "is congruent to, modulo 5." The proofs of a number of theorems given without proof in a former article (Proceedings of the National Academy of Sciences, vol. 23 (1937), pp. 552–555) are included in this paper. A theorem is also proved concerning semi-groups, which includes Dedekind's well known theorem of inversion. (Received December 30, 1939.)

198. H. S. Vandiver: *On general methods for obtaining congruences involving Bernoulli numbers.*

This paper has been published (this Bulletin, vol. 46 (1940), pp. 121–123).

199. Abraham Wald: *Asymptotically most powerful tests of statistical hypotheses.*

Let $F(x, \theta)$ be the distribution of a variate X involving an unknown parameter θ. For testing the hypothesis $\theta=\theta_0$, it is necessary to choose a region of rejection W_n in the n-dimensional sample space. Denote by $P(W_n|\theta)$ the probability that the sample point will fall in W_n under the assumption that θ is the true value of the parameter. For any region U_n denote by $g(U_n)$ the greatest lower bound of $P(U_n|\theta)$. For any pair of regions U_n and T_n denote by $L(U_n, T_n)$ the least upper bound of $P(U_n|\theta)-P(T_n|\theta)$. An infinite sequence $\{W_n\}$ of regions is said to be an asymptotically most powerful test of the hypothesis $\theta=\theta_0$ on the level of significance α if $P(W_n|\theta_0)=\alpha$ and if for any sequence $\{Z_n\}$ of regions for which $P(Z_n|\theta_0)=\alpha$, lim sup $L(Z_n, W_n)\leqq 0$ holds. A sequence of regions $\{W_n\}$ is said to be an asymptotically most powerful unbiased test of the hypothesis $\theta=\theta_0$ on the level of significance α if $P(W_n|\theta_0)=\alpha$, lim inf $g(W_n)\geqq\alpha$, and if for any sequence $\{Z_n\}$ of regions for which $P(Z_n|\theta_0)=\alpha$ and lim inf $g(Z_n)\geqq\alpha$ the inequality lim sup $L(Z_n, W_n)\leqq 0$ holds. It has been shown that the commonly used tests, based on the maximum likelihood estimate, are asymptotically most powerful. (Received February 5, 1940.)

200. William Wernick: *Complete sets of logical functions.*

H. M. Sheffer has shown that the 16 functions of the calculus of sentences can all be defined by the single function "stroke" which may have a dual interpretation. Other authors have shown that certain pairs of undefined functions can define this calculus, and that certain other pairs cannot. The set of all functions which can be defined in terms of members of a given undefined set of functions is called the field of that given set. Fields of all single functions and of pairs, triads, and tetrads of functions are discussed, using known properties of symmetry, duality, and transitivity. A set of undefined functions is called "*complete*" if it is ***sufficient*** (all the 16 functions can be defined in terms of members of the set) and ***non-redundant*** (no member of the set can be defined in terms of other members of the set). The existence of 2 complete sets of single functions, 24 complete pairs, and 6 complete triads is shown. It is proved that there exists no complete set of four or more functions, by showing that if such a set is sufficient it must be redundant. (Received January 24, 1940.)

201. Hermann Weyl: *Theory of reduction for arithmetical equivalence.*

In the frame of Minkowski's geometry of numbers, reduction appears as the problem to normalize a basis for the n-dimensional lattice of vectors with integral components in terms of a given convex body. The theory in this generality can be carried through much further than Minkowski seems to have assumed, by means of his geometric methods. He limited himself to the special case of ellipsoids (quadratic forms), at the same time abandoning his own geometric approach. Even for this special case the new treatment induces considerable simplification, closes certain gaps, and results in sharper estimates. The theory is carried over to vectors whose components are complex numbers or quaternions, with an integrity domain of class number 1 serving for the definition of lattice vectors. The quadratic forms are then replaced by Hermitian and "Hamiltonian" forms respectively. (Received January 24, 1940.)

202. John Williamson: *An algebraic problem involving the involutory integrals of linear dynamical systems.*

It is known that a conservative Hamiltonian system with n degrees of freedom has exactly n independent conservative integrals which form an involutory system. It is here shown that, if the system is linear, these n integrals may be chosen as quadratic forms. It is also shown that, if $2H$ is the symmetric matrix, which is the Hessian of the Hamiltonian function, and G the matrix of the covariant bilinear form, that is, the standard canonical form of a nonsingular skew symmetric matrix, the total number of independent conservative quadratic integrals is $2n-m$, where either $2m$ or $2m+1$ is the order of the minimal equation of HG^{-1}. In case HG^{-1} is nonderogatory, the n independent quadratic integrals may be so chosen that their matrices are the symmetric matrices $(HG^{-1})^{2k}H$, $(k=0, 1, 2, \cdots, n-1)$. These n integrals in this case form a set in involution. (Received January 30, 1940.)

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A NOTE ON MEASURE FUNCTIONS IN A LATTICE

M. F. SMILEY

We give first an equivalent statement of the measurability criterion of Carathéodory[2] which is applicable to an arbitrary lattice. We then study the closure with respect to finite and denumerable sums and products of the subset of measurable elements of a *modular* lattice. The case of regular[3] "outer measure functions" is then briefly discussed. The elements of the theory of lattices are presupposed.[4]

Let us consider a lattice L on which is defined a real-valued function $\mu(a)$. The elements $a \varepsilon L$ which satisfy

$$\mu(a+b)+\mu(ab)=\mu(a)+\mu(b) \tag{1}$$

for every $b \varepsilon L$ will be called μ-measurable. The totality of μ-measurable elements will be denoted by $L(\mu)$.

REMARK 1. *If L is a Boolean algebra and $\mu(0)=0$, then $a \varepsilon L(\mu)$ if and only if $a \varepsilon L$ and satisfies the condition of Carathéodory,*[5] *that is,*

$$\mu(b)=\mu(ab)+\mu(b-ab) \tag{2}$$

for every $b \varepsilon L$. For, if $a \varepsilon L$ satisfies (1), *the equation* (1) *and*

$$\mu(a+(b-ab))+\mu(0)=\mu(a)+\mu(b-ab)$$

yield (2). *The converse is proved by Carathéodory.*[6]

THEOREM 1. *If L is a modular lattice, then $L(\mu)$ is a sublattice of L.*

PROOF. Let $a, c \varepsilon L(\mu)$, $b \varepsilon L$. We obtain successively

$$\begin{aligned}\mu(a+(c+b))+\mu(a(c+b)) &= \mu(a)+\mu(c+b)\\ &= \mu(a)+\mu(c)+\mu(b)-\mu(cb)\\ &= \mu(a+c)+\mu(b)+\mu(ac)-\mu(cb).\end{aligned}$$

Since $c \varepsilon L(\mu)$ we have

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[1] Presented to the Society, September 5, 1939.

[2] *Vorlesungen über Reelle Funktionen*, 2d edition, p. 246.

[3] Ibid., p. 258.

[4] See, for example, G. Birkhoff, *On the combination of subalgebras*, Proceedings of the Cambridge Philosophical Society, vol. 29 (1933), pp. 441–464; O. Ore, *On the foundations of abstract algebra* I, Annals of Mathematics, (2), vol. 36 (1935), pp. 406–437. The terminology and notation are those used by L. R. Wilcox and the author, *Metric lattices*, Annals of Mathematics, (2), vol. 40 (1939), pp. 309–327.

[5] Op. cit., p. 246.

[6] Ibid., p. 252.

$$\mu(c + a(c + b)) + \mu(ac) = \mu(c) + \mu(a(c + b)),$$
$$\mu(c + (a + c)b) + \mu(cb) = \mu(c) + \mu((a + c)b).$$

Using the modular law we see that $(a+c)(c+b)=c+a(c+b)=c+(a+c)b$. It is then clear that

$$\mu(ac) - \mu(cb) = \mu(a(c + b)) - \mu((a + c)b),$$

and (1) with a replaced by $a+c$ follows easily. Thus $a+c \in L(\mu)$. By duality, $ac \in L(\mu)$. This completes the proof.

DEFINITION 1. *If, for each increasing (decreasing) sequence* $(a_i; i=1, 2, \cdots)$ *of elements of* $L(\mu)$ *with a sum (product)* $a \in L$, *we have* $\lim \mu(a_i)=\mu(a)$ *as* $i\rightarrow\infty$ *we say that*[7] L *satisfies* $B^+(\mu)$ $(B^-(\mu))$; *if moreover* $\lim \mu(a_i+b)=\mu(a+b)$ *and* $\lim \mu(a_ib)=\mu(ab)$ *as* $i\rightarrow\infty$ *for each* $b \in L$, *we say that* L *satisfies* B^+ (B^-).

REMARK 2. *If* L *satisfies* B^+ (B^-), *then* L *satisfies* $B^+(\mu)$ $(B^-(\mu))$. *It suffices to take* $b=a$ *in the definition of* B^+ (B^-).

We shall assume throughout the remainder of this note that L is modular and that $\mu(a)$ is monotone increasing.

THEOREM 2. *A sufficient condition for closure of* $L(\mu)$ *with respect to denumerable sums (products) in* L *is that* L *satisfy* B^+ (B^-). *This condition is necessary if* L *satisfies* $B^+(\mu)$ $(B^-(\mu))$.

PROOF. To show that B^+ is sufficient, consider a sequence $(a_i; i=1, 2, \cdots)$ of elements of $L(\mu)$ with a sum $a \in L$. Define $c_i \equiv \sum(a_j; j=1, 2, \cdots, i)$. Clearly $a=\sum c_i$, and $(c_i; i=1, 2, \cdots)$ is increasing. By Theorem 1, $c_i \in L(\mu)$ for each $i=1, 2, \cdots$, and hence $\mu(c_i+b)+\mu(c_ib)=\mu(c_i)+\mu(b)$ for each $b \in L$. On taking the limit and using B^+ we see that $\mu(a+b)+\mu(ab)=\mu(a)+\mu(b)$. Thus $a \in L(\mu)$, and B^+ is sufficient. For the necessity, consider an increasing sequence $(a_i; i=1, 2, \cdots)$ of elements of $L(\mu)$ with a sum $a \in L$. For each $b \in L$ and each $i=1, 2, \cdots$, $a+b\geqq a_i+b$ and $ab\geqq a_ib$; and hence $\mu(a+b)\geqq\mu(a_i+b)$, $\mu(ab)\geqq\mu(a_ib)$. Define $\alpha\equiv\lim \mu(a_i+b)$ and $\beta\equiv\lim \mu(a_ib)$ as $i\rightarrow\infty$. Since $a_i \in L(\mu)$ we have $\mu(a_i+b)+\mu(a_ib)=\mu(a_i)+\mu(b)$. On taking the limit and using $B^+(\mu)$ and the fact that $a \in L(\mu)$ we obtain $\alpha+\beta=\mu(a)+\mu(b)$. It follows that $\alpha=\mu(a+b)$, $\beta=\mu(ab)$. Thus B^+ is necessary when L satisfies $B^+(\mu)$. The alternate reading is dual. The proof is complete.

DEFINITION 2. (1) *For each* $a \in L$ *we define* $\mu^+(a)\equiv$g.l.b.$[\mu(c); c \in L(\mu), c\geqq a]$, $\mu^-(a)\equiv$l.u.b.$[\mu(c); c \in L(\mu), c\leqq a]$.

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[7] Cf. L. R. Wilcox and the author, op. cit., p. 317.

(2) *We say that* $\mu(a)$ *is outer (inner) regular*[8] *in case* $\mu(a)=\mu^+(a)$ $(\mu(a)=\mu^-(a))$ *for every* $a \in L$.

LEMMA 1. *If* $\mu(a)$ *is outer regular, then*

$$\mu(a + b) + \mu(ab) \leqq \mu(a) + \mu(b)$$

for every $a, b \in L$.

PROOF. Consider $a, b \in L$. For each $c, d \in L(\mu)$ for which $c \geqq a$, $d \geqq b$ we have $c+d \geqq a+b$, $cd \geqq ab$; and, by Theorem 1, $c+d$, $cd \in L(\mu)$. Consequently, since $\mu(a)$ is outer regular, $\mu(a+b)+\mu(ab) \leqq \mu(c+d) + \mu(cd) = \mu(c)+\mu(d)$. The lemma follows by applying a simple property of the greatest lower bound.

THEOREM 3. *If L satisfies* $B^+(\mu)$ $(B^-(\mu))$ *and* $\mu(a)$ *is outer (inner) regular, then L satisfies* B^+ (B^-).

PROOF. This follows from Lemma 1 and its dual by the method used in proving Theorem 2.

We now assume that L is closed with respect to denumerable sums and products.

LEMMA 2. *If L satisfies* B^- (B^+), *then for each* $a \in L$ *there is an element* $c \in L(\mu)$ *such that* $c \geqq a$ $(c \leqq a)$ *and* $\mu(c)=\mu^+(a)$ $(\mu(c)=\mu^-(a))$.

PROOF. This is an easy consequence of Theorem 2.

REMARK 3. *It is now clear that when L satisfies* B^- *and* $\mu(a)$ *is outer regular the distance function*[9] $\delta(a, b) = 2\mu(a+b) - \mu(a) - \mu(b)$ *identifies each* $a \in L$ *with an element* $c \in L(\mu)$.

THEOREM 4. *If L satisfies* B^+ (B^-) *and* $\mu(a)$ *is outer (inner) regular, then an element* $a \in L$ *belongs to* $L(\mu)$ *if and only if* $\mu^-(a)=\mu(a)$ $(\mu^+(a)=\mu(a))$.

PROOF. Consider an element $a \in L$ for which $\mu^-(a)=\mu(a)$. By hypothesis and Lemma 2 there is an element $c \in L(\mu)$ such that $c \leqq a$ and $\mu(c)=\mu^-(a)$. Thus, for each $b \in L$, $\mu(a)+\mu(b)=\mu^-(a)+\mu(b)=\mu(c) + \mu(b) = \mu(c+b)+\mu(cb) \leqq \mu(a+b)+\mu(ab)$. Consequently, by Lemma 1, $a \in L(\mu)$. The converse is trivial. The alternate reading is dual. The proof is complete.

LEHIGH UNIVERSITY

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[8] Cf. Carathéodory, op. cit., p. 258.

[9] See L. R. Wilcox and the author, op. cit., p. 311.

THE INEQUALITIES OF MORSE WHEN THE MAXIMUM TYPE IS AT MOST THREE[1]

ROSS H. BARDELL

1. **Introduction.** The theory of critical points has been developed by Morse[2] [1, 2] by the use of combinatorial topology. In particular the theory developed by Morse is applicable to simple integral problems in the calculus of variations. The author, in his doctoral dissertation [3], studied the theory of types of extremals for a simple integral problem in the plane without the use of combinatorial topology. In the present paper a new and interesting property of extremal arcs joining two fixed points in the plane will be proved. This property makes possible a proof of the inequalities of Morse [3, p. 30] in the special case where the maximum type of each extremal arc is at most three. This is the first proof of the inequalities without the use of topology and without assuming the problem to be reversible [3, p. 25].

The hypotheses of this paper are those made in §1 of the dissertation [3] referred to above and likewise the notation used here is that of the earlier paper.

2. **Properties of the extremal arcs joining two fixed points of R.** In the paper mentioned above [3, p. 17] it was proved that every point 2 of R, which is not on an envelope arc of the family of extremal arcs though the point 1, is joined to 1 by $2r+1$ (r a positive integer) extremal arcs of which r are of odd type and $r+1$ are of even type, one of which at least is of type zero.

Consider now a point 2 of R which is not on an envelope arc of the family of extremals through the point 1. Let E_{a_0} designate an extremal arc joining the point 2 to 1 and such that the arc 12 of E_{a_0} is of type zero. It can be shown [3, p. 10] that the extremal arc E_a for $a>a_0$ and near a_0 and also for $a<a_0+2\pi$ and near $a_0+2\pi$ has no intersections with the arc 12 of E_{a_0}. Then as a increases from a_0 to $a_0+2\pi$ all intersections of the extremal arc E_a with the arc 12 of E_{a_0} which move onto this arc must move off again. In fact it can easily be proved [3, p. 18] that an intersection 3 of E_a with the arc 12 of E_{a_0} which moves onto the arc 12 of E_{a_0} for an extremal arc of odd (even) type must move off of the arc 12 of E_{a_0} for an extremal arc of even

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[1] Presented to the Society, April 8, 1938.

[2] The numbers in brackets here and elsewhere refer to the bibliography at the end of this paper.

(odd) type. While the preceding statement is true for all extremal arcs, the following much stronger property can be proved in the special case where the extremal arcs in question are of type zero or one.

LEMMA 2.1. *Let* 2 *be any point of the region* R *which is not on an envelope arc of the family of extremals through* 1, *and let* 3 *designate a particular intersection of* E_a *with the arc* 12 *of* E_{a_0} *which is moving through* 2 *onto the arc* 12 *of* E_{a_0} *as* a *increases through* a' *and off of the arc* 12 *of* E_{a_0} *as* a *increases through* a'', *where* $a'<a''$. *Then, if the arc* 12 *of* $E_{a'}$ *is of type zero, the arc* 12 *of* $E_{a''}$ *is of type one. Moreover, if the arc* 12 *of* $E_{a''}$ *is of type zero, then the arc* 12 *of* $E_{a'}$ *is of type one.*

It will suffice to prove the first statement of the above lemma since the last statement is exactly equivalent to the first when the parameter is allowed to decrease from a'' to a'. We will now proceed to prove the first statement.

Since, by hypotheses, the arc 12 of $E_{a'}$ is of type zero, it follows that the arc 13 of E_a for $a>a'$ and near a' has no intersections with the arc 12 of $E_{a'}$ [3, p. 10]. Moreover, as a increases from a' to a'' the point 3 never passes through 2, and hence no intersection of the arc 13 of E_a can move onto the arc 12 of $E_{a'}$. Therefore the arc 13 of E_a for $a'<a<a''$ has no intersections with the arc 12 of $E_{a'}$ and hence the arc 12 of $E_{a''}$ has only the point 2 in common with the arc 12 of $E_{a'}$.

Let the arc 12 of $E_{a''}$ be of type k. Then as a decreases from a'' and is near a'', the arc 13 of E_a will have k intersections with the arc 12 of $E_{a''}$. We wish to show that the number of intersections of the arc 13 of E_a with the arc 12 of $E_{a''}$ is also equal to k for $a\leqq a'$ and near a'. In that case the preceding paragraph shows that $k=1$ and hence the lemma is proved. To show this let 4 and 6 designate the intersections of $E_{a'}$ and $E_{a''}$, respectively, with E_{a_0} which are adjacent to and preceding 2 on E_{a_0} and let 5 designate the intersection of E_a with E_{a_0} which is adjacent to and preceding 3 on E_{a_0}. Further let the intersection of $E_{a''}$ with E_{a_0} which is adjacent to and preceding 6 on E_{a_0} be designated by 7. We have then the following two cases to consider.

CASE 1. Sin $(\theta_0-\theta)$ has opposite signs at the two points 4 and 2, where θ_0 and θ are the inclinations of the tangents to E_{a_0} and E_a, respectively, at their particular intersection in question. In this case the intersections 5 and 3 can move only on the interval 42 of E_{a_0} for $a'<a<a''$. This follows since, as was seen above, the arc 13 of E_a has no intersections with the arc 12 of $E_{a'}$, for $a'<a<a''$. Therefore the intersection 6 must also be on the interval 42 of E_{a_0} and by exactly

the same reasoning it follows that the intersection 7 must be on the interval 14 of E_{a_0}. Hence as a decreases from a'' to a' the intersection 3, which must stay on the interval 42, may pass through 6 but not through 7. When it does pass through 6 an intersection of the arc 13 of E_a with the arc 12 of $E_{a''}$ is lost, but is regained when 3 reverses its direction and moves again through 6 as it must since its final position is 2. Therefore in this case the arc 13 of E_a for $a \leqq a'$ and near a' has k intersections with the arc 12 of $E_{a''}$.

CASE 2. Sin $(\theta_0 - \theta)$ has the same sign at the two intersections 4 and 2. In this case the point 3 can move only on the interval 42 of E_{a_0} and the point 5 can move only on the interval 14 of E_{a_0} as a decreases from a'' to a'. This follows from the fact that, as above, the arc 13 of E_a has no intersections with the arc 12 of $E_{a'}$ for $a' < a < a''$. Therefore 6 must remain on the interval 14 of E_{a_0} as a decreases from a'' to a' and hence 3 cannot pass through 6 since 3 must remain on the interval 42 of E_{a_0}. Hence in this case also the arc 13 of E_a for $a' \leqq a < a''$ always has k intersections with the arc 12 of $E_{a''}$.

3. **The inequalities.** For every a on the interval $a_0 < a < a_0 + 2\pi$ defining an extremal arc E_a which joins the point 2 to 1 an intersection of E_a with the arc 12 of E_{a_0} either moves onto or off of this arc. Let the extremal arcs joining the points 1 and 2 be paired so that the arcs of each pair are those defined in the family of extremals through the point 1 by the parameter values for which a particular intersection 3 of E_a with the arc 12 of E_{a_0} moves onto and off of this latter arc.

LEMMA 3.1. *Every extremal arc of the set of extremals joining the points* 1 *and* 2, *except* E_{a_0}, *belongs to one and only one of the pairs of extremal arcs defined above. Moreover, if the maximum type of each extremal arc joining the points* 1 *and* 2 *is assumed to be not greater than* 3, *then the extremal arcs of each pair differ in type by unity.*

The first statement of the lemma is an immediate consequence of the definition of the pairs of extremal arcs given in the first paragraph of this section. The last statement in the lemma is a consequence of Lemma 2.1 and the statement in the paragraph immediately preceding this lemma.

Let p_i represent the number of pairs of extremal arcs joining the points 1 and 2, such that one arc of each pair is of type i and the other of type $i-1$. Obviously $p_i \geqq 0$, $(i=1, 2, 3)$. Let M_i represent the number of extremal arcs of type i joining the points 1 and 2. Then equations (4:1) and (4:2), on page 24 of the dissertation [3] referred to above, with $S_i = M_i$ are easily obtained. Moreover, the inequalities (4:3) on the same page, with $S_i = M_i$, $(i=0, 1, 2, 3)$, follow as before except that the $>$ must be replaced by $\geqq$ since now $p_i \geqq 0$.

Summarizing the results of the preceding pages we have the following theorem.

THEOREM 3.1. *For a calculus of variations problem and a region R satisfying the hypotheses made in* §1 *above, let* 1 *and* 2 *represent two fixed points of R such that* 2 *is not an envelope arc of the family of extremals through* 1. *Further let M_i represent the number of extremal arcs joining the points* 1 *and* 2 *and each of type i and let* 3 *be the maximum type of each arc. Then the numbers M_i, $(i=0, 1, 2, 3)$, so defined must satisfy the relations* $M_0 \geqq 1$, $M_1 \geqq M_0 - 1$, $M_2 \geqq M_1 - M_0 + 1$, $M_3 = M_2 - M_1 + M_0 - 1$.

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UNIVERSITY OF WISCONSIN AT MILWAUKEE

A METHOD FOR PROVING CERTAIN ABSTRACT GROUPS TO BE INFINITE[1]

H. S. M. COXETER

1. **Introduction.** I have stated elsewhere[2] that the group (3, 3, 4; 4), defined by

$$R^3 = S^3 = (RS)^4 = (R^{-1}S^{-1}RS)^4 = 1,$$

is infinite. This fact will now be proved by showing that there is a factor group of order $24n^4$ for every positive integer n.

We shall find a closely related group of order $48n^4$, satisfying the relations $S^3 = T^2 = (ST)^8 = (S^{-1}TST)^6 = 1$, which have been studied by Brahana;[3] but there is no overlapping, since his "subgroup H" is not invariant in our case, although there still is an abelian invariant subgroup of index 48. In fact, it was the search for such a subgroup that led to the simple treatment given here.

Section 7 is inserted for its intrinsic interest, and can be omitted without impairing the proof of the main result (§8).

2. **A group of order n^4.** Consider the direct product of two cyclic groups of order n. Since the defining relations $M_1^n = M_2^n = M_1^{-1}M_2^{-1}M_1M_2 = 1$ imply $(M_1M_2)^n = 1$, they may be put into the form[4]

$$M_1^n = M_2^n = M_3^n = M_1M_2M_3 = M_3M_2M_1 = 1. \tag{1}$$

Hence the direct product of four cyclic groups of order n is defined by

$$\begin{aligned} M_i^n = M_1M_2M_3 = M_3M_2M_1 = N_j^n = N_1N_2N_3 = N_3N_2N_1 = 1, \\ M_iN_j = N_jM_i, \qquad i, j = 1, 2, 3. \end{aligned} \tag{2}$$

3. **A group of order $4n^4$.** These relations continue to hold when M_i is replaced by N_i, and N_j by M_j^{-1}. We now enlarge the group of order n^4 by adjoining an operator A, of period four, which transforms it according to this automorphism. The extra relations that have to be added to (2) are

$$A^4 = 1, \qquad A^{-1}M_iA = N_i, \qquad A^{-1}N_jA = M_j^{-1}.$$

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[1] Presented to the Society, September 6, 1938. The enumerative method described in the abstract (this Bulletin, 44-9-331) seems to be effective only in those cases where more orthodox methods are equally effective.

[2] Coxeter [2, p. 101, second footnote].

[3] Brahana [1].

[4] In the notation of Coxeter [2, p. 87], this is $(n, n, n; 1)$.

The enlarged group,[5] of order $4n^4$, may be put into a symmetrical form by defining $B=AM_3$, $C=N_2A$, and eliminating the M's and N's by means of the relations

$$B^{-1}C = M_3^{-1}M_2^{-1} = M_1, \qquad C^{-1}A = A^{-1}N_2^{-1}A = M_2, \qquad A^{-1}B = M_3,$$
$$BC^{-1} = N_3^{-1}N_2^{-1} = N_1, \qquad CA^{-1} = N_2, \qquad AB^{-1} = AM_3^{-1}A^{-1} = N_3.$$

The result is

$$\begin{aligned} A^4 = B^4 = C^4 = (BC)^2 = (CA)^2 = (AB)^2 &= A^{-1}BC^{-1}AB^{-1}C \\ = (B^{-1}C)^n = (C^{-1}A)^n = (A^{-1}B)^n &= 1. \end{aligned} \tag{3}$$

These relations imply

$$\begin{aligned} (A^2B^{-1}C)^2 &= (B^2C^{-1}A)^2 = (C^2A^{-1}B)^2 = (A^{-1}B^2C)^2 = (B^{-1}C^2A)^2 \\ &= (C^{-1}A^2B)^2 = (A^2B^2C^2)^2 = (ABC)^4 = 1. \end{aligned}$$

In detail,

$$\begin{aligned} (A^2B^{-1}C)^2 &= A^{-1}\cdot A^{-1}B^{-1}\cdot CA\cdot AB^{-1}C = A^{-1}\cdot BA\cdot A^{-1}C^{-1}\cdot AB^{-1}C \\ &= A^{-1}BC^{-1}AB^{-1}C = 1, \\ (A^{-1}B^2C)^2 &= A^{-1}B\cdot BC\cdot A^{-1}B^{-1}\cdot B^{-1}C = A^{-1}B\cdot C^{-1}B^{-1}\cdot BA\cdot B^{-1}C \\ &= A^{-1}BC^{-1}AB^{-1}C = 1, \\ (A^2B^2C^2)^2 &= A^2B\cdot BC\cdot CA^2B^2C^2 = A^2BC^{-1}\cdot B^{-1}CA^2\cdot B^2C^2 \\ &= CB^{-1}A^{-2}\cdot A^{-2}C^{-1}B\cdot B^2C^2 = C\cdot B^{-1}C^{-1}B^{-1}\cdot C^2 = C^4 = 1, \\ (ABC)^4 &= (AB\cdot CA\cdot BC)^2 = (B^{-1}A^{-1}\cdot A^{-1}C^{-1}\cdot C^{-1}B^{-1})^2 \\ &= (BC^2A^2B)^{-2} = 1. \end{aligned}$$

As one of the relations $M_i^n=1$ is superfluous in (1), so one of the consequent relations $(B^{-1}C)^n=(C^{-1}A)^n=(A^{-1}B)^n=1$ is superfluous in (3), say $(C^{-1}A)^n=1$. In terms of A, B, C and $(ABC)^{-1}$, (3) takes the form

$$\begin{aligned} A^4=B^4=C^4=D^4 = ABCD = (BC)^2 = (CA)^2 = (AB)^2 &= (BD)^2 \\ = (A^{-1}B)^n = (B^{-1}C)^n = (C^{-1}D)^n = (D^{-1}A)^n &= 1, \end{aligned} \tag{4}$$

implying

$$BC^{-1}AB^{-1}CA^{-1} = BC^{-1}\cdot AB\cdot B\cdot BC\cdot A^{-1} = (BC^{-1}B^{-1}A^{-1})^2 = (BD)^2 = 1.$$

Of course, the relations $(C^{-1}D)^n=(D^{-1}A)^n=1$ (inserted for the sake of symmetry) are superfluous; in fact, the rest of (4) implies $AB^{-1}\cdot A^{-1}C=A^{-1}C\cdot AB^{-1}$, whence $(AD^{-1})^n=(A^2BC)^n=(AB^{-1}\cdot A^{-1}C)^n=(AB^{-1})^n(A^{-1}C)^n=1$, and similarly $(D^{-1}C)^n=(ABC^2)^n=(AC^{-1}\cdot B^{-1}C)^n=(AC^{-1})^n(B^{-1}C)^n=1$.

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[5] There is an intermediate group, of order $2n^4$, generated by A^2, AB, B^2, BC, C^2, CA, and defined by $T_1^2=T_2^2=T_3^2=T_4^2=T_5^2=T_6^2=T_1T_2T_3T_4T_5T_6=(T_1T_2)^n=(T_2T_3)^n=(T_3T_4)^n=(T_4T_5)^n=(T_5T_6)^n=(T_6T_1)^n=(T_iT_jT_k)^2=1$, $(i<j<k)$.

4. **A group of order** $8n^4$. The relations (4) continue to hold when A, B, C and D are replaced by C^{-1}, D^{-1}, D^2A $(=DA^{-1}D^{-1})$ and BC^2 $(=C^{-1}B^{-1}C)$. For, BC, CA, AB, BD, $A^{-1}B$, $B^{-1}C$, $C^{-1}D$, $D^{-1}A$ are then replaced by DA, DB, DC, AC, CD^{-1}, $D^{-1}A$, BA^{-1}, $C^{-1}B$. Let $\mathfrak{P}$ denote this automorphism. The repeated automorphism $\mathfrak{P}^2$ replaces A, B, C, D by DAD^{-1}, $C^{-1}BC$, BCB^{-1}, $A^{-1}DA$, and so is equivalent to transformation by BC or DA. We now enlarge the group of order $4n^4$ by adjoining an operator P, whose square is BC, and which transforms the group according to the automorphism $\mathfrak{P}$. The extra relations that have to be added to (4) are

$$P^2 = BC,\ P^{-1}AP = C^{-1},\ P^{-1}BP = D^{-1},\ P^{-1}CP = D^2A,\ P^{-1}DP = BC^2.$$

Defining $Q=AP$, so that $QP^{-1}=A$, $PQ=B$, $Q^{-1}P=C$, $P^{-1}Q^{-1}=D$, we obtain the enlarged group, of order $4n^4$, in the form

$$\begin{aligned}(5)\qquad P^4 = Q^4 = (PQ)^4 = (P^{-1}Q)^4 &= (P^{-1}Q^{-1}PQ)^2\\ &= (PQ^{-1}PQ)^n = (P^{-1}QPQ)^n = 1.\end{aligned}$$

Since

$$\begin{aligned}(PQ^2)^2 &= PQ\cdot QP\cdot Q^2 = BD^{-1}AB = BAB\cdot CA\cdot B\\ &= A^{-1}\cdot A^{-1}C^{-1}\cdot B = (B^{-1}CA^2)^{-1},\\ (P^2Q)^2 &= P^2\cdot QP\cdot PQ = BCD^{-1}B = B\cdot CA\cdot BCB\\ &= B\cdot A^{-1}C^{-1}\cdot C^{-1} = (C^2AB^{-1})^{-1},\\ P^{-1}Q^2P^{-1} &= C^{-1}A,\end{aligned}$$

the relations (5) must imply $(PQ^2)^4=(P^2Q)^4=(P^2Q^2)^n=1$. In terms of P, Q and $(PQ)^{-1}$, we have, therefore,

$$\begin{aligned}(6)\qquad O^4 = P^4 = Q^4 = OPQ = (QPO)^2 &= (P^{-1}Q)^4 = (Q^{-1}O)^4\\ = (O^{-1}P)^4 = (P^2Q^2)^n &= (Q^2O^2)^n = (O^2P^2)^n = 1.\end{aligned}$$

5. **A group of order** $24n^4$. To this group of order $8n^4$ we adjoin an operator R, of period three, which transforms the three generators according to a cyclic permutation.[6] The substitution $O=RQR^{-1}$, $P=R^{-1}QR$ gives us the enlarged group, of order $24n^4$, in the form

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[6] Compare Coxeter [2, p. 96].

$$(7)\quad Q^4 = R^3 = (QR)^3 = (Q^{-1}R)^6 = (Q^{-1}R^{-1}QR)^4 = (Q^2R^{-1}Q^2R)^n = 1.$$

In terms of R and $(QR)^{-1}$, this becomes

$$(8)\quad R^3 = S^3 = (RS)^4 = (R^{-1}S)^6 = (R^{-1}S^{-1}RS)^4 = (R^{-1}SRS^{-1}RS)^n = 1.$$

6. **A group of order** $48n^4$. Finally, to the group of order $24n^4$ we adjoin an involutory operator T which interchanges R and S, obtaining

$$(9)\quad \begin{aligned} S^3 = T^2 = (ST)^8 = (S^{-1}TST)^6 &= [(S^{-1}T)^2(ST)^2]^4 \\ &= [(ST)^4T]^{2n} = 1. \end{aligned}$$

For, if $T^2=1$ and $R=TST$, we have $R^{-1}S^{-1}RS=TS^{-1}TS^{-1}TSTS$ and $R^{-1}SRS^{-1}RS=(TS^{-1}TSTS)^2=(TST\cdot T\cdot STSTS)^2$.

In terms of ST and T, (9) takes the form

$$(10)\quad U^8 = T^2 = (UT)^3 = (U^{-1}TUT)^6 = (U^{-2}TU^2T)^4 = (U^4T)^{2n} = 1.$$

7. **Other related groups.** Several further groups with simple defining relations can be derived from those obtained above. For instance, adjoining to (3) an operator V which cyclically permutes A, B, C, we obtain the group

$$(11)\quad V^3 = A^4 = (V^{-1}AVA)^2 = (VA^{-1}VA)^3 = (V^{-1}A^{-1}VA)^n = 1,$$

of order $12n^4$, and we deduce that these relations imply $(VA)^{12} =(VA^2)^6=1$.

Again, adjoining to (4) an operator X which cyclically permutes A, B, C, D, we obtain

$$X^4 = A^4 = (XA)^4 = (X^2A)^4 = (X^{-1}AXA)^2 = (X^{-1}A^{-1}XA)^n = 1,$$

of order $16n^4$. In terms of X and XA, this becomes

$$(12)\quad X^4 = Y^4 = (XY)^4 = (X^{-1}Y)^4 = (X^2Y^2)^2 = (X^{-1}Y^{-1}XY)^n = 1.$$

Concerning (5), it is natural to ask whether the periods of $PQ^{-1}PQ$ and $P^{-1}QPQ$ are inevitably equal. The rather surprising answer is, as we shall see, that by leaving one of them unrestricted we only double the order of the group. Since O, P, Q are interchangeable, this means that the group

$$(13)\quad P^4 = Q^4 = (PQ)^4 = (P^{-1}Q)^4 = (P^{-1}Q^{-1}PQ)^2 = (P^2Q^2)^n = 1$$

is of order $16n^4$, like (12).

To build up such a group, we begin with the direct product of two cyclic groups of orders $2n$ and n (generated by M_2 and M_3), which can be written in a form resembling (1):

$$M_1^n = M_2^n, \qquad M_3^n = M_1M_2M_3 = M_3M_2M_1 = 1.$$

Instead of (2) we take the group

$$\begin{gathered}M_1^n = M_2^n = N_1^n = N_2^n,\\ M_3^n = M_1M_2M_3 = M_3M_2M_1 = N_3^n = N_1N_2N_3 = N_3N_2N_1 = 1,\\ M_iN_j = N_jM_i, \qquad i, j = 1, 2, 3,\end{gathered}$$

whose order is $2n^4$ since its general operator can be expressed as

$$M_1^pM_2^qN_1^rN_2^sZ^t, \qquad 0 \leqq p, q, r, s < n;\ t = 0 \text{ or } 1,$$

where $Z = M_1^n$.

Instead of (3) and (4), we derive two equivalent definitions for a certain group of order $8n^4$: first[7]

$$\begin{aligned}A^4 = B^4 = C^4 = (BC)^2 = (CA)^2 = (AB)^2 = A^{-1}BC^{-1}AB^{-1}C\\ = (AC^{-1})^n(B^{-1}C)^n = (A^{-1}B)^n = 1,\end{aligned}$$

and second

$$\begin{aligned}A^4 = B^4 = C^4 = D^4 = ABCD = (BC)^2 = (CA)^2 = (AB)^2 = (BD)^2\\ = (A^{-1}B)^n = (C^{-1}D)^n = 1.\end{aligned}$$

Finally, instead of (5) we obtain the group, of order $16n^4$,

$$P^4 = Q^4 = (PQ)^4 = (P^{-1}Q)^4 = (P^{-1}Q^{-1}PQ)^2 = (PQ^{-1}PQ)^n = 1.$$

8. **Conclusions regarding infinite groups.** The consistency of (8) for all values of n shows that the group $(3, 3 \mid 4, 6; 4)$, defined by $R^3 = S^3 = (RS)^4 = (R^{-1}S)^6 = (R^{-1}S^{-1}RS)^4 = 1$, is infinite. The "larger" groups[8] $(3, 3 \mid 4, 6)$, $(3, 3, 4; 4)$ are infinite *a fortiori*. Similarly, (5) establishes infinite order for $(4, 4 \mid 4, 4; 2)$, and thence for[9] $(4, 4, 4; 2)$.

9. **Comparison with Brahana's groups.** The infinite group $(2, 3, 8; 6)$, of which (9) is a factor group, has been investigated by Brahana.[10] His operators T_1, T_2, T_3 are easily recognized in our factor group as BC, AB, D^2. Since $T_2T_3T_1 = CD \cdot D^2 \cdot DA = CA$, the subgroup

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[7] These relations imply $(B^{-1}C)^n = (C^{-1}A)^n = (BC^{-1})^n = (CA^{-1})^n$. In detail, $(C^{-1}A)^{-n}(B^{-1}C)^n = (C^{-1}A)^{-n}(CA^{-1})^n = A^{-1}(AC^{-1})^{-n}(ACA^2)^nA = A^{-1}(B^{-1}C)^n(C^{-1}A)^nA = A^{-1}(B^{-1}A)^nA = 1$.

[8] Coxeter [**2**, pp. 86, 101].

[9] Coxeter [**2**, p. 97]. By the method of Coxeter [**2**, p. 90, §2.5], (12) establishes infinite order for $(4, 8 \mid 2, 4; 4)$. This raises an interesting question as to the finite or infinite order of $(4, 7 \mid 2, 4; 4)$.

[10] Brahana [**1**, p. 892].

$\{T_1, T_2, T_3\}$ is $\{BC, CA, AB\}$. This subgroup, being[11] $((n, n, n; 2))$, of order $2n^2$, is of index $24n^2$. It is not invariant,[12] since, if it were, its index would be just 24. Hence (9) is not one of the groups treated in Brahana's main theorems, but is a first step towards the "large undertaking" mentioned in his final paragraph.[13]

REFERENCES

1. H. R. Brahana, *On the groups generated by two operators of orders two and three whose product is of order eight*, American Journal of Mathematics, vol. 53 (1931), pp. 891–901.
2. H. S. M. Coxeter, *The abstract groups* $G^{m,n,p}$, Transactions of this Society, vol. 45 (1939), pp. 73–150.

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[11] Coxeter [2, p. 143].

[12] Brahana [1, p. 892].

[13] Brahana [1, p. 901].

ANOMALOUS PLANE CURVE SYSTEMS ASSOCIATED WITH SINGULAR SURFACES[1]

T. R. HOLLCROFT

1. Introduction. A plane curve system of order n and genus p, with δ nodes, κ cusps, ι statangents, τ bitangents, is of first virtual dimension d_0, of virtual dimension d_0', and of effective dimension d. For such a system, $d_0' = 3n+p-\kappa-1 = 3m+p-\iota-1$. If the curve system has only distinct nodes and cusps, $d_0 = d_0'$; if higher singularities, $d_0 < d_0'$.

For any irreducible, continuous, complete curve system, $d \geqq d_0$. If $d = d_0$, the curve system is called homalous, if $d > d_0$, anomalous. The anomaly A of a plane curve system is defined by the relation $A = d - d_0$. The above definitions were introduced by B. Segre.[2]

The sections by a plane π of the tangent cones to a nonsingular surface of order ν from the points of S_3 constitute a continuous plane curve system with the characteristics:

$$n = \nu(\nu - 1), \qquad \kappa = \nu(\nu - 1)(\nu - 2),$$
$$\delta = (1/2)\nu(\nu - 1)(\nu - 2)(\nu - 3).$$

B. Segre[3] has shown that for this curve system,

$$d_0 = (5/2)\nu(\nu - 1), \qquad d = (1/6)(\nu + 1)(\nu + 2)(\nu + 3) - 5,$$
$$A = d - d_0 = (1/6)(\nu - 2)(\nu - 3)(\nu - 4).$$

Hence, for $\nu \geqq 5$, this plane curve system is anomalous.

The purpose of the present paper is to ascertain the dimensions and anomaly of plane sections of tangent cones to certain singular surfaces.

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[1] Presented to the Society, April 16, 1938.

[2] B. Segre, *Esistenza e dimensione di sistemi continui distinti di curve piane algebriche con dati caratteri*, Rendiconti dell'Accademia dei Lincei, (6), vol. 10 (1929), pp. 31–38. The adjective "irregolare" used by B. Segre to describe the plane curve system for which $d > d_0$ has two English equivalents "irregular" and "anomalous." The first translation is the more natural one. Since it has been found, however, that similar systems of surfaces exist and since the term "irregular" has long had a different definite meaning for surfaces, another term is necessary for surfaces and, therefore, should also be used for plane curves. Homalous and anomalous are Greek antonyms. Only the latter has heretofore been used in English.

[3] B. Segre, *Sulla caratterizzazione delle curve di diramazione dei piani multipli generali*, Memorie delle Reale Accademia d'Italia, Classe di Scienze Fisische, Matematiche e Naturali, vol. 1 (1930), pp. 5–31.

2. **Surfaces with multiple points.** Let the surface F of order ν have the point P of multiplicity k, such that the plane section of the tangent cone to F at P is a nonsingular curve of order k. The tangent cone to F from an arbitrary point has a multiple generator of order $k(k-1)$ through P. This multiple generator reduces the number of distinct cuspidal generators by $k(k-1)(k-2)$ and the number of distinct nodal generators by $(1/2)k(k-1)(k-2)(k-3)$.

The plane sections C by a plane π of the tangent cones to F from points of S_3 consist of plane curves each of which has a $k(k-1)$-fold point at the intersection of π with the multiple generator. This plane curve system C has the additional characteristics

$$\begin{aligned} n &= \nu(\nu-1), \qquad m = \nu(\nu-1)^2 - k(k-1)^2, \\ p &= (1/2)\nu(\nu-1)(2\nu-5) - (1/2)k(k-1)(2k-3), \\ \kappa &= \nu(\nu-1)(\nu-2) - k(k-1)(k-2), \\ \delta &= (1/2)\nu(\nu-1)(\nu-2)(\nu-3) - (1/2)k(k-1)(k-2)(k-3), \\ d_0 &= (5/2)\nu(\nu-1) - (3/2)k(k-1) + 2. \end{aligned}$$

In the above, δ and κ are the numbers of distinct nodes and cusps of the plane sections. The $k(k-1)$-fold point contains also the equivalent of $(1/2)k(k-1)(k^2-k-1)$ nodes.

To obtain the effective dimension of C, the effective dimension of F must be found. The invariant postulation[4] of a k-fold point on F is $(1/6)k(k+1)(k+2)-3$. Since the above plane curve system may be the plane section of the tangent cone to ∞^4 surfaces, the effective dimension d of C is

$$d = (1/6)(\nu+1)(\nu+2)(\nu+3) - (1/6)k(k+1)(k+2) - 2.$$

Hence

$$A = (1/6)(\nu-2)(\nu-3)(\nu-4) - (1/6)(k-1)(k-2)(k-3).$$

The above formulas are derived for one multiple point of order k. If the surface has α distinct, independent, multiple points of orders k_i, the above formulas hold with the sum sign. For example, for such a surface,

$$A = (1/6)(\nu-2)(\nu-3)(\nu-4) - (1/6)\sum_{i=1}^{\alpha}(k_i-1)(k_i-2)(k_i-3).$$

From the above expression for A, there results:

(1) A surface may have any number of independent double or

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[4] T. R. Hollcroft, *Invariant postulation*, this Bulletin, vol. 36 (1930), p. 421.

triple points without affecting the anomaly of plane sections of the tangent cone.

(2) The plane sections of the tangent cones to a monoid ($\alpha=1$, $k=\nu-1$) are always homalous plane curve systems.

Since $A\neq 0$, this expression for A ($k_i=k$) defines the following limit to the number α of distinct independent k-fold points, $k\geqq 4$, on an irreducible surface of order ν:

$$\text{(A)} \qquad \alpha \leqq \frac{(\nu-2)(\nu-3)(\nu-4)}{(k-1)(k-2)(k-3)} \cdot$$

The smallest limit obtained heretofore results from the fact that the genus of the plane section of the tangent cone to an irreducible surface cannot be negative. This limit[5] is

$$\text{(B)} \qquad \alpha \leqq \frac{(\nu-2)(2\nu^2-3\nu-1)}{k(k-1)(2k-3)} \cdot$$

Limit (A) is smaller than limit (B) for a given value of $k\geqq 4$ and $\nu=k+\beta$, where

$$\beta \leqq \frac{3(6k-k^2-6)+(121k^4-832k^3+1988k^2-1920k+612)^{1/2}}{2(7k-12)} \cdot$$

A very close approximation to this limiting value of β is given by $\beta\leqq(1/7)(4k-3)$.

It has been assumed that the multiple points are unrelated in position on the surface and that they are associated with sets of independent invariants among the coefficients of the surface.

The nodes and cusps of the system C lie on adjoint curves. The adjoint curves of the lowest order are the following:

(1) The adjoint curves of order $(\nu-1)(\nu-2)$ pass simply through the $\delta+\kappa$ distinct double points and have a multiple point of order $(k_i-1)(k_i-2)$ at each $k_i(k_i-1)$-fold point.

(2) The adjoint curves of order $\nu(\nu-2)$ pass simply through the nodes, are tangent to C at each cusp, and have a multiple point of order $k_i(k_i-2)$ at each $k_i(k_i-1)$-fold point.

If the multiple points have cones whose plane sections are singular curves, the values of d and d_0 can be found in a given case, but a general formula that will include all such cases is not feasible.

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[5] T. R. Hollcroft, *Limits for multiple points and curves of surfaces*, Tôhoku Mathematical Journal, vol. 30 (1928), pp. 116–117. Limit (1) derived on page 116 depends upon the Lefschetz postulate and, therefore, does not hold.

3. **Surfaces with singular curves.** Consider the surface F of order ν and class ν' with a nodal curve of order b and a cuspidal curve of order c. The other surface characteristics to be used are the following:

a, order of tangent cone = class of plane section;
δ, number of nodal generators of tangent cone;
κ, number of cuspidal generators of tangent cone;
σ, class of cuspidal developable;
σ', order of parabolic curve;
j, number of pinch points on nodal curve;
ω, number of tacnodal points on cuspidal curve;
χ, number of cubic nodes on cuspidal curve;
Δ, effective dimension of surface.

From the Cayley-Zeuthen equations, there results

$$(1)\quad a = \nu(\nu - 1) - 2b - 3c, \quad 2\delta + 4\kappa = a(a - 2) + \sigma + 2j + 3\chi + 3\omega.$$

Since, for the plane sections C of the tangent cone,

$$d_0 = (1/2)a(a + 3) - \delta - 2\kappa,$$

there results, from the above,

$$(2)\qquad d_0 = (1/2)(5a - \sigma - 2j - 3\chi - 3\omega).$$

For a nonsingular surface, $d_0 = 5a/2$.

No such general formula exists for the effective dimension d of this curve system C. Further and more specific characteristics of the double curves must be given. For example, the dimension of a space curve may involve properties of the two surfaces of which it is the partial or complete intersection. In order to determine the effective dimension Δ of F, the intersection characteristics of the nodal and cuspidal curves must be known.

In particular cases, when sufficient data is given, Δ can be found, and therefore d. Nodal and cuspidal lines will be investigated.

4. **Surfaces with a nodal line.** The surface F of order ν has a nodal line both of whose tangent planes are torsal. The characteristics of F are

$$a = \nu(\nu - 1) - 2, \quad b = 1, \quad j = 2(\nu - 2), \quad \nu' = (\nu - 2)(\nu^2 - 6).$$

The virtual dimension of the plane section C of the tangent cone is $d_0 = (1/2)(5a - 2j) = (1/2)(\nu - 2)(5\nu + 1)$.

The postulation of a nodal line on F is $3\nu + 1$. Its invariant postulation[6] is $3(\nu - 1)$.

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[6] *Invariant postulation*, loc. cit., p. 424.

Hence, for F

$$\Delta = (1/6)(\nu + 1)(\nu + 2)(\nu + 3) - 3\nu + 2.$$

The effective dimension of C is $d = \Delta - 4$, hence

$$d = (1/6)(\nu + 1)(\nu + 2)(\nu + 3) - 3\nu - 2.$$

The anomaly of the system C is

$$A = d - d_0 = (1/6)\nu(\nu - 4)(\nu - 5).$$

If one tangent plane of the nodal line is fixed and the other torsal, the line has $2(\nu-2)$ pinch points that coincide in pairs. The computation is the same as above for both planes torsal.

5. **Surfaces with a cuspidal line.** A surface F may have two types of cuspidal lines, (1) the tangent plane to F is fixed, that is, is the same for all points of the line; (2) the tangent plane to F along the line is torsal.

(1) The surface F of order ν has a cuspidal line, such that the same plane is tangent to F at every point of this line. F has the characteristics

$$a = \nu(\nu - 1) - 3, \quad \nu' = (\nu - 3)(\nu^2 + \nu - 8),$$
$$\omega = \nu - 2, \quad \chi = \nu - 3, \quad \kappa = (\nu - 3)(\nu^2 - 4), \quad d_0 = (1/2)\nu(5\nu - 11),$$
$$d = \Delta - 4 = (1/6)(\nu + 1)(\nu + 2)(\nu + 3) - 4\nu - 1,$$
$$A = (1/6)\nu(\nu - 4)(\nu - 5).$$

(2) The surface F of order ν with a cuspidal line whose tangent plane is torsal has the characteristics

$$a = \nu(\nu - 1) - 3, \quad \nu' = (\nu - 3)(\nu^2 + \nu - 8),$$
$$\chi = \nu, \quad \omega = \nu - 4, \quad \sigma = 1, \quad \kappa = (\nu - 3)(\nu^2 - 4) + 4,$$
$$d_0 = (1/2)\nu(5\nu - 11) - 2,$$
$$d = \Delta - 4 = (1/6)(\nu + 1)(\nu + 2)(\nu + 3) - 4\nu - 3,$$
$$A = (1/6)\nu(\nu - 4)(\nu - 5).$$

In this case, the cuspidal developable is a point, and is, therefore, of order zero and class $\sigma = 1$.

Although the values of certain characteristics are different, the value of A is the same for both types of cuspidal lines.

6. **Summary for double lines.** The results of the two preceding sections lead to the following conclusion:

If an algebraic surface F of order $\nu \geqq 4$ has a double line (either nodal or cuspidal), the anomaly A of the plane sections C of a tangent cone to F is $A = (1/6)\nu(\nu-4)(\nu-5)$. As is evident from this expression for A, plane curve systems C associated with a surface having a double line are anomalous only for $\nu \geqq 6$.

When $\nu = 3$, the above formula yields $A = 1$. However, a cubic surface with a nodal line is ruled, and a cubic surface with a cuspidal line is a cone. The treatment given in the two preceding sections and the resulting expression for A do not apply to such surfaces.

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ON THE STABILITY OF THE LAMINAR FLOW OF A VISCOUS FLUID[1]

RUDOLPH E. LANGER

The problem of the effect of two-dimensional first-order disturbances upon the laminar flow of an incompressible viscous fluid is known to be fundamental to the analysis of the phenomenon of turbulence. This discussion is concerned with such a problem in the case of a flow which takes place parallel to and between parallel plane boundaries. If the direction normal to these boundaries is designed by y, and that of the flow by x, the unit of length and the origin of coordinates may be chosen so that the boundary planes are given by the equations $y = 1$ and $y = -1$. It is to be assumed then that the velocity of the undisturbed flow depends only on y, and is given by a function $U(y)$, which is suitably differentiable and nonnegative, which is nonincreasing as to $|y|$, and which vanishes at the boundaries. If the maximum velocity is chosen as the unit, as will be assumed, it follows that

$$U(0) = 1, \qquad U(1) = 0, \qquad U(-y) \equiv U(y).$$

The disturbance imposed upon this flow is to be taken as an elementary wave of length $2\pi/\alpha$, in the direction of flow.

The problem as stated is known[2] to admit of formulation as the differential boundary problem

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[1] Presented to the Society, September 7, 1939.

[2] C. L. Pekeris, *On the stability problem in hydrodynamics*, I, Proceedings of the Cambridge Philosophical Society, vol. 32 (1936), p. 55, and II, Journal of the Aeronautical Sciences, vol. 5 (1938), p. 236.

$$(1)\qquad \begin{aligned} &f^{iv}(y) - 2\alpha^2 f''(y) + \alpha^4 f(y) \\ &\qquad + i\alpha R[\{U(y) - C\}\{f''(y) - \alpha^2 f(y)\} - U''(y)f(y)] = 0, \\ &\qquad\qquad f(1) = f(-1) = f'(1) = f'(-1) = 0. \end{aligned}$$

Of the constants involved in this, α has already been defined and R is the Reynold's number. C is a complex parameter for characteristic values of which the boundary problem admits of solution. In terms of such a solution the stream function of the associated flow is given by the formula $f(y)e^{i\alpha(Ct-x)}$. This flow is accordingly stable if and only if the imaginary component of the characteristic value C is positive.

In the special case of a parabolic flow, that is, $U(y)=1-y^2$, the problem has recently been considered by Pekeris.[3] His deductions apply in the main to cases in which the Reynold's number is suitably small, it being shown in particular that stability always obtains if this number does not exceed the smaller of the numbers 2α and α^2. The developments of the smaller characteristic values in series of powers of αR are analyzed, and it is also shown that a symmetrical flow is stable in the limiting case as $\alpha \to 0$. The case of more general velocity profiles $U(y)$ has been considered, among others by Tollmien.[4] His work has special reference to cases in which the Reynold's number is large enough to make $1/\alpha R$ generally negligible, and is focused upon characteristic values with imaginary components that are small.

The present discussion differs widely from those cited. It is of an asymptotic nature and is concerned with the characteristic values for which the product $\alpha R|C|$ is large. The results may be regarded, therefore, as applying to flows associated with large characteristic values if αR is small or moderate, and to characteristic values which are large, moderate, or even small if αR is large. The values of α and R, which are real and positive, are regarded as either fixed or variable. In the latter case the ranges of α, and of $1/\alpha R$, are to be taken as bounded. It will be shown that any flow for which these hypotheses are fulfilled, and for which the imaginary component of C is not very small, is stable.

In the complex plane of the parameter C, let the region consisting of the infinite strip

$$(2)\qquad 0 < \Re(C) < 1, \qquad \Im(C) > 0,$$

plus an arbitrarily small but fixed neighborhood of the segment of the axis of reals from 0 to 1, be shaded, as is shown in Fig. 1. It will be assumed throughout the pending discussion that C lies in the unshaded region, and it will be shown that under this assumption no solution of the boundary problem (1) exists. The inference will then be clear that all characteristic values lie in the shaded region.

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[3] Loc. cit. Reference is made to these papers for further citations of the literature.

[4] W. Tollmien, *Ein allgemeines Kriterium der Instabilität laminarer Geschwindigkeitsverteilungen*, Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, 1935, p. 79.

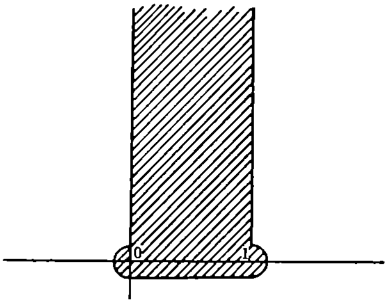

FIG. 1

In terms of the abbreviations

$$
\begin{aligned}
\rho &= |\alpha RC|^{1/2} \exp\left\{i(\pi/4 + (1/2)\arg C)\right\}, \\
\phi(y) &= [1 - U(y)/C]^{1/2}, \qquad \phi(1) = 1, \qquad (3)\\
\kappa(y) &= U''(y)/C,
\end{aligned}
$$

the differential equation (1) takes the form

$$(4) \quad f^{\text{iv}} - \left\{\rho^2\phi^2(y) + 2\alpha^2\right\}f'' + \left\{\rho^2[\alpha^2\phi^2(y) - \kappa(y)] + \alpha^4\right\}f = 0.$$

The range of the variable is $-1 \leqq y \leqq 1$. Since $U(y)$ is real and takes only values from 0 to 1, whereas C is located in the unshaded portion of the complex plane, the functions $\kappa(y)$, $\phi(y)$, $1/\phi(y)$ are bounded.

If an expression

$$(5) \qquad \exp\left\{\rho\int_0^y \phi(y)dy\right\}P(y, \rho),$$

in which $P(y, \rho)$ is an undetermined power series in $1/\rho$, that is,

$$(6) \qquad P(y, \rho) \equiv p_0(y) + p_1(y)/\rho + p_2(y)/\rho^2 + \cdots,$$

is substituted for f, the left-hand member of (4) takes the form of an exponential multiplied by a power series in $1/\rho$ with coefficients that are functions of y. By equating these coefficients to zero a sequence of differential equations is obtained which involve the functions $p_j(y)$, $j = 0, 1, 2, \cdots$, and from which these functions are successively determinable by integrations. The constants which enter by virtue of these integrations may be chosen, and are to be chosen, so that the functions $p_j(y)$ with even subscripts are even functions of y, and those with odd subscripts are odd functions of y, and so that $p_0(1)=1$. Those constants which remain undetermined by these stipulations may be chosen at pleasure, since their choice has no effect upon the deductions which are to be made. With the determinations described, the expression (5) is a formal solution of the differential equation (4), and formally also $P(-y,\ \rho)\equiv P(y,\ -\rho)$. Inasmuch as the equation (4) involves ρ only in the form ρ^2, it is clear also that with this determination of $P(y,\ \rho)$ the expression

$$\exp\left\{-\rho\int_0^y \phi(y)dy\right\}P(y,\ -\rho) \tag{7}$$

is likewise a formal solution of (4).

If, now, in a similar manner, an undetermined series

$$S(y,\ \rho)\equiv s_0(y)+s_1(y)/\rho^2+s_2(y)/\rho^4+\cdots$$

is substituted for f, the left-hand member of (4) takes the form of a power series in $1/\rho^2$. Equating its coefficients to zero leads to the sequence of relations

$$\begin{aligned} &\phi^2 s_0'' - (\alpha^2\phi^2-\kappa)s_0 = 0,\\ &\phi^2 s_j'' - (\alpha^2\phi^2-\kappa)s_j = s_{j-1}^{\text{iv}} - 2\alpha^2 s_{j-1}'' + \alpha^4 s_{j-1},\quad j=1,\ 2,\ 3,\ \cdots . \end{aligned} \tag{8}$$

These equations may be solved by functions $s_j(y)$ determined successively as odd functions of y, say as functions $s_{j,0}(y)$, $j=0,\ 1,\ 2,\ \cdots$. They may, however, also be solved by functions determined successively as even functions of y, say as functions $s_{j,e}(y)$, $j=0,\ 1,\ 2,\ \cdots$. If this is done, the resulting series

$$\begin{aligned} S_0(y,\ \rho) &\equiv s_{0,0}(y)+s_{1,0}(y)/\rho^2+s_{2,0}(y)/\rho^4+\cdots,\\ S_e(y,\ \rho) &\equiv s_{0,e}(y)+s_{1,e}(y)/\rho^2+s_{2,e}(y)/\rho^4+\cdots \end{aligned} \tag{9}$$

are respectively odd and even, and are formal solutions of the equation (4).

The function $U(y)$ takes on only real values from 0 to 1. For C located anywhere in the unshaded part of the plane in Fig. 1, it accordingly follows that

$$-3\pi/2 \leqq \arg\,[C-U(y)] \leqq \pi/2,$$

and hence, by (3), that

$$-\pi/2 \leqq \arg\,\rho\phi(y) \leqq \pi/2. \tag{10}$$

This, however, is a familiar sufficient condition[5] that the formal expressions (5), (7), and (9), which are in general divergent, have the property of each representing asymptotically some solution of the differential equation (4). There exist, therefore, solutions $f_j(y, \rho)$, $j=1, 2, 3, 4$, of this equation, for which the representations

$$
\begin{aligned}
f_1(y, \rho) &\sim \exp\left\{\rho\int_0^y \phi(y)dy\right\} P(y, \rho) \\
&\qquad\qquad - \exp\left\{-\rho\int_0^y \phi(y)dy\right\} P(y, -\rho), \\
f_2(y, \rho) &\sim \exp\left\{\rho\int_0^y \phi(y)dy\right\} P(y, \rho) \\
&\qquad\qquad + \exp\left\{-\rho\int_0^y \phi(y)dy\right\} P(y, -\rho), \\
f_3(y, \rho) &\sim S_0(y, \rho), \qquad f_4(y, \rho) \sim S_e(y, \rho)
\end{aligned}
\tag{11}
$$

are valid. Of these solutions f_1 and f_3 are odd functions of y, and f_2 and f_4 are even.

Any solution of the boundary problem (1) is either even or odd; that is, the flow associated with it is either symmetrical or anti-symmetrical. Let it be assumed, to begin with, that there exists such an *odd* solution. Since it satisfies the differential equation (1), it may be represented in terms of the solutions (11) in the manner $c_1f_1(y)+c_3f_3(y)$, with coefficients c_1, c_3 which are independent of y, and, since it satisfies the boundary conditions (1), this and its derivative vanish at $y=1$. Inasmuch as c_1 and c_3 are not both zero if the solution is nontrivial, it follows that the determinant

$$
\begin{vmatrix} f_1(1, \rho) & f_3(1, \rho) \\ f_1'(1, \rho) & f_3'(1, \rho) \end{vmatrix}
$$

must be zero, a relation which in virtue of the formulas (11), and the evaluations $\phi(1)=p_0(1)=1$, takes the form

$$
\exp\left\{2\rho\int_0^1 \phi(y)dy\right\} \cdot \left\{\frac{[\rho P(1, \rho) + P'(1, \rho)]S_0(1, \rho) - P(1, \rho)S_0'(1, \rho)}{[-\rho P(1, \rho) + P'(1, \rho)]S_0(1, \rho) - P(1, \rho)S_0'(1, \rho)}\right\} \sim 1.
$$

---

[5] H. L. Turrittin, *Asymptotic solutions of certain ordinary differential equations* · · · , American Journal of Mathematics, vol. 58 (1936), p. 364.

This, it is found, is reducible to the form

$$(12) \qquad \rho \int_0^1 \phi(y)dy + A_1/\rho + A_2/\rho^2 + \cdots \sim B_n i,$$

in which n is an appropriate integer, and in which

$$A_1 = p_0'(1) - s_{0,0}'(1)/s_{0,0}(1), \qquad B_n = (n + 1/2)\pi,$$

if for the case in hand $s_{0,0}(1) \neq 0$, while

$$A_1 = - s_{1,0}(1)/s_{0,0}'(1), \qquad B_n = n\pi,$$

if $s_{0,0}(1) = 0$ (and hence $s_{0,0}'(1) \neq 0$). Since $|\rho|$ is assumed to be large, the numerical values of n and B_n are evidently also large.

Let M be chosen as any sufficiently large real positive constant. The values of C may be considered then in the categories $|C| \geqq M$, and $|C| < M$, and it is convenient to do this. If $|C| \geqq M$, the expansions

$$\phi(y) = 1 - U(y)/2C + \cdots, \qquad A_1 = a_0 + a_1/C + \cdots$$

may be used, and simple calculations show that the constant a_0 is real. The relation obtained by squaring the members of (12) is then found to be

$$C - \int_0^1 U(y)dy - 2ia_0/\alpha R + O(1/C) + O(1/\rho) = B_n^2 i/\alpha R.$$

To satisfy this the value C must evidently have a positive imaginary component, and, if M has been chosen suitably large, a real component which differs so little from the value $\int_0^1 U(y)dy$ that it lies between the values 0 and 1. The relation is, therefore, impossible for the values of C here in question since they lie in the unshaded part of the plane of Fig. 1.

If $|C| < M$ on the other hand, let y_1 and y_2 be chosen in any way such that $0 < y_1 < y_2 < 1$ and $U(y_1) < 1$, $U(y_2) > 0$, and let ϵ designate the smaller of the values $1 - U(y_1)$, $U(y_2)$. Then since C lies in the unshaded part of the plane, it follows that when $y_1 \leqq y \leqq y_2$ then

$$- \pi - \tan^{-1} M/\epsilon \leqq \arg [C - U(y)] < \tan^{-1} M/\epsilon,$$

namely that

$$|\arg \rho\phi(y)| \leqq \pi/2 - (1/2) \cot^{-1} M/\epsilon.$$

Thus, on the interval (y_1, y_2)

$$\Re(\rho\phi(y)) \geqq |\rho\phi(y)| \sin ((1/2) \cot^{-1} M/\epsilon),$$

and since by (10) this real part is not negative even outside the interval concerned, it follows that

$$\Re\left\{\rho\int_0^1 \phi(y)dy\right\} \geqq |\rho| \sin((1/2)\cot^{-1} M/\epsilon)\int_{y_1}^{y_2} |\phi(y)|\, dy.$$

The leading term in the left-hand member of (12) accordingly has a real part which in magnitude is of the order of $|\rho|$. Inasmuch as the right-hand member is pure imaginary, the relation is clearly impossible.

It has thus been shown that all characteristic values for which the flow of the fluid is anti-symmetrical must lie in the shaded part of the plane of Fig. 1. In like manner the same result may be obtained for the characteristic values associated with symmetrical flows. Except for possible flows in the immediate border region between stability and instability, that is, for which the imaginary component of C is very small, it must be inferred, therefore, that any flow for which $\alpha R|C|$ is large is stable.

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A NOTE ON HERMITIAN FORMS[1]

N. JACOBSON

In this note we effect a reduction of the theory of hermitian forms of two particular types (coefficients in a quadratic field or in a quaternion algebra with the usual anti-automorphism) to that of quadratic forms. The main theorem (§2) enables us to apply directly the known results on quadratic forms. This is illustrated in the discussion in §3 of a number of special cases.

Let Φ be an arbitrary quasi-field of characteristic different from 2 in which an involutorial anti-automorphism $\alpha\rightarrow\bar{\alpha}$ is defined. For the present we do not exclude the cases where Φ is commutative and $\bar{\alpha}\equiv\alpha$ or Φ is a quadratic field with $\alpha\rightarrow\bar{\alpha}$ as its automorphism. Suppose $\mathfrak{R}$ is an n-dimensional vector space over Φ. We define a bilinear form (x, y) as a function of pairs of vectors with values in Φ, such that

$$(1)\quad \begin{aligned} (x_1+x_2, y) &= (x_1, y)+(x_2, y), & (x, y_1+y_2) &= (x, y_1)+(x, y_2),\\ (x, y\alpha) &= (x, y)\alpha, & (x\alpha, y) &= \bar{\alpha}(x, y), \end{aligned}$$

for all x, y in $\mathfrak{R}$ and α in Φ. If $x_1, x_2, \cdots, x_n$ is a basis for $\mathfrak{R}$ and $(x_i, x_j)=\alpha_{ij}$, the matrix $A=(\alpha_{ij})$ is called the matrix of (x, y) relative to this basis. By (1) it determines (x, y) as $\sum\bar{\xi}_i\alpha_{ij}\eta_j$, if $x=\sum x_i\xi_i$ and $y=\sum x_i\eta_i$. If $y_1, y_2, \cdots, y_n$ where $y_i=\sum x_j\rho_{ji}$ is a second basis for $\mathfrak{R}$ where $R=(\rho_{ij})$ is nonsingular, the matrix of (x, y) relative to this basis is $\bar{R}'AR$. We call A and $\bar{R}'AR$ cogredient. The form (x, y) is hermitian (skew-hermitian), if $(y, x)=\overline{(x, y)}$ ($(y, x)=-\overline{(x, y)}$). This is equivalent to the condition $\bar{A}'=A$ ($\bar{A}'=-A$).

It is readily seen that we may pass from the basis y_i to the x's by a sequence of substitutions of the following two types:

I. $y_i\rightarrow y_i$, $(i\neq r)$, $y_r\rightarrow y_r+y_s\theta$, $(s\neq r)$.

II. $y_i\rightarrow y_i$, $(i\neq r)$, $y_r\rightarrow y_r\theta$, $(\theta\neq 0)$.

It follows that we may pass from a matrix to any other matrix cogredient to it by a sequence of transformations of the corresponding types:

I. Addition of the sth column multiplied on the right by θ to the rth together with addition of the sth row multiplied on the left by $\bar{\theta}$ to the rth.

II. Multiplication of the rth column on the right by $\theta\neq 0$ together with multiplication of the rth row on the left by $\bar{\theta}$.

We showed in an earlier paper that any hermitian form or skew-hermitian form with $\bar{\alpha}\neq\alpha$ has a matrix in diagonal form; that is, there is a basis $u_1, u_2, \cdots, u_n$ for $\mathfrak{R}$ such that[2] $(u_i, u_j)=0$, if $i\neq j$. We call a basis of this type orthogonal and u, v orthogonal, if $(u, v)=0$. If $(u_i, u_i)=\beta_i\neq 0$ for $i\leq r$ and $(u_i, u_i)=0$ for $i>r$, we obtain the diagonal matrix

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[1] Presented to the Society, October 28, 1939.

$$(2)\qquad [\beta_1, \beta_2, \cdots, \beta_r, 0, \cdots, 0]$$

for our form.[3] The element β_1 may be taken to be any nonzero element represented by the form, that is, any element for which a u_1 exists such that $(u_1, u_1)=\beta_1$, β_2 is any element represented by (x, y) restricted to the space of vectors orthogonal to u_1, and so on. We note also that β_i may be replaced by $\bar{\gamma}_i\beta_i\gamma_i$, $(\gamma_i\neq 0)$.

The space $\mathfrak{R}_0$ generated by $u_{r+1}, u_{r+2}, \cdots, u_n$ may be characterized as the totality of vectors z, such that $(x, z)=0$ for all x. The space $\mathfrak{R}_1$ generated by $u_1, u_2, \cdots, u_r$ satisfies the condition $\mathfrak{R}=\mathfrak{R}_0+\mathfrak{R}_1$, $\mathfrak{R}_0 \cap \mathfrak{R}_1=0$. If $\mathfrak{R}_2$ is a second space of this sort, it has a basis of the form u_i+z_i, $(i=1, \cdots, r)$, and hence the matrices of (x, y) in $\mathfrak{R}_1$ and in $\mathfrak{R}_2$ are cogredient. We may therefore restrict our attention to nondegenerate forms $(\mathfrak{R}_0=0)$ and shall do so in the remainder of this note.

Two nondegenerate forms $(x, y)_1$ and $(x, y)_2$ in $\mathfrak{R}$ and $\mathfrak{R}'$ respectively are *cogredient* if there is a (1-1) correspondence $x\rightarrow x'$ between $\mathfrak{R}$ and $\mathfrak{R}'$ such that[4] $(x, y)_1=(x', y')_2$. It follows that

$$(x', (y_1+y_2)')_2 = (x', y_1'+y_2')_2$$

and hence that $(y_1+y_2)'=y_1'+y_2'$. Similarly $(y\alpha)'=y'\alpha$ and so $x\rightarrow x'$ is a linear transformation and $\mathfrak{R}$ and $\mathfrak{R}'$ have the same dimensionality. If $x_1, x_2, \cdots, x_n$ is a basis for $\mathfrak{R}$, then $x_1', x_2', \cdots, x_n'$ is one for $\mathfrak{R}'$. The matrix of $(x, y)_1$ relative to the first basis is the same as that of $(x', y')_2$ relative to the second. Hence the matrices of $(x, y)_1$ and $(x', y')_2$ relative to any bases are cogredient and conversely cogredience of the matrices implies that of the forms.

We shall suppose from now on that Φ is either a quadratic field $\Phi_0(i)$, $i^2=-\lambda$ and $\bar{\alpha}=\alpha_0-i\alpha_1$ for $\alpha=\alpha_0+i\alpha_1$ or that $\Phi=\Phi_0(i, j)$ is a quaternion algebra in which $i^2=-\lambda$, $j^2=-\mu$, $k=ij=-ji$ and $\bar{\alpha}=\alpha_0-i\alpha_1-j\alpha_2-k\alpha_3$ for $\alpha=\alpha_0+i\alpha_1+j\alpha_2+k\alpha_3$. We suppose also that (x, y) is hermitian. Then $(x, x)\ \varepsilon\ \Phi_0$ and any β in (2) may be replaced by $\beta N(\gamma)$, $N(\gamma)=\bar{\gamma}\gamma$. Let Φ_0' be the multiplicative group of nonzero elements in Φ_0, Φ_0^* the subgroup of norms, and $\Gamma=\Phi_0'/\Phi_0^*$. A determinant for any hermitian matrix A has been defined by E. H. Moore.[5] We recall that, if a matrix B has the form (2) with $r=n$, then det $B=\beta_1\beta_2\cdots\beta_n$ and, if $A=\bar{R}'BR$, det $A=N(\rho)$ det B. Thus the coset of det A in Γ is an invariant of the class of matrices cogredient to A (or an invariant of the form). We shall call this coset the discriminant of A (or of the form).

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[2] *Simple Lie algebras over a field of characteristic zero*, Duke Mathematical Journal, vol. 4 (1938), p. 542.

[3] The above notation for diagonal matrices will be used throughout this note.

[4] $\mathfrak{R}$ and $\mathfrak{R}'$ have the same quasi-field and anti-automorphism.

$\mathfrak{R}$ may be regarded as a vector space of $2n$ or $4n$ dimensions over Φ_0 and

$$\{x, y\} = (1/2)[(x, y) + (y, x)] = (1/2)\ \mathrm{tr}\ (x, y) \tag{3}$$

is a symmetric form in $\mathfrak{R}$ over Φ_0. The symmetric form $\{x, y\}$ satisfies the special condition

$$\{x\alpha, y\alpha\} = \{x, y\}N(\alpha), \tag{4}$$

whence

$$\{x\bar{\alpha}, y\} = \{x\bar{\alpha}, y\bar{\alpha}^{-1}\bar{\alpha}\} = \{x, y\bar{\alpha}^{-1}\}N(\alpha) = \{x, y\alpha\}.$$

Hence, if $\bar{\alpha}=-\alpha$, $\{x, x\alpha\}=-\{x\alpha, x\}=0$. Conversely, if $\{x, y\}$ is any symmetric bilinear form in $\mathfrak{R}$ over Φ_0 such that (4) holds, (x, y) defined by

$$(x, y) = \begin{cases} \{x, y\} - (i/\lambda)\{x, yi\}, & \text{if} \quad \Phi = \Phi_0(i), \\ \{x, y\} - (i/\lambda)\{x, yi\} - (j/\mu)\{x, yj\} & \\ \qquad - (k/\lambda\mu)\{x, yk\}, & \text{if} \quad \Phi = \Phi_0(i, j), \end{cases} \tag{5}$$

is hermitian in $\mathfrak{R}$ over Φ. The relation between (x, y) and $\{x, y\}$ is a reciprocal one and $\{x, y\}$ is nondegenerate if (x, y) is.[6]

Evidently, if $(x, y)_1$ in $\mathfrak{R}$ over Φ and $(x', y')_2$ in $\mathfrak{R}'$ over Φ are cogredient, then $\{x, y\}_1$ and $\{x', y'\}_2$ are cogredient also. Suppose now that $\{x, y\}_1$ and $\{x', y'\}_2$ are cogredient. Then we have u_1 and u_1', such that[7] $(u_1, u_1)_1=\{u_1, u_1\}_1=\{u_1', u_1'\}_2=(u_1', u_1')_2=\beta_1\neq 0$. Let $\mathfrak{R}_1$ and $\mathfrak{R}_1'$ respectively denote the spaces of vectors orthogonal to u_1 and u_1' relative to $(x, y)_1$ and $(x', y')_2$. The space $\mathfrak{R}_1$ may also be characterized as the set of vectors orthogonal to u_1, u_1i if $\Phi=\Phi_0(i)$, or to u_1, u_1i, u_1j, u_1k, if $\Phi=\Phi_0(i, j)$, with respect to $\{x, y\}_1$. A similar characterization holds for $\mathfrak{R}_1'$. The matrix of $\{x, y\}_1$ relative to u_1, $u_1 i$ or $u_1, u_1 i, u_1 j, u_1 k$ and of $\{x', y'\}_2$ relative to $u_1', u_1' i$ or $u_1', u_1' i, u_1' j, u_1' k$ is

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[5] *General Analysis*, I, American Philosophical Society Publication, Philadelphia, 1935.

[6] We make use of the relation $a=\alpha_0+i\alpha_1+j\alpha_2+k\alpha_3=(1/2)[\mathrm{tr}\ a-(i/\lambda)\ \mathrm{tr}\ ai-(j/\mu)\ \mathrm{tr}\ aj-(k/\lambda\mu)\ \mathrm{tr}\ ak]$.

[7] There exists a vector u_1 such that $(u_1, u_1)\neq 0$. Cf. Jacobson, loc. cit.

$$(6) \qquad [\beta_1, \lambda\beta_1] \quad \text{or} \quad [\beta_1, \lambda\beta_1, \mu\beta_1, \lambda\mu\beta_1].$$

Hence it follows from a theorem of Witt[8] that $\{x, y\}_1$ and $\{x', y'\}_2$ are cogredient when restricted to $\mathfrak{R}_1$ and $\mathfrak{R}_1'$. By induction $(x, y)_1$ and $(x', y')_2$ are cogredient. Thus we have proved the following theorem:

THEOREM. *A necessary and sufficient condition that two hermitian forms* $(x, y)_1$ *and* $(x, y)_2$ *be cogredient is that the corresponding symmetric forms* $\{x, y\}_1$ *and* $\{x, y\}_2$ *be cogredient.*

If $u_1, u_2, \cdots, u_n$ is an orthogonal basis, $(u_i, u_i) = \beta_i$, then $u_1, u_1 i, u_2, u_2 i, \cdots, u_n, u_n i$ or $u_1, u_1 i, u_1 j, u_1 k, \cdots, u_n, u_n i, u_n j, u_n k$ is an orthogonal basis for $\mathfrak{R}$ over Φ relative to $\{x, y\}$ and the corresponding matrix, where B_i is as in (6), is

$$(7) \qquad [B_1, B_2, \cdots, B_n].$$

We consider now some special cases:

(1) $\Phi_0(i)$, *where* Φ_0 *is a field in which every nondegenerate symmetric form in* 5 *or more variables is a null-form.* Examples of such fields are (a) any p-adic field, (b) an algebraic function field of one variable over a finite constant field.[9] In these cases any nondegenerate symmetric form in 4 or more variables represents every $\alpha \neq 0$ in Φ_0. For, if $\{x, y\}$ represents 0, say $\{u, u\} = 0$, we choose v such that $\{u, v\} = \beta \neq 0$. Then $\{u\xi + v\eta, u\xi + v\eta\} = \eta(2\beta\xi + \gamma\eta)$, $\gamma = \{v, v\}$ and the equation $\eta(2\beta\xi + \gamma\eta) = \alpha$ can be solved for ξ, η in Φ_0. If $\{x, y\}$ does not represent 0, we form the vector space of $(n+1)$ dimensions by adjoining z to $\mathfrak{R}$, and define $\{x\xi + z\eta, x\rho + z\sigma\} = \{x, x\}\xi\rho - \alpha\eta\sigma$. Since this form represents 0, we have $\{x, x\}\xi^2 - \alpha\eta^2 = 0$ for $\eta \neq 0$ since $\{x, x\} \neq 0$. Thus $\{x\xi\eta^{-1}, x\xi\eta^{-1}\} = \alpha$. It follows that any hermitian form in a space of 2 or more dimensions represents any α in Φ_0. Hence we may choose $\beta_1 = \beta_2 = \cdots = \beta_{n-1} = 1$ in (2). Thus two forms are cogredient, if, and only if, they have the same discriminant.

(2) $\Phi_0(i, j)$, Φ_0 *of the same type as in case* (1). Here we may take $\beta_1 = \cdots = \beta_n = 1$ and hence all nondegenerate forms are cogredient.

(3) $\Phi_0(i)$, Φ_0 *a real closed field.* Here we may suppose $\lambda = 1$ and we may suppose $\beta_1 = \cdots = \beta_p = 1$, $\beta_{p+1} = \cdots = \beta_n = -1$. For $\{x, y\}$ we

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[8] *Theorie der quadratischen Formen in beliebigen Körpern*, Journal für die reine und angewandte Mathematik, vol. 176 (1936–1937), p. 34.

[9] Witt, loc. cit., p. 40, and Albert, *Quadratic null forms over a function field*, Annals of Mathematics, (2), vol. 39 (1938), pp. 494–505.

obtain $2p$ values $+1$ and $(2n-2p)$ values -1 in the diagonal form. Since the signature is an invariant for bilinear forms it is invariant also for the hermitian form (x, y).

(4) $\Phi_0(i, j)$, Φ_0 *a real closed field.* The considerations are similar to case (3). We find that two nondegenerate hermitian forms are cogredient if and only if they have the same signatures.[10]

(5) $\Phi_0(i)$, Φ_0 *an algebraic number field.* As is well known, the symmetric forms $\{x, y\}_1$ and $\{x, y\}_2$ in $\mathfrak{R}$ over Φ_0 are cogredient, if, and only if, they are cogredient in every p-adic extension of Φ_0. Suppose first that p is a prime spot such that $(-\lambda/p)=1$, that is, $-\lambda$ is a square in the p-adic field[11] Φ_{0p}. Then the matrix B_i in (7) is cogredient in Φ_{0p} to $[\beta_i, -\beta_i]$ and hence also[12] to $[1, -1]$. Thus any two matrices of the form (7) are cogredient. If $(-\lambda/p)=-1$, $\Phi_{0p}(i)$ is a quadratic field over Φ_{0p}. Hence $\{x, y\}_1$ and $\{x, y\}_2$ are cogredient, if, and only if, $(x, y)_1$ and $(x, y)_2$ are cogredient in $\mathfrak{R}$ over $\Phi_{0p}(i)$. The condition for this is that the discriminants be the same when p is finite and the signatures be the same when p is infinite. Combining these results, we see that a necessary and sufficient condition that two nondegenerate hermitian forms in $\mathfrak{R}$ over Φ be cogredient is that they have the same discriminant and the same signature at the infinite prime spots for which λ is positive.[13]

(6) $\Phi_0(i, j)$, Φ_0 *an algebraic number field.* To obtain conditions for cogredience of $(x, y)_1$ and $(x, y)_2$ we again consider $\{x, y\}_1$ and $\{x, y\}_2$. Let p be a prime spot at which $((-\lambda, -\mu)/p)=1$, that is, $\Phi_{0p}(i, j)$ is a matrix algebra. Then either $-\lambda$ is a square in Φ_{0p} or $-\mu$ is a norm in $\Phi_{0p}(i)$. In the first case B_i is cogredient to $[\beta_i, -\beta_i, \mu\beta_i, -\mu\beta_i]$ and hence to $[1, -1, 1, -1]$. If $-\mu$ is a norm in $\Phi_{0p}(i)$, the bilinear form with matrix $[\beta_i, \lambda\beta_i, \mu\beta_i]$ represents 0 and hence is cogredient to $[1, -1, -\lambda\mu\beta_i]$, and again (6) is cogredient to (8). If p is a prime spot for which $\Phi_{0p}(i, j)$ is a division algebra, $(x, y)_1$ and $(x, y)_2$ are always cogredient, if p is finite, and these forms are cogredient for p infinite, if, and only if, they have the same signatures. Thus a necessary and sufficient condition that $(x, y)_1$ and $(x, y)_2$ in $\mathfrak{R}$ over Φ be cogredient is that these forms have the same signatures for all infinite prime spots for which $\Phi_{0p}(i, j)$ is a division algebra.

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[10] E. H. Moore, loc. cit., p. 193.

[11] See Witt and the references cited there to Hasse's papers.

[12] This is a consequence of Witt's theorem that any two symmetric forms in two variables which are nonsingular and represent 0 are cogredient (Witt, p. 34).

[13] Cf. Landherr, *Äquivalenz Hermitescher Formen über einem beliebigen algebraischen Zahlkörper*, Abhandlungen aus dem mathematischen Seminar der Hansischen Universität, vol. 11 (1936), pp. 245–248.

AN EXTENSION OF A COVARIANT DIFFERENTIATION PROCESS[1]

MARIE M. JOHNSON

Craig[2] has considered tensors $T^{\alpha\cdots}_{\beta\cdots}$ whose components are functions of n variables represented by x and their m derivatives $x', x'', \cdots, x^{(m)}$. He obtained the covariant derivative

$$T^{\alpha\cdots}_{\beta\cdots x^{(m-1)\gamma}} - mT^{\alpha\cdots}_{\beta\cdots x^{(m)\lambda}}\left\{{\lambda \atop \gamma}\right\}, \qquad m \geqq 2, \tag{1}$$

where

$$\left\{{\lambda \atop \gamma}\right\} \equiv x'^{\alpha}\Gamma^{\lambda}_{\gamma\alpha} + (1/2)x''^{\beta}f_{\gamma\delta\beta}f^{\delta\lambda}, \tag{2}$$

and partial differentiation in (1) is denoted by the added subscript. Throughout, a repeated letter in one term indicates a sum of n terms. The purpose of this note is to derive another tensor from $T^{\alpha\cdots}_{\beta\cdots}$ whose covariant rank is one larger. The general process will be shown clearly by using $T^{\alpha}(x, x', x'', x''')$.

The extended point transformation

$$x^{\alpha} = x^{\alpha}(y), \qquad x'^{\alpha} = \frac{\partial x^{\alpha}}{\partial y^{i}} y'^{i},$$

$$x''^{\alpha} = \frac{\partial x^{\alpha}}{\partial y^{i}} y''^{i} + \frac{\partial^2 x^{\alpha}}{\partial y^{i}\partial y^{j}} y'^{i}y'^{j}, \cdots, \qquad \alpha = 1, \cdots, n,$$

gives the tensor equations of transformation of the tensor T^{α} as

$$\overline{T}^{i}(y, y', y'', y''') = T^{\alpha}(x, x', x'', x''')\partial y^{i}/\partial x^{\alpha}, \tag{3}$$

where y stands for the n variables $y^1, y^2, \cdots, y^n$ and a similar notation is used for the derivatives y', y'', and y'''. On differentiating equations (3) with respect to y'^k it is found that

$$\overline{T}^{i}_{y'^k} = \left(T^{\alpha}_{x'^\beta}\frac{\partial x^{\beta}}{\partial y^{k}} + T^{\alpha}_{x''^\beta}\frac{\partial x''^{\beta}}{\partial y'^{k}} + T^{\alpha}_{x'''^\beta}\frac{\partial x'''^{\beta}}{\partial y'^{k}}\right)\partial y^{i}/\partial x^{\alpha}. \tag{4}$$

The derivatives can be expressed by using the following general formulas:

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[1] Presented to the Society, April 15, 1939.

[2] H. V. Craig, *On a covariant differentiation process*, this Bulletin, vol. 37 (1931), pp. 731–734.

$$(5)\qquad \frac{\partial x^{(m-1)\beta}}{\partial y^{(m-2)k}} = (m-1)\,\frac{\partial x'^{\beta}}{\partial y^{k}}, \qquad \frac{\partial x^{(m)\beta}}{\partial y^{(m-2)k}} = \frac{m(m-1)}{2}\,\frac{\partial x''^{\beta}}{\partial y^{k}},$$

in which $\partial x'^{\beta}/\partial y^{k}$ are eliminated by[3]

$$(6)\qquad \overline{\left\{ {l \atop k} \right\}}\,\frac{\partial x^{\beta}}{\partial y^{l}} = \frac{\partial x'^{\beta}}{\partial y^{k}} + \left\{ {\beta \atop \gamma} \right\}\frac{\partial x^{\gamma}}{\partial y^{k}}.$$

The derivatives $\partial x''^{\beta}/\partial y^{k}$ are simplified by first writing

$$(7)\qquad x''^{\beta} = \frac{\partial x^{\beta}}{\partial y^{j}}\,y''^{j} + \overline{\Gamma}^{r}_{jk}\,y'^{j}y'^{k}\,\frac{\partial x^{\beta}}{\partial y^{r}} - \Gamma^{\beta}_{\alpha\delta}\,x'^{\alpha}x'^{\delta},$$

with the help of (2), (6) and[4] $f_{\alpha\beta\gamma}x'^{\beta}=0$. It is necessary also to have

$$(8)\qquad \frac{\partial^{2}x^{\beta}}{\partial y^{j}\partial y^{k}} = \overline{\Lambda}^{t}_{jk}\,\frac{\partial x^{\beta}}{\partial y^{t}} - \Lambda^{\beta}_{\alpha\delta}\,\frac{\partial x^{\alpha}}{\partial y^{j}}\,\frac{\partial x^{\delta}}{\partial y^{k}},$$

where

$$\Lambda^{\beta}_{\alpha\delta} = \Gamma^{\beta}_{\alpha\delta} - (1/2)f^{\beta\gamma}\left(f_{\delta\gamma\tau}\left\{ {\tau \atop \alpha} \right\} + f_{\gamma\alpha\tau}\left\{ {\tau \atop \delta} \right\} - f_{\alpha\delta\tau}\left\{ {\tau \atop \gamma} \right\}\right).$$

This is obtained from Taylor's[5] formula (19) in the following way. Multiply this formula by $(\partial y^{k}/\partial x^{\epsilon})f^{\beta\epsilon}=(\partial x^{\beta}/\partial y^{l})\bar{f}^{kl}$, and sum for k. Use the tensor equations for $f_{\alpha\beta\gamma}$ and substitute from (6) for $\partial x'^{\gamma}/\partial y^{j}$.

By means of formulas (6) and (8) and the tensor $Q^{\beta}(x, x', x'') \equiv x''^{\beta}+\Gamma^{\beta}_{\alpha\delta}x'^{\alpha}x'^{\delta}$ the partial derivatives of (7) have the form

$$(9)\qquad \frac{\partial x''^{\beta}}{\partial y^{k}} = -\left| {\beta \atop \gamma} \right|\frac{\partial x^{\gamma}}{\partial y^{k}} + \overline{\left| {r \atop k} \right|}\,\frac{\partial x^{\beta}}{\partial y^{r}} - 2\left\{ {\beta \atop \alpha} \right\}\overline{\left\{ {l \atop k} \right\}}\,\frac{\partial x^{\alpha}}{\partial y^{l}} + 2\,\overline{\left\{ {r \atop i} \right\}}\,\overline{\left\{ {i \atop k} \right\}}\,\frac{\partial x^{\beta}}{\partial y^{r}},$$

in which we have the nontensor form

$$(10)\qquad \left| {\beta \atop \gamma} \right| = Q^{\beta}_{x^{\gamma}} - Q^{\beta}_{x'^{\alpha}}\left\{ {\alpha \atop \gamma} \right\} + Q^{\alpha}\Lambda^{\beta}_{\alpha\gamma}.$$

If formulas (6) and (9) are substituted in equations (5) and the results used in (4), we find

$$\begin{aligned}\overline{T}^{i}_{y'^{k}} = {}& \left(T^{\alpha}_{x'^{\beta}} - 2T^{\alpha}_{x''^{\delta}}\left\{ {\delta \atop \beta} \right\} - 3T^{\alpha}_{x'''^{\delta}}\left| {\delta \atop \beta} \right|\right)\frac{\partial x^{\beta}}{\partial y^{k}}\,\frac{\partial y^{i}}{\partial x^{\alpha}} \\ & - \left(-\,2\overline{T}^{i}_{y''^{l}}\,\overline{\left\{ {l \atop k} \right\}} - 3\overline{T}^{i}_{y'''^{l}}\,\overline{\left| {l \atop k} \right|}\right).\end{aligned}$$

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[3] J. H. Taylor, *A generalization of Levi-Civita's parallelism and the Frenet formulas'* Transactions of this Society, vol. 27 (1925), p. 255.

[4] J. H. Taylor, loc. cit., p. 248.

[5] J. H. Taylor, loc. cit., p. 254.

Hence the new tensor whose covariant rank has been increased by one is

$$T^{\alpha}_{x',\beta} - 2T^{\alpha}_{x'',\delta}\begin{Bmatrix}\delta\\ \beta\end{Bmatrix} - 3T^{\alpha}_{x''',\delta}\begin{vmatrix}\delta\\ \beta\end{vmatrix}, \tag{11}$$

where $\begin{Bmatrix}\delta\\ \beta\end{Bmatrix}$ and $\begin{vmatrix}\delta\\ \beta\end{vmatrix}$ are defined in (2) and (10).

Because of the general relations in (5) it is easy to verify that the tensor

$$T^{\alpha\cdots}_{\gamma\cdots x^{(m-2)}\beta} - (m-1)T^{\alpha\cdots}_{\gamma\cdots x^{(m-1)}\delta}\begin{Bmatrix}\delta\\ \beta\end{Bmatrix} - \frac{m(m-1)}{2}T^{\alpha\cdots}_{\gamma\cdots x^{(m)}\delta}\begin{vmatrix}\delta\\ \beta\end{vmatrix}, \qquad m \geqq 3, \tag{12}$$

has a covariant rank which is one larger than that of $T^{\alpha\cdots}_{\gamma\cdots}$ whose components are functions of $(x, x', \cdots, x^{(m)})$.

If the components of the tensor $T^{\alpha}(x, x', x'', x''')$ do not contain the derivatives x''', then (11) reduces to Craig's covariant derivative (1), and if there are no x'' or x''' derivatives, then the result is a partial differentiation with respect to x'.

The usual rules for the derivative of a sum of tensors of the same type and rank and for the product of any tensors are preserved by this provess.

If $m=2$, a scalar $T(x, x', x'')$ will give a covariant tensor which is similar to that in (11) when the tensor equations for $\overline{T}(y, y', y'')$ are differentiated with respect to y instead of y'. The tensor is

$$T_{x\beta} - T_{x',\delta}\begin{Bmatrix}\delta\\ \beta\end{Bmatrix} - T_{x'',\delta}\begin{vmatrix}\delta\\ \beta\end{vmatrix}. \tag{13}$$

However, if $m=2$ and a tensor $T^{\alpha}(x, x', x'')$ is used, an extra term $T^{\delta}\Delta^{\alpha}_{\delta\beta}$ has to be added to three terms similar to those in (13). If this process is performed on the tensor $Q^{\alpha}(x, x', x'')$, the result is the zero tensor.

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INDEFINITELY DIFFERENTIABLE FUNCTIONS WITH PRESCRIBED LEAST UPPER BOUNDS[1]

IRWIN E. PERLIN

1. Introduction. Let $F(x)$ be a real indefinitely differentiable function of the real variable x defined on the interval $a \leqq x \leqq b$, and let M_n denote the least upper bound of $|F^{(n)}(x)|$ on that interval.[2] In this paper we shall establish sufficient conditions that there exist an indefinitely differentiable function $F(x)$ taking on certain prescribed M_n.

It is easy to see that M_0 and M_1 can be assigned arbitrarily. However, the first three upper bounds M_0, M_1, and M_2 must satisfy certain inequalities.[3] Let us consider the interval (0, 1). Let t_1 be the value of x for which $|F^{(1)}(x)|$ attains its maximum. Then

$$F(1) - F(t_1) = (1 - t_1)F^{(1)}(t_1) + (1/2!)F^{(2)}(\theta_1)(1 - t_1)^2,$$

where $t_1 < \theta_1 < 1$. And similarly

$$F(0) - F(t_1) = -t_1F^{(1)}(t_1) + (1/2!)F^{(2)}(\theta_2)t_1^2,$$

where $0 < \theta_2 < t_1$. On subtracting these equations we obtain

$$F^{(1)}(t_1) = F(1) - F(0) + (1/2!)\{F^{(2)}(\theta_2)t_1^2 - F^{(2)}(\theta_1)(1 - t_1)^2\},$$
$$M_1 \leqq 2M_0 + M_2/2!.$$

By the same procedure we can obtain for the interval $(0, a)$

$$M_1 \leqq 2M_0/a + M_2a/2!. \tag{1}$$

In the case of the interval $(0, \infty)$ we can replace (1) by a more precise inequality. Since a is arbitrary, we can replace a by the positive value which minimizes the right side of (1), and obtain

$$M_1 \leqq 2(M_0M_2)^{1/2}.$$

Ore[4] in a recent paper employed the results of W. Markoff to obtain certain inequalities connecting the least upper bounds of $|F^{(i)}(x)|$, $(1 \leqq i \leqq n)$, with those of $|F(x)|$ and $|F^{(n+1)}(x)|$ where $F(x)$ is a function with bounded $(n+1)$th derivative. For the first derivative the inequality (1) is slightly better than that obtained by Ore.

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[1] Presented to the Society, September 8, 1939.

[2] By $F^{(0)}(x)$ we shall mean $F(x)$.

[3] Hadamard, Comptes Rendus des Séances de la Société Mathématique de France, 1914, pp. 68–72; T. Carlman, *Les fonctions quasi analytiques*, Paris, 1926.

[4] O. Ore, *On functions with bounded derivatives*, Transactions of this Society, vol. 43 (1938), pp. 321–326.

2. Construction of an indefinitely differentiable function with prescribed least upper bounds.

We now prove the following theorem.

THEOREM. *If the sequence* $\{M_i a^i\}$ *is monotone decreasing, then there exists an indefinitely differentiable function* $F(x)$ *defined on* $(0 \leqq x \leqq a)$ *such that* M_i *is the least upper bound of* $|F^{(i)}(x)|$ *on* $(0 \leqq x \leqq a)$.

Define

$$0 \leqq S_i = \sum_{j=0}^{\infty} (-1)^j \frac{M_{i+j} a^j}{j!} \leqq M_i, \qquad F(x) = \sum_{i=0}^{\infty} S_i \frac{x^i}{i!}.$$

Now the function $F(x)$ so defined is an entire function. Let $x = b$. Then

$$F(b) = \sum_{i=0}^{\infty} S_i \frac{b^i}{i!} = \sum_{i=0}^{\infty} (-1)^i \frac{M_i}{i!} \left(\sum_{j=0}^{i} (-1)^j C_{i-j,j} a^{i-j} b^j \right),$$
$$= \sum_{i=0}^{\infty} (-1)^i \frac{M_i (a-b)^i}{i!}.$$

And since the series

$$\sum_{i=0}^{\infty} \frac{M_i a^i |(1-b/a)^i|}{i!} < M_0 \sum_{i=0}^{\infty} \frac{|(1-b/a)^i|}{i!}$$

converges, the series $\sum_{i=0}^{\infty} (S_i/i!) b^i$ converges.[5] Further,

$$F^{(i)}(a) = \sum_{j=0}^{\infty} (-1)^j M_{i+j} a^j \left(\sum_{k=0}^{j} \frac{(-1)^k}{k!(j-k)!} \right) = M_i,$$

and

$$|F^{(i)}(x)| \leqq F^{(i)}(a) = M_i, \qquad 0 \leqq x \leqq a.$$

Thus we have given explicitly a function $F(x)$ satisfying the conditions of the theorem.

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[5] See T. J. I'A. Bromwich, *Theory of Infinite Series*, London, 1931, p. 266.

A DECOMPOSITION OF ADDITIVE SET FUNCTIONS[1]

R. S. PHILLIPS

This paper is concerned with a decomposition theorem for additive functions on an additive family of sets to either real numbers or a Banach space. Additive bounded set functions have as yet been little studied. However the recent paper of Hildebrandt[2] illustrates their importance.

We shall use the following notation:

(a) T: an abstract class of arbitrary elements t.

(b) $\mathfrak{J}$: a completely additive family of subsets τ of T; that is, $T \varepsilon \mathfrak{J}$, $\tau \varepsilon \mathfrak{J}$ implies $T-\tau \varepsilon \mathfrak{J}$, and $\tau_n \varepsilon \mathfrak{J}$ for $n=1, 2, \cdots$ implies $\sum\tau_n \varepsilon \mathfrak{J}$.

(c) α: a set function on $\mathfrak{J}$ to real numbers.

(d) A: the subclass of set functions on $\mathfrak{J}$ to real numbers which are additive and bounded; that is, $\tau_1, \tau_2 \varepsilon \mathfrak{J}$ and $\tau_1\cdot\tau_2=0$ implies $\alpha(\tau_1+\tau_2)=\alpha(\tau_1)+\alpha(\tau_2)$.

(e) C: the subclass of set functions on $\mathfrak{J}$ to real numbers which are completely additive (c.a.), that is, $\tau_n \varepsilon \mathfrak{J}$ for $n=1, 2, \cdots$ and $\tau_i\cdot\tau_j=0$ if $i\neq j$ implies $\alpha(\sum\tau_n)=\sum\alpha(\tau_n)$. The functions in C are bounded.[3]

The notations A_P and C_P refer to the subclasses of A and C respectively whose elements are nonnegative.

(f) x: a set function on $\mathfrak{J}$ to a Banach space[4] X. The definitions of additive and c.a. set functions are formally retained. If $\{\tau_n\}$ is a sequence of disjoint sets of $\mathfrak{J}$ and $x(\tau)$ is c.a., then $\sum x(\tau_n)$ is unconditionally convergent.[5]

(g) C_X: the class of c.a. set functions on $\mathfrak{J}$ to X.

In the statement of the following theorems, D will designate any one of the classes A, A_P, C, C_P, and $\bar{\tau}$ will denote the cardinal number of τ.

THEOREM 1. *Let $\aleph$ be an infinite cardinal number not greater than $\bar{\bar{T}}$. For every $\alpha \varepsilon D$ there exists an unique decomposition $\alpha=\alpha_1+\alpha_2$ and a set $R(\alpha) \varepsilon \mathfrak{J}$ of cardinal number not greater than $\aleph$ such that $\alpha_1, \alpha_2 \varepsilon D$,*

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[1] Presented to the Society April 15, 1939, under the title *On additive set functions.*

[2] T. H. Hildebrandt, *On bounded linear functional operations*, Transactions of this Society, vol. 36 (1934), pp. 868–875.

[3] S. Saks, *Theory of the Integral*, Monografje Matematyczne, Warsaw, 1937, p. 10, Theorem 6.1.

[4] S. Banach, *Théorie des Opérations Linéaires*, Monografje Matematyczne, Warsaw, 1932, chap. 5.

[5] If x_n is a series of elements of X and if every subseries $\sum x_n$ is convergent, then $\sum x_n$ is said to be unconditionally convergent.

$$\alpha_1(\tau) = \alpha(R\cdot\tau), \qquad \alpha_2(\tau) = 0 \quad \textit{if} \quad \overline{\overline{\tau}} \leqq \aleph .$$

Let $\Sigma \equiv E_\tau[\tau \in \mathfrak{I}, \overline{\overline{\tau}} \leqq \aleph, \alpha(\tau) \neq 0]$. We define a transfinite sequence $(\tau_1, \tau_2, \cdots; \tau_\omega, \cdots, \tau_\lambda, \cdots)$ as follows: τ_1 is an arbitrary element of Σ. Suppose τ_λ have been defined for all $\lambda < \mu$. If there exists τ such that $\tau\cdot\sum_{\lambda<\mu}\tau_\lambda = 0$ and $\tau \in \Sigma$, then we set $\tau = \tau_\mu$. .

As $\alpha(\tau)$ is bounded, $\alpha(\tau)$ cannot differ from zero on a nondenumerable number of disjoint sets. The sequence therefore contains only a denumerable set of elements.

Let $R = \sum_\lambda \tau_\lambda$. Then $R \in \mathfrak{I}$ and $\overline{\overline{R}} \leqq \aleph$. We define $\alpha_1(\tau) = \alpha(R\cdot\tau)$, $\alpha_2(\tau) = \alpha(\tau) - \alpha_1(\tau) = \alpha(\tau - R\cdot\tau)$. The $\alpha_1(\tau)$, $\alpha_2(\tau)$ are clearly elements of D. If $\overline{\overline{\tau}} \leqq \aleph$, then by the definition of R, $\alpha_2(\tau) = \alpha(\tau - R\cdot\tau) = 0$.

Although the set R is not unique, the function decomposition is unique: Suppose there exist two different sets R_1, R_2 having the properties of the R defined above. The set identity $R_1\cdot\tau + (R_2 - R_1)\cdot\tau = R_2\cdot\tau + (R_1 - R_2)\cdot\tau$ and $\alpha[(R_1 - R_2)\cdot\tau] = 0 = \alpha[(R_2 - R_1)\cdot\tau]$ imply that $\alpha(R_1\cdot\tau) = \alpha(R_2\cdot\tau)$.

A set function α on $\mathfrak{I}$ will be said to be *nonsingular* if for every $t \in \mathfrak{I}$, $\alpha(t) = 0$. A set function α on $\mathfrak{I}$ will be called $\aleph$*-homogeneous* if there exists a set R such that $R \in \mathfrak{I}$, $\overline{\overline{R}} = \aleph$, $\alpha(\tau) = \alpha(R\cdot\tau)$, and $\alpha(\tau) = 0$ if $\overline{\overline{\tau}} < \aleph$.

Without loss of generality we may consider only nonsingular set functions because for every $\alpha \in D$ there exists a unique decomposition $\alpha = \alpha_1 + \alpha_2$ and a denumerable set $\{t_i\}$ of elements of T, such that $\alpha_1, \alpha_2 \in D$, $\alpha_1(\tau) = \sum_{i=1}^{\infty}\alpha(\tau\cdot t_i)$, and α_2 is nonsingular. We omit the proof.

THEOREM 2. *For every nonsingular* $\alpha \in D$, *there exists an unique decomposition* $\alpha = \sum_i \alpha_i$, *the sum being absolutely convergent, and such that* α_i *is* $\aleph_i$*-homogeneous and* $\aleph_i \neq \aleph_j$ *if* $i \neq j$.

In the proof of this theorem an induction is made on the infinite cardinals not exceeding that of T, well-ordered according to magnitude. We define a transfinite sequence of set functions $(\alpha_1, \alpha_2, \cdots; \alpha_\omega, \cdots, \alpha_\lambda, \cdots)$ as follows: Suppose α_λ have been defined for all $\lambda < \mu$ and (1) only a denumerable number of the α_λ are not identically zero; (2) $\sum_{\lambda \leqq \lambda_0} |\alpha_\lambda(\tau)| < \infty$; and (3) $\alpha_\lambda \in D$ and is $\aleph_\lambda$-homogeneous. By Theorem 1 there exist $R_\mu \in \mathfrak{I}$ and a decomposition $\alpha = \alpha_\mu^1 + \alpha_\mu^2$ such that $\overline{\overline{R}}_\mu \leqq \aleph_\mu$, $\alpha_\mu^1(\tau) = \alpha(R_\mu\cdot\tau)$, $\alpha_\mu^2(\tau) = 0$ if $\overline{\overline{\tau}} \leqq \aleph_\mu$, and α_μ^1, $\alpha_\mu^2 \in D$. Clearly $\alpha_\lambda(\tau) = \alpha(R_\mu\cdot R_\lambda\cdot\tau)$ if $\lambda < \mu$.

Let $\alpha_\mu(\tau) = \alpha_\mu^1(\tau) - \sum_{\lambda<\mu}\alpha_\lambda(\tau)$. We consider the following cases:

I. $\alpha \in C, C_P$. Let $\overline{R}_\mu = R_\mu - \sum_{\pi_\mu}\overline{R}_\lambda$ where $\pi_\mu \equiv E_\lambda[\lambda < \mu, \alpha_\lambda \neq 0]$. The sets $\overline{R}_\mu$ are disjoint. Suppose $\alpha_\lambda(\tau) = \alpha(\overline{R}_\lambda\cdot\tau)$ for $\lambda < \mu$. Then by (1)

$$\alpha_\mu(\tau) = \alpha(R_\mu\cdot\tau) - \sum_{\pi_\mu}\alpha_\lambda(\tau) = \alpha(R_\mu\cdot\tau) - \sum_{\pi_\mu}\alpha(R_\mu\cdot\overline{R}_\lambda\cdot\tau)$$
$$= \alpha\left[\left(R_\mu - \sum_{\pi_\mu} R_\mu\cdot\overline{R}_\lambda\right)\cdot\tau\right] = \alpha(\overline{R}_\mu\cdot\tau).$$

It is clear that (1), (2), and (3) are satisfied for $\mu+1$. $\alpha_\lambda\not\equiv 0$ implies that $\alpha(\tau)\neq 0$ for some subset of $\overline{R}_\lambda$. As the $\overline{R}_\lambda$ are disjoint, the sequence will contain only a denumerable number of functions not identically zero.

II. $\alpha\ \varepsilon\ A_P$. For $\lambda_0<\mu$, $\alpha(T)\geqq\overset{1}{\alpha}_{\lambda_0}(T)\equiv\sum_{\lambda\leqq\lambda_0}\alpha_\lambda(T)\geqq\sum_{\lambda\leqq\lambda_0}\alpha_\lambda(\tau)$. Clearly (1) and (2) are satisfied for $\mu+1$, and the sequence contains only a denumerable number of functions not identically zero. Let λ_i be a spanning sequence for $E_\lambda[\lambda<\mu, \alpha_\lambda\not\equiv 0]$. Then

$$\alpha_\mu(\tau) = \overset{1}{\alpha}_\mu(\tau) - \sum_{\lambda<\mu}\alpha_\lambda(\tau) = \alpha(R_\mu\cdot\tau) - \lim_{i\to\infty}\overset{1}{\alpha}_{\lambda_i}(\tau)$$
$$= \alpha(R_\mu\cdot\tau) - \lim_{i\to\infty}\alpha(R_\mu\cdot R_{\lambda_i}\cdot\tau).$$

Hence (3) is likewise satisfied.

III. $\alpha\ \varepsilon\ A$. Every $\alpha\ \varepsilon\ A$ has a decomposition $\alpha=\alpha_1-\alpha_2$ where $\alpha_1, \alpha_2\ \varepsilon\ A_P$. An application of II to α_1 and α_2 gives the desired decomposition.

The decomposition is unique: Any two sequences of homogeneous functions differ in a first function, α_μ. But this is contrary to $\overset{1}{\alpha}_\mu=\sum_{\lambda\leqq\mu}\alpha_\lambda$ being unique.

In these theorems the restriction that the additive bounded set function be defined over an additive family $\mathfrak{J}$ is optional, since the range of definition of such a function can always be extended to an additive family. The type of argument used by Pettis[6] will prove this statement.

We next consider the possibility of extending these theorems to functions $x(\tau)$ on $\mathfrak{J}$ to a Banach space. The theorem is not in general valid for additive bounded set functions of this type. This is illustrated by $x(\tau)$ defined on all subsets of $T\equiv(0, 1)$ to the space X of bounded functions on $S\equiv(0, 1)$ where $x(\tau)$ is the characteristic function of the subset of S which has the same coordinate values as τ. Clearly there exists no denumerable set R such that $x(\tau-R\tau)=0$ for all denumerable sets τ.

However analogous theorems are obtained for c.a. set functions on $\mathfrak{J}$ to X.

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[6] B. J. Pettis, *Linear functionals and completely additive set functions*, Duke Mathematical Journal, vol. 4 (1938), p. 554, Theorem 1.1.

THEOREM 3. *Let $\aleph$ be an infinite cardinal number not greater than $\overline{\overline{T}}$. For every $x \varepsilon C_X$ there exists a unique decomposition $x = x_1 + x_2$ and a set $R(x) \varepsilon \mathfrak{T}$ of cardinal power not greater than $\aleph$ such that $x_1, x_2 \varepsilon C_X$, $x_1(\tau) = x(R \cdot \tau)$, $x_2(\tau) = 0$ if $\overline{\overline{\tau}} \leqq \aleph$.*

$x(\tau) \neq 0$ on at most a denumerable number of disjoint sets of $\mathfrak{T}$. Suppose the contrary. Then there exists a denumerable sequence of disjoint sets $\{\tau_i\}$ and an $e > 0$ such that $\|x(\tau_i)\| > e$, $(i = 1, 2, \cdots)$. As $x(\tau)$ is c.a., $\sum_i x(\tau_i)$ converges. The supposition is therefore false.

The argument used in Theorem 1 will now prove the theorem.

THEOREM 4. *For every nonsingular $x \varepsilon C_X$, there exists an unique decomposition $x = \sum_i x_i$, the sum being unconditionally convergent, and such that x_i is $\aleph_i$-homogeneous and $\aleph_i \neq \aleph_j$ if $i \neq j$.*

The proof is identical with that of I in Theorem 2. Again there will exist disjoint $\overline{R}_\mu$'s such that $x_\mu(\tau) = x(\overline{R}_\mu \cdot \tau)$.

UNIVERSITY OF MICHIGAN

ON THE COMPLETENESS OF A CERTAIN METRIC SPACE WITH AN APPLICATION TO BLASCHKE'S SELECTION THEOREM[1]

G. BALEY PRICE

1. **Introduction.** The purpose of this note is to prove that the metric space whose elements are the closed, bounded, non-null subsets of a complete metric space, and whose metric is the Hausdorff distance, is complete; and, using this result and others already known, to give a simple proof of Blaschke's selection theorem.

2. **Preliminaries.** Let K be a metric space with elements $x, y, \cdots$ and distance function $d(x, y)$. A sequence $x_1, x_2, \cdots$ in K such that $\sum_1^\infty d(x_i, x_{i+1})$ converges has been called an absolutely convergent sequence by MacNeille[2] [7, p. 192]. Every absolutely convergent sequence is a Cauchy sequence, and every Cauchy sequence contains absolutely convergent subsequences.

Let K^* be a metric space whose elements $X, Y, \cdots$ are the closed, bounded, and non-null subsets of K, and whose distance function $D(X, Y)$ is the Hausdorff distance between the sets X and Y (see Hausdorff [5, pp. 145–146] and Kuratowski [6, pp. 89–90]).

3. **The theorem.** *If K is complete, then K^* is also complete.*

Let $X_1, X_2, \cdots$ be any Cauchy sequence in K^*; without loss of generality we can assume that it is absolutely convergent. We shall define a set X and show that it is the limit of the given sequence. Let x_1 be any point in X_1, x_2 any point in X_2 such that $d(x_1, x_2) < D(X_1, X_2) + 2^{-1}$, x_3 any point in X_3 such that $d(x_2, x_3) < D(X_2, X_3) + 2^{-2}$, and so on. The existence of points $x_2, x_3, \cdots$ with the properties stated follows from the definition of the Hausdorff distance. Every point x_i in X_i is a member of a sequence $x_1, x_2, \cdots$ of the kind described. The sequence $x_1, x_2, \cdots$ is absolutely convergent and hence a Cauchy sequence; since K is complete, it has a limit x_0 in K. Let X_0 be the projali locus of all the points x_0 obtained as the limits of all possible sequences formed in the manner stated; let X be the closure of X_0. Then X is closed, bounded, and non-null, and X is in K^*. We shall show that $\lim X_k = X$. Let any $\epsilon > 0$ be given. Choose $n = n(\epsilon)$ so that $\sum_n^\infty [D(X_i, X_{i+1}) + 2^{-i}] < \epsilon/2$. Let $x^* \in X$, and let x_0 be the limit of a sequence $x_1, x_2, \cdots, x_k, \cdots$ and such that $d(x^*, x_0) < \epsilon/2$. Then the distance from x^* to X_k is equal to or less than

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[1] Presented to the Society, December 28, 1938, under the title *Spaces whose elements are sets.*

[2] Numbers in square brackets refer to the references at the end.

$$d(x^*, x_0) + \sum_k^\infty d(x_i, x_{i+1}) < d(x^*, x_0) + \sum_k^\infty [D(X_i, X_{i+1}) + 2^{-i}]$$
$$< \epsilon/2 + \epsilon/2 = \epsilon$$

if $k \geqq n$. Since every point x_k in X_k belongs to a sequence $x_1, x_2, \cdots$, the distance from any point x_k to X, which does not exceed the distance from x_k to the limit x_0 of the sequence $x_1, x_2, \cdots, x_k, \cdots$, is equal to or less than

$$\sum_k^\infty d(x_i, x_{i+1}) < \sum_k^\infty [D(X_i, X_{i+1}) + 2^{-i}] < \epsilon/2$$

if $k \geqq n$. From these facts it follows that $D(X_k, X) < \epsilon$ for $k \geqq n$, and hence that $\lim X_k = X$. Thus the (absolutely convergent) Cauchy sequence $X_1, X_2, \cdots$ in K^* has the limit X in K^*, and the proof of the theorem is complete.

4. **The space K^* when K is a Banach space.** The space K^* has additional properties when K is a Banach space, that is, a space which is linear, normed, and complete (see Banach [1, p. 53]). Let aX denote the set of elements ax, $x \in X$, when a is a real number; let $X+Y$ denote the set of elements $x+y$, $x \in X$ and $y \in Y$; let $C[X]$ denote the closed convex extension of X; and let $\rho(X)$ denote the diameter of X. Then K^* has, in addition to its elementary properties as a metric space, the following ones:

(4.1) $$D(aX, aY) = |a| D(X, Y) \quad \text{for every real number } a;$$

(4.2) $$D(X_1 + \cdots + X_n, Y_1 + \cdots + Y_n) \leqq D(X_1, Y_1) + \cdots + D(X_n, Y_n);$$

(4.3) $$D(C[X], C[Y]) \leqq D(X, Y);$$

(4.4) $$D(X + Y_1, X + Y_2) \leqq D(Y_1, Y_2);$$

(4.5) $$\rho(X_i) \leqq \rho(X_1 + X_2) \leqq \rho(X_1) + \rho(X_2), \ i = 1, 2;$$

(4.6) $$\rho(C[X]) = \rho(X), \quad D(C[X], 0) = D(X, 0).$$

The last two of these relations have been given by Birkhoff [2, pp. 368, 360]. The proofs of the others will be given elsewhere. It can be shown by means of examples that the inequality may hold in (4.4).

If the limit of a sequence $X_1, X_2, \cdots$ of convex sets in K^* is a set X, it follows from (4.3) that X also is convex. For

$D(X_i, C[X]) = D(C[X_i], C[X]) \leqq D(X_i, X)$; and since $D(X_i, X) \to 0$, $\lim X_i = C[X]$. But since a sequence has a unique limit, we have $C[X] = X$, and X is convex.

5. **Blaschke's selection theorem.** Let E be a closed and compact subset of a Banach space K, and let E^* denote the subset of K^* which consists of the closed, non-null subsets of E. Then both E and E^* are totally bounded, and E^* is closed and compact in K^* (see Hausdorff [5, pp. 107–108] and Kuratowski [6, p. 91]). Let E^*_C denote the subset of E^* which consists of convex sets. Since E^* is totally bounded, any infinite set of elements in $E^* \subset E^*$ contains a Cauchy sequence; since K^* is complete and E^* is closed, this sequence has a limit in E^*. By the result at the end of the last section, this limit element is itself a closed, convex set and therefore belongs to E^*_C. We have thus shown that E^* is closed and compact. This result is Blaschke's selection theorem extended to a Banach space (see Blaschke [3] and Bonnesen and Fenchel [4, p. 34]).

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SOME PROBLEMS IN INTERPOLATION BY CHARACTERISTIC FUNCTIONS OF LINEAR DIFFERENTIAL SYSTEMS OF THE FOURTH ORDER

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In this paper we consider the convergence to $f(x)$, defined on $[0, 1]$, of

$$\Sigma_p[f(x)] = \alpha_{0p}u_0(x) + \alpha_{1p}u_1(x) + \cdots + \alpha_{pp}u_p(x),$$

where $u_n(x)$, $(n=0, 1, \cdots, p)$, are characteristic functions of certain self-adjoint linear differential systems of fourth order,

$$\alpha_{np} = \sum_{k=0}^{p}{}' f(x_k)u_n(x_k)\left\{\sum_{k=0}^{p}{}' u_n^2(x_k)\right\}^{-1}, \qquad n = 0, 1, \cdots, p,$$

and the symbol $\sum'$ is used in the sense $\sum'^{p}_{k=0} y_k = y_0/2 + \sum^{p}_{k=1} y_k$. Throughout the discussion, $x_k = 2k/(2p+1)$, $(k=0, 1, \cdots, p)$. The differential systems considered are

$$u^{(\text{iv})} - \rho^4 u = 0,$$

with boundary conditions

I. $u'(0)=0$, $u'''(0)=0$, $u'(1)=0$, $u'''(1)+u(1)=0$,
II. $u'(0)=0$, $u'''(0)=0$, $u'(1)+u(1)=0$, $u'''(1)+u''(1)=0$,
III. $u(0)=0$, $u''(0)=0$, $u(1)=0$, $u''(1)+u'(1)=0$,
IV. $u'(0)=0$, $u'''(0)=0$, $u(1)=0$, $u''(1)+u'(1)=0$,
V. $u(0)=0$, $u'(0)=0$, $u(1)=0$, $u'(1)=0$,
VI. $u'(0)=0$, $u'''(0)=0$, $u(1)=0$, $u'(1)=0$.

The following theorems may be proved for these systems respectively.

I, II. *If $f(x)$ is continuous and of bounded variation in* $[0, 1]$, *then* $\lim_{p\to\infty}\Sigma_p[f(x)]=f(x)$ *uniformly in* $[0, 1]$.

III. *If $f(x)$ is continuous and of bounded variation in* $[0, 1]$ *and* $f(0)=f(1)=0$, *then* $\lim_{p\to\infty}\Sigma_p[f(x)]=f(x)$ *uniformly in* $[0, 1]$.

IV. *If $f(x)$ is continuous and of bounded variation in* $[0, 1]$ *and* $f(1)=0$, *then* $\lim_{p\to\infty}\Sigma_p[f(x)]=f(x)$ *uniformly in* $[0, 1-\eta]$.

V, VI. *If $f(x)$ satisfies a Lipschitz condition in* $[0, 1]$ *and* $f(0)=f(1)=0$, *then* $\lim_{p\to\infty}\Sigma_p[f(x)]=f(x)$ *uniformly in* $[\eta, 1-\eta]$.

Here and hereafter $\eta>0$ is arbitrarily small but fixed.

The method of proof for these theorems, as well as for those to follow, is essentially the same as that of C. M. Jensen [1], who considered convergence properties of a somewhat similar sum using Sturm-Liouville functions.

The following equivalence theorems (the term "equivalence" is used in the sense of Jensen) may be proved for systems I, II, III, IV, respectively.

I, II. *If $f(x)$ is continuous in* $[0, 1]$, *then*

$$\lim_{p\to\infty} \{\Sigma_p[f(x)] - T_p[f(x)]\} = 0$$

uniformly in $[0, 1]$, *where* $T_p[f(x)]$ *is the cosine interpolation formula*

$$T_p[f(x)] = a_{0p} + a_{1p}\cos \pi x + \cdots + a_{pp}\cos p\pi x,$$

$$a_{np} = \sum_{k=0}^{p}{}' f(x_k)\cos n\pi x_k \left\{ \sum_{k=0}^{p}{}' \cos^2 n\pi x_k \right\}^{-1}$$

$$= \begin{cases} \dfrac{4}{2p+1}\displaystyle\sum_{k=0}^{p}{}' f(x_k)\cos n\pi x_k, & n = 1, 2, \cdots, p, \\ \dfrac{2}{2p+1}\displaystyle\sum_{k=0}^{p}{}' f(x_k), & n = 0. \end{cases}$$

III. *If $f(x)$ is continuous in* $[0, 1]$, *then*

$$\lim_{p\to\infty} \{\Sigma_p[f(x)] - \overline{T}_p[f(x)]\} = 0$$

uniformly in $[\eta, 1-\eta]$, *where* $\overline{T}_p[f(x)]$ *is the sine interpolation formula*

$$\overline{T}_p[f(x)] = \bar{a}_{1p}\sin \pi x + \bar{a}_{2p}\sin 2\pi x + \cdots + \bar{a}_{pp}\sin p\pi x,$$

$$\bar{a}_{np} = \sum_{k=0}^{p}{}' f(x_k)\sin n\pi x_k \left\{ \sum_{k=0}^{p}{}' \sin^2 n\pi x_k \right\}^{-1}$$

$$= \frac{4}{2p+1}\sum_{k=0}^{p}{}' f(x_k)\sin n\pi x_k, \qquad n = 1, 2, \cdots, p.$$

IV. *If $f(x)$ is continuous in* $[0, 1]$, *then*

$$\lim_{p\to\infty} \{\Sigma_p[f(x)] - U_p[f(x)]\} = 0$$

uniformly in $[\eta, 1-\eta]$, *where* $U_p[f(x)]$ *is given by*

$$U_p[f(x)] = b_{0p}\cos(\pi/2)x + b_{1p}\cos(3\pi/2)x + \cdots$$
$$+ b_{pp}\cos(p+1/2)\pi x,$$

$$b_{np} = \sum_{k=0}^{p}{}' f(x_k)\cos(n+1/2)\pi x_k \left\{ \sum_{k=0}^{p}{}' \cos^2(n+1/2)\pi x_k \right\}^{-1}$$

$$= \begin{cases} \dfrac{4}{2p+1} \sum_{k=0}^{p}{}' f(x_k) \cos (n + 1/2)\pi x_k, & n = 0, 1, \cdots, p-1, \\[2ex] \dfrac{2}{2p+1} \sum_{k=0}^{p}{}' f(x_k) \cos (n + 1/2)\pi x_k, & n = p. \end{cases}$$

The first set of theorems also holds using characteristic functions of $u^{(\mathrm{iv})} - [\rho^4 + \lambda(x)]u = 0$, with the above boundary conditions, $\lambda(x)$ being an arbitrary continuous function. Here, provided ${}_0u(x) \not\equiv 0$, the term ${}_0u(x)\sum_{k=0}^{\prime p} f(x_k)\ {}_0u(x_k) \{\sum_{k=0}^{\prime p} {}_0u^2(x_k)\}^{-1}$ is adjoined to $\Sigma_p[f(x)]$, where ${}_0u(x)$ is the characteristic function corresponding to $\rho = 0$. (For $\lambda(x) \equiv 0$, $\rho = 0$ is not a characteristic number.)

We give here the proof of the theorem for system V. System V is the problem of lateral vibrations of an elastic homogeneous rod clamped at both ends [2]. We first state two lemmas.

LEMMA 1. *If $f(x)$ is continuous and of bounded variation in* $[0, 1]$ *and $f(0) = f(1) = 0$, then $W_p[f(x)]$ and $w_p[f(x)]$ tend to the same limit $M(x)$ uniformly in* $[\eta, 1-\eta]$ *as $p \to \infty$, where*

$$\begin{aligned} W_p[f(x)] = {} & c_{0p}[\cos (\pi/2)x - \sin (\pi/2)x] \\ & + c_{1p}[\cos (3\pi/2)x - \sin (3\pi/2)x] + \cdots \\ & + c_{pp}[\cos (p + 1/2)\pi x - \sin (p + 1/2)\pi x], \end{aligned}$$

$$c_{np} = \frac{2}{2p+1} \sum_{k=0}^{p}{}' f(x_k) [\cos (n + 1/2)\pi x_k - \sin (n + 1/2)\pi x_k], \qquad n = 0, 1, \cdots, p,$$

$$\begin{aligned} w_p[f(x)] = {} & c_0[\cos (\pi/2)x - \sin (\pi/2)x] + c_1[\cos (3\pi/2)x - \sin (3\pi/2)x] \\ & + \cdots + c_p[\cos (p + 1/2)\pi x - \sin (p + 1/2)\pi x], \end{aligned}$$

$$c_n = \int_0^1 f(t) [\cos (n + 1/2)\pi t - \sin (n + 1/2)\pi t] dt, \qquad n = 0, 1, \cdots, p.$$

LEMMA 2. *If $f(x)$ satisfies a Lipschitz condition in* $[0, 1]$ *and $f(0) = f(1) = 0$, then there exists a constant C, depending only on η, such that for n sufficiently great, $p \geqq n$, and x in* $[\eta, 1-\eta]$,

$$|\alpha_{np}u_n(x) - c_{np}[\cos (n + 1/2)\pi x - \sin (n + 1/2)\pi x]| < C/n^2.$$

To prove Lemma 1, write

$$\begin{aligned} W_p[f(x)] &= {}_1W_p[f(x)] + {}_2W_p[f(x)] - {}_3W_p[f(x)], \\ w_p[f(x)] &= {}_1w_p[f(x)] + {}_2w_p[f(x)] - {}_3w_p[f(x)], \end{aligned}$$

where

$$ {}_1W_p[f(x)] = \frac{2}{2p+1}\sum_{k=0}^{p}{}' f(x_k)\sum_{n=0}^{p}\cos\,(n+1/2)\pi x_k\cos\,(n+1/2)\pi x, $$

$$ {}_2W_p[f(x)] = \frac{2}{2p+1}\sum_{k=0}^{p}{}' f(x_k)\sum_{n=0}^{p}\sin\,(n+1/2)\pi x_k\sin\,(n+1/2)\pi x, $$

$$ {}_3W_p[f(x)] = \frac{2}{2p+1}\sum_{k=0}^{p}{}' f(x_k)\sum_{n=0}^{p}\sin\,(n+1/2)\pi(x_k+x), $$

$$ {}_1w_p[f(x)] = \int_0^1 f(t)\sum_{n=0}^{p}\cos\,(n+1/2)\pi t\cos\,(n+1/2)\pi x dt, $$

$$ {}_2w_p[f(x)] = \int_0^1 f(t)\sum_{n=0}^{p}\sin\,(n+1/2)\pi t\sin\,(n+1/2)\pi x dt, $$

$$ {}_3w_p[f(x)] = \int_0^1 f(t)\sum_{n=0}^{p}\sin\,(n+1/2)\pi(t+x) dt. $$

We first show that if $f(x)$ is continuous and of bounded variation in $[0, 1]$, then $\lim_{p\to\infty} {}_1W_p[f(x)] = (1/2)f(x)$ uniformly in $[0, 1-\eta]$. We employ the cosine interpolation formula $T_p[f(x)]$. We shall use $r_n(x)$ as generic notation for a function uniformly bounded in n and for x in $[0, 1]$, unless the range for x is otherwise stated, and r_n $[r_{np}]$ for a quantity depending on n $[n$ and $p]$ and uniformly bounded in n $[n$ and $p]$. We have

$$ \begin{aligned} T_p[f(x)] &= \frac{2}{2p+1}\sum_{k=0}^{p}{}' f(x_k) + \frac{4}{2p+1}\sum_{k=0}^{p}{}' f(x_k)\sum_{n=1}^{p}\cos n\pi x_k\cos n\pi x \\ &= \frac{1}{2p+1}\sum_{k=0}^{p}{}' f(x_k)\left[\frac{\sin\,(p+1/2)\pi(x_k-x)}{\sin\,(\pi/2)(x_k-x)} + \frac{\sin\,(p+1/2)\pi(x_k+x)}{\sin\,(\pi/2)(x_k+x)}\right] \\ &= \frac{1}{2p+1}\sum_{k=0}^{p}{}' f(x_k)\left[\frac{\sin p\pi(x_k-x)\cos\,(\pi/2)(x_k-x)}{\sin\,(\pi/2)(x_k-x)} + \frac{\sin p\pi(x_k+x)\cos\,(\pi/2)(x_k+x)}{\sin\,(\pi/2)(x_k+x)}\right] \\ &\quad + \frac{2}{2p+1}\cos p\pi x\sum_{k=0}^{p}{}' f(x_k)\cos p\pi x_k. \end{aligned} $$

Using auxiliary Lemma A stated below, we have

$$T_p[f(x)] = \frac{1}{2p+1}\sum_{k=0}^{p}{}' f(x_k)\left[\frac{\sin p\pi(x_k - x)\cos(\pi/2)(x_k - x)}{\sin(\pi/2)(x_k - x)} + \frac{\sin p\pi(x_k + x)\cos(\pi/2)(x_k + x)}{\sin(\pi/2)(x_k + x)}\right] + \frac{r_p(x)}{p},$$

and

$$\begin{aligned} {}_1W_p[f(x)] &= \frac{2}{2p+1}\sum_{k=0}^{p}{}' f(x_k)\sum_{n=0}^{p-1}\cos(n+1/2)\pi x_k\cos(n+1/2)\pi x \\ &\quad + r_p(x)/p \\ &= \frac{1}{2(2p+1)}\sum_{k=0}^{p}{}' f(x_k)\left[\frac{\sin p\pi(x_k - x)}{\sin(\pi/2)(x_k - x)} \right. \\ &\quad \left. + \frac{\sin p\pi(x_k + x)}{\sin(\pi/2)(x_k + x)}\right] + \frac{r_p(x)}{p}. \end{aligned}$$

Thus

$$\begin{aligned} &{}_1W_p[f(x)] - (1/2)T_p[f(x)] \\ &= \frac{1}{2(2p+1)}\sum_{k=0}^{p}{}' f(x_k)\left[\frac{1-\cos(\pi/2)(x_k - x)}{\sin(\pi/2)(x_k - x)}\sin p\pi(x_k - x)\right. \\ &\qquad \left. + \frac{1-\cos(\pi/2)(x_k + x)}{\sin(\pi/2)(x_k + x)}\sin p\pi(x_k + x)\right] + \frac{r_p(x)}{p} \\ &= \frac{1}{2(2p+1)}\sum_{k=0}^{p}{}' f(x_k)[\tan(\pi/4)(x_k - x)\sin p\pi(x_k - x) \\ &\qquad + \tan(\pi/4)(x_k + x)\sin p\pi(x_k + x)] + r_p(x)/p. \end{aligned}$$

By reason of the nature of Lemma A and the fact that for t in $[0, 1]$ the functions $\tan(\pi/4)(t-x)$ and $\tan(\pi/4)(t+x)$ are of uniform bounded variation with respect to x in $[0, 1-\eta]$, we have ${}_1W_p[f(x)]-(1/2)T_p[f(x)]=r_p(x)/p$ for x in $[0, 1-\eta]$. Hence [3] $\lim_{p\to\infty} {}_1W_p[f(x)]=(1/2)f(x)$ uniformly in $[0, 1-\eta]$.

Employing $t_p[f(x)]$, the partial sum of order p in the Fourier cosine series,

$$t_p[f(x)] = a_0 + a_1\cos\pi x + \cdots + a_p\cos p\pi x,$$

$$a_n = \frac{\int_0^1 f(x)\cos n\pi x\,dx}{\int_0^1 \cos^2 n\pi x\,dx} = \begin{cases} 2\int_0^1 f(x)\cos n\pi x\,dx, & n = 1, 2, \cdots, p, \\ \int_0^1 f(x)\,dx, & n = 0, \end{cases}$$

we may show similarly that if $f(x)$ is continuous and of bounded variation in $[0, 1]$, then $\lim_{p\to\infty} {}_1w_p[f(x)]=(1/2)f(x)$ uniformly in $[0, 1-\eta]$. Employing $\overline{T}_p[f(x)]$, the sine interpolation formula, and $\bar{t}_p[f(x)]$, the partial sum of order p in the Fourier sine series,

$$\bar{t}_p[f(x)] = \bar{a}_1 \sin \pi x + \bar{a}_2 \sin 2\pi x + \cdots + \bar{a}_p \sin p\pi x,$$

$$\bar{a}_n = \frac{\int_0^1 f(x) \sin n\pi x dx}{\int_0^1 \sin^2 n\pi x dx} = 2\int_0^1 f(x) \sin n\pi x dx, \; n = 1, 2, \cdots, p,$$

we may likewise show that if $f(x)$ is continuous and of bounded variation in $[0, 1]$ and $f(0)=f(1)=0$, then

$$\lim_{p\to\infty} {}_2W_p[f(x)] = (1/2)f(x), \qquad \lim_{p\to\infty} {}_2w_p[f(x)] = (1/2)f(x)$$

uniformly in $[0, 1-\eta]$. In connection with ${}_1w_p[f(x)]$ and ${}_2w_p[f(x)]$ we use Lemma B, stated below.

Finally, we show that if $f(x)$ is of bounded variation in $[0, 1]$, then ${}_3W_p[f(x)]$ and ${}_3w_p[f(x)]$ both tend to $(1/2)\int_0^1 f(t)\{\sin (\pi/2)(t+x)\}^{-1}dt$ uniformly in $[\eta, 1-\eta]$. We have

$${}_3w_p[f(x)] = \frac{1}{2}\int_0^1 f(t)\, \frac{1-\cos (p+1)\pi(t+x)}{\sin (\pi/2)(t+x)}\, dt.$$

For x in $[\eta, 1-\eta]$,

$${}_3w_p[f(x)] = \frac{1}{2}\int_0^1 \frac{f(t)}{\sin (\pi/2)(t+x)}\, dt - \frac{1}{2}\int_0^1 f(t)\, \frac{\cos (p+1)\pi(t+x)}{\sin (\pi/2)(t+x)}\, dt.$$

Effecting some trigonometric reductions on the second integrand and using Lemma B, we have

$${}_3w_p[f(x)] = \frac{1}{2}\int_0^1 \frac{f(t)}{\sin (\pi/2)(t+x)}\, dt + \frac{r_p(x)}{p}\cdot$$

We may then prove that

$$\lim_{p\to\infty} {}_3W_p[f(x)] = \frac{1}{2}\int_0^1 \frac{f(t)}{\sin (\pi/2)(t+x)}\, dt$$

uniformly in $[\eta, 1-\eta]$, using the fact just established that

$$\lim_{p\to\infty} {}_3w_p[f(x)] = \frac{1}{2}\int_0^1 \frac{f(t)}{\sin (\pi/2)(t+x)}\,dt$$

uniformly in $[\eta, 1-\eta]$, together with Lemma C, stated below.

LEMMA A. *If* $f(x)$ *is of bounded variation in* $[0, 1]$, *then for* $n=1, 2, \cdots, 2p$,

$$\frac{1}{2p+1}\left|\sum_{k=0}^{p}{}' f(x_k)\cos n\pi x_k\right| = \frac{r_{np}}{n},$$

$$\frac{1}{2p+1}\left|\sum_{k=0}^{p}{}' f(x_k)\sin n\pi x_k\right| = \frac{r_{np}}{n}\cdot$$

Also

$$\frac{1}{2p+1}\left|\sum_{k=0}^{p}{}' f(x_k)\sin (2p+1)\pi x_k\right| = 0.$$

LEMMA B. *If* $f(x)$ *is of bounded variation in* $[0, 1]$, *then for* $n>0$,

$$\left|\int_0^1 f(x)\cos n\pi x dx\right| = \frac{r_n}{n}, \qquad \left|\int_0^1 f(x)\sin n\pi x dx\right| = \frac{r_n}{n}\cdot$$

LEMMA C. *If* $f(x)$ *is of bounded variation in* $[0, 1]$, *then for any preassigned* $\epsilon>0$ *there exists* Q *such that for* $p>q\geqq Q$ *and* x *in* $[\eta, 1]$,

$$\frac{2}{2p+1}\left|\sum_{k=0}^{p}{}' f(x_k)\sum_{n=q+1}^{p}\sin (n+1/2)\pi(x_k+x)\right| < \epsilon.$$

Lemma 2 is proved by means of auxiliary Lemmas D, E, and A.

LEMMA D. *For* n *sufficiently great, and* $p\geqq n$,

$$\left[\sum_{k=0}^{p}{}' u_n^2(x_k)\right]^{-1} = \frac{2}{2p+1}\left(1+\frac{r_{np}}{n}\right).$$

LEMMA E. *If* $f(x)$ *satisfies a Lipschitz condition in* $[0, 1]$ *and if* $f(0)=f(1)=0$, *then for* $n=1, 2, \cdots, p$,

$$\frac{1}{2p+1}\left|\sum_{k=0}^{p}{}' f(x_k)\exp\{-(n+1/2)\pi x_k\}\right| = \frac{r_{np}}{n^2},$$

$$\frac{1}{2p+1}\left|\sum_{k=0}^{p}{}' f(x_k)\exp\{-(n+1/2)\pi(1-x_k)\}\right| = \frac{r_{np}}{n^2}\cdot$$

The following asymptotic expression is known [4] for the characteristic functions of system V:

$$\begin{aligned} u_n(x) = {} & \cos (n + 1/2)\pi x - \sin (n + 1/2)\pi x \\ & + (-1)^n \exp \left\{- (n + 1/2)\pi(1 - x)\right\} \\ & - \exp \left\{- (n + 1/2)\pi x\right\} + \exp \left\{- (n + 1/2)\pi\right\} r_n(x). \end{aligned}$$

Define

$$\sigma_p[f(x)] = \alpha_0 u_0(x) + \alpha_1 u_1(x) + \cdots + \alpha_p u_p(x),$$

$$\alpha_n = \frac{\int_0^1 f(x) u_n(x) dx}{\int_0^1 u_n^2(x) dx}, \qquad n = 0, 1, \cdots, p.$$

Denote by $\Sigma_p^{(r,s)}$ the sum of the terms in $\Sigma_p[f(x)]$ with subscripts r to s inclusive; similarly in the other sums.

For x in $[\eta, 1-\eta]$,

$$\begin{aligned} \left| f(x) - \Sigma_p[f(x)] \right| \leqq {} & \left| \sigma^{(0,N)} - \Sigma_p^{(0,N)} \right| + \left| W_p^{(N+1,p)} - \Sigma_p^{(N+1,p)} \right| \\ & + \left| W_p^{(0,N)} - w^{(0,N)} \right| + \left| M(x) - W_p^{(0,p)} \right| \\ & + \left| w^{(0,N)} - M(x) \right| + \left| f(x) - \sigma^{(0,N)} \right|. \end{aligned}$$

The right-hand member can be made arbitrarily small by choosing p sufficiently large. Call the six terms $D_1, D_2, \cdots, D_6$. Given ϵ, first choose N sufficiently large so that $D_2 < \epsilon$, $D_5 < \epsilon$, $D_6 < \epsilon$, for $p \geqq N+1$. Having fixed N, choose P sufficiently large so that $D_1 < \epsilon$, $D_3 < \epsilon$, $D_4 < \epsilon$ for $p \geqq P$. It remains to justify these statements. For D_6 we use a result in a paper by J. D. Tamarkin [5]. For D_2 we use Lemma 2, and for D_4 and D_5 Lemma 1. In D_1 and D_3 we are dealing essentially with integrals of continuous functions and the sums which tend to the integrals as limits, the number of terms being finite.

Now consider $u^{(\mathrm{iv})} - [\rho^4 + \lambda(x)]u = 0$, $\lambda(x)$ being an arbitrary continuous function. A fundamental system of solutions of $u^{(\mathrm{iv})} - \rho^4 u = 0$ is $u_1 = \cos \rho x$, $u_2 = \sin \rho x$, $u_3 = e^{\rho x}$, $u_4 = e^{-\rho x}$. By the method of variation of constants we have, as an equation satisfied by u,

$$\begin{aligned} u = {} & A \cos \rho x + B \sin \rho x + Ce^{\rho x} + De^{-\rho x} \\ & + \frac{1}{2\rho^3} \int_0^x \lambda(t) u(t) \left[- \sin \rho(x - t) + \frac{e^{\rho(x-t)}}{2} - \frac{e^{-\rho(x-t)}}{2}\right] dt, \end{aligned}$$

where A, B, C, D are arbitrary constants. Applying the boundary conditions of system V and choosing the multiplicative constant so that $A=1$, we have, assuming $\rho\neq 0$,

$$\begin{aligned}
u = {} & \cos \rho x - \sin \rho x \\
& + \Big[\sin \rho x \Big\{ -2e^{-\rho} \sin \rho + \frac{1}{2\rho^3}\int_0^1 \lambda(t)u(t)\,[-e^{-\rho} \sin \rho(1-t) \\
& \qquad\qquad - e^{-\rho} \cos \rho(1-t) + e^{-\rho t}]dt \Big\} \\
& + e^{-\rho(1-x)} \sin \rho - \frac{1}{4\rho^3}\int_0^1 \lambda(t)u(t)\,[-e^{-\rho(1-x)} \sin \rho(1-t) \\
& \qquad\qquad - e^{-\rho(1-x)} \cos \rho(1-t) + e^{\rho(x-t)}]dt \\
& - e^{-\rho x} + e^{-\rho(1+x)} \cos \rho + \frac{1}{4\rho^3}\int_0^1 \lambda(t)u(t)\,[-e^{-\rho(1+x)} \sin \rho(1-t) \\
& - e^{-\rho(1+x)} \cos \rho(1-t) + e^{-\rho(x+t)}]dt \Big] \{1 - e^{-\rho} \sin \rho - e^{-\rho} \cos \rho\}^{-1} \\
& + \frac{1}{2\rho^3}\int_0^x \lambda(t)u(t)\Big[-\sin \rho(x-t) + \frac{e^{\rho(x-t)}}{2} - \frac{e^{-\rho(x-t)}}{2}\Big] dt,
\end{aligned}$$

and, as an equation satisfied by the characteristic numbers $\rho=\rho_n$,

$$\begin{aligned}
\cos \rho = {} & 2e^{-\rho} - e^{-2\rho} \cos \rho \\
& + \frac{\sin \rho}{2\rho^3}\int_0^1 \lambda(t)u(t)\Big[e^{-\rho} \cos \rho(1-t) - \frac{e^{-\rho t}}{2} - \frac{e^{-\rho(2-t)}}{2}\Big] dt \\
& + \frac{\cos \rho}{2\rho^3}\int_0^1 \lambda(t)u(t)\Big[-e^{-\rho} \sin \rho(1-t) + \frac{e^{-\rho t}}{2} - \frac{e^{-\rho(2-t)}}{2}\Big] dt \\
& + \frac{1}{4\rho^3}\int_0^1 \lambda(t)u(t)\,[\sin \rho(1-t) - \cos \rho\,(1-t) + e^{-\rho(1-t)}]dt \\
& - \frac{e^{-\rho}}{4\rho^3}\int_0^1 \lambda(t)u(t)\,[-e^{-\rho} \sin \rho(1-t) - e^{-\rho} \cos \rho(1-t) + e^{-\rho t}]dt.
\end{aligned}$$

For each $\rho=\rho_n$, u is a continuous function; we show u uniformly bounded in n. In u, the terms

$$-\frac{1}{4\rho^3}\int_0^1 \lambda(t)u(t)e^{\rho(x-t)}dt\{1 - e^{-\rho} \sin \rho - e^{-\rho} \cos \rho\}^{-1},$$

$$\frac{1}{4\rho^3}\int_0^x \lambda(t)u(t)e^{\rho(x-t)}dt$$

may be combined to give

$$\left[-\frac{1}{4\rho^3}\int_x^1 \lambda(t)u(t)e^{\rho(x-t)}dt - \frac{\sin\rho+\cos\rho}{4\rho^3}\int_0^x \lambda(t)u(t)e^{-\rho(1-x+t)}dt\right]$$

$$\cdot\left\{1 - e^{-\rho}\sin\rho - e^{-\rho}\cos\rho\right\}^{-1}.$$

For t in $[x, 1]$, $x-t\leqq 0$; for t in $[0, x]$, $1-x+t\geqq 0$. Also, there exists a constant $c>0$ such that $1-e^{-\rho}\sin\rho-e^{-\rho}\cos\rho\geqq c$ for $\rho=\rho_n$. Call $M_n=\max_{[0,1]}|u_n(x)|$, $K=\int_0^1|\lambda(t)|dt$. Then

$$|u_n(x)| \leqq 2+\frac{5}{c}+\frac{17M_nK}{4c\rho_n^3}, \qquad M_n \leqq 2+\frac{5}{c}+\frac{17M_nK}{4c\rho_n^3},$$

$$M_n \leqq \frac{2+5/c}{1-17K/(4c\rho_n^3)}.$$

Thus for n sufficiently great, $M_n\leqq 4+10/c$. The remaining n's form a finite set. Hence $u_n(x)$ is uniformly bounded in n. Thus

$$u_n(x) = \cos\rho_n x - \sin\rho_n x + e^{-\rho_n(1-x)}\sin\rho_n - e^{-\rho_n x} + r_n(x)/\rho_n^3,$$

and $\cos\rho_n=\phi(\rho_n)$, where $\lim_{n\to\infty}\phi(\rho_n)=0$. Thus $\rho_n=(n+1/2)\pi+\epsilon_n$, where $\lim_{n\to\infty}\epsilon_n=0$. It follows that

$$\begin{aligned} u_n(x) = {} & \cos(n+1/2)\pi x - \sin(n+1/2)\pi x \\ & + (-1)^n\exp\left\{-(n+1/2)\pi(1-x)\right\} \\ & - \exp\left\{-(n+1/2)\pi x\right\} + r_n(x)/n^3. \end{aligned}$$

The theorem for system V follows.

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BROWN UNIVERSITY

AN ALMOST UNIVERSAL FORM

GORDON PALL

P. R. Halmos[1] obtained the 88 possible forms (a, b, c, d), $0 < a \leqq b \leqq c \leqq d$, which represent all positive integers with one exception, and proved that property for all except for the form $h = (1, 2, 7, 13)$. A proof for h follows.

The forms $f = (1, 2, 7)$ and $g = (1, 1, 14)$ constitute the reduced forms of a genus.[2] Between them they represent all positive integers not of the form[3] $\Lambda = 7^{2k+1}(7m+3, 5, 6)$. The identities

$$\begin{aligned} x^2 + y^2 + 14z^2 &= x^2 + 2((y + 7z)/3)^2 + 7((y - 2z)/3)^2 \\ &= y^2 + 2((x + 7z)/3)^2 + 7((x - 2z)/3)^2 \end{aligned}$$

show that every number represented by g with either $y \equiv -z$ or $x \equiv -z \pmod 3$ is also represented by f. Hence every number $3n$ and $3n+1$ not of the form Λ is represented by f. For, $x \equiv y \equiv 0$, $z \not\equiv 0$, and $x, y \not\equiv 0$, $z \equiv 0 \pmod 3$ both imply $g \equiv 2$. If $N = 3n$ or $3n+1$ is of the form Λ, then $7 \mid N$, so that $N - 13 \cdot 3^2 \neq \Lambda$. Similarly, one of $3n+2-13$ and $3n+2-52$ is not of the form Λ; but neither of these is congruent to 2 (mod 3). These linear forms are positive if $n \geqq 39$; h represents all integers not less than 119. The only number less than 119 not represented in (1, 2, 7, 13) is found to be 5.

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[1] This Bulletin, vol. 44 (1938), pp. 141–144.

[2] See any table of positive ternaries.

[3] For example, see B. W. Jones, Transactions of this Society, vol. 33 (1931), pp. 111–124.

THE GENERAL LIMIT THEOREM IN THE THEORY OF PROBABILITY[1]

WOLFGANG STERNBERG

Let $x_1, x_2, \cdots, x_n$ be independent chance variables with any distribution functions $V_1, V_2, \cdots, V_n$.

We put[2] for all $i=1, 2, \cdots, n$

$$M[x_i] = \int \xi dV_i(\xi) = a_i \qquad \text{(mean value)},$$

$$\text{St. D. } (x_i) = \int (\xi - a_i)^2 dV_i(\xi) = b_i \qquad \text{(standard deviation)},$$

$$\int |\xi - a_i|^3 dV_i(\xi) = c_i \qquad \text{(absolute moment of the 3rd order)}.$$

(The term "standard deviation" is used by some authors for $b_i^{1/2}$ instead of b_i.)

Denoting mean value and standard deviation of the sum $x_1 + \cdots + x_i$ $(i=2, \cdots, n)$ by A_i and B_i respectively, it is known that

$$A_i = a_1 + \cdots + a_i, \qquad B_i = b_1 + \cdots + b_i.$$

For the distribution function W_i of the sum $x_1 + \cdots + x_i$ the recursion formula

$$(1) \qquad W_i(x) = \int W_{i-1}(x - \xi) dV_i(\xi), \qquad i > 1,$$

holds true.

Let a_i and B_n be as follows:

$$(2) \qquad a_i = 0,$$

$$(3) \qquad B_n = 1.$$

These conditions are only preliminarily stated in order to abbreviate the explanation. From (2) we have that $A_i = 0$, in particular, $A_n = 0$. (The case $b_i = 0$ may be excluded, since then x_i assumes the value 0 only with the probability 1 and thus does not appear in the sum $x_1 + \cdots + x_n$.)

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[1] Presented to the Society, October 28, 1939.

[2] The following integrals are Stieltjes integrals. The lower and upper limits always are $-\infty$ and $+\infty$ respectively.

We have to prove that the function $W_n(x)$ lies uniformly in x near the Gaussian function having also the mean value 0 and the standard deviation 1, that is,

$$\phi(x/2^{1/2}) = \frac{1}{\pi^{1/2}} \int_{-\infty}^{x/2^{1/2}} e^{-t^2} dt,$$

if very general suppositions with respect to the V are made. We suppose that the quotients c_i/b_i are "very small" for all i and state the general limit theorem in the following form:

To every positive ϵ *there is a positive* δ, *so that uniformly in* x

$$\left| W_n(x) - \phi(x/2^{1/2}) \right| < \epsilon$$

holds true if the conditions

$$c_i/b_i < \delta, \qquad i = 1, 2, \cdots, n, \tag{4}$$

are satisfied.

Our proof is suggested by that of A. Khintchine (see below). We remark first of all that

$$b_i < \delta, \qquad i = 1, 2, \cdots, n, \tag{5}$$

follows from (4) according to the well known inequality $b_i^3 < c_i^3$. (Of course we can suppose $\delta < 1$.) Since $B_n = 1$ but the b_i are small, it is obvious that n must be a large number.

Now we note that the function of two variables

$$\phi(x/(2y)^{1/2}) = \frac{1}{\pi^{1/2}} \int_{-\infty}^{x/(2y)^{1/2}} e^{-t^2} dt = U(x, y),$$

that is, the Gaussian function with the variable standard deviation y represents a solution of the differential equation of the conduction of heat

$$\frac{\partial \phi}{\partial y} = \frac{1}{2} \frac{\partial^2 \phi}{\partial x^2}. \tag{6}$$

Here and in what follows it is always supposed that $y \geqq \eta$, η being a positive small but constant magnitude. The equation (1) suggests that we compare the two expressions $U(x, B_i)$ and

$$\int U(x - \xi, B_{i-1}) d V_i(\xi).$$

(Later the variable y will be put equal to B_{i-1}.) According to Taylor's formula we have

$$U(x, y + b_i) = U(x, y) + b_i \frac{\partial U(x, y)}{\partial y} + R_1,$$

$$R_1 = \frac{b_i^2}{2} \frac{\partial^2 U(x, y + \theta b_i)}{\partial y^2}, \qquad 0 < \theta < 1.$$

Let $M(\eta)$ be an upper limit of the absolute values of the derivatives of the 2d and 3d order of $U(x, y)$ in the half-plane $y \geqq \eta$; then

$$| R_1 | < b_i^2 M(\eta)/2.$$

On the other hand,

$$U(x - \xi, y) = U(x, y) - \xi \frac{\partial U(x, y)}{\partial x} + \frac{\xi^2}{2} \frac{\partial^2 U(x, y)}{\partial x^2}$$

$$- \frac{\xi^3}{6} \frac{\partial^3 U(x - \theta_1 \xi, y)}{\partial x^3}, \qquad 0 < \theta_1 < 1.$$

Therefore, in view of (2) and (6)

$$\int U(x - \xi, y) dV_i(\xi) = U(x, y) + b_i \frac{\partial U(x, y)}{\partial y} + R_2,$$

where

$$R_2 = -\frac{1}{6} \int \xi^3 \frac{\partial^3 U(x - \theta_1 \xi, y)}{\partial x^3} dV_i(\xi);$$

thus

$$| R_2 | < c_i M(\eta)/6.$$

It follows that

$$U(x, y + b_i) = \int U(x - \xi, y) dV_i(\xi) + R \tag{7}$$

in which $R = R_1 - R_2$; accordingly

$$| R | \leqq | R_1 | + | R_2 | < M(\eta) \{ b_i^2/2 + c_i/6 \} < [M(\eta)/2] \{ b_i^2 + c_i \}. \tag{8}$$

Now note that all $b_i < \eta$ because of (5), since of course $\delta < \eta$ may be supposed. On the contrary there are numbers $B_i > \eta$, as for instance $B_n = 1 > \eta$. There must be consequently a number B_r satisfying $\eta < B_r < 2\eta$. Now we put $y = B_{i-1}$ in (7); and we suppose $i \geqq r+1$ in order that $y \geqq B_r > \eta$. Besides we write with regard to what follows

$x+2\alpha$ instead of x, α representing a positive parameter. Then it follows from (7) that

$$U(x+2\alpha, B_i) = \int U(x+2\alpha-\xi, B_{i-1})dV_i(\xi) + R \tag{9}$$

in which R satisfies (8). Putting the difference

$$W_i(x) - U(x+2\alpha, B_i) = W_i(x) - \phi\left(\frac{x+2\alpha}{(2B_i)^{1/2}}\right) = D_i(x), \qquad i = 1, 2, \cdots, n, \tag{10}$$

and subtracting (9) from (1), we get the fundamental formula, valid for all x,

$$D_i(x) = \int D_{i-1}(x-\xi)dV_i(\xi) - R \tag{11}$$

together with (8), where $i=r+1, r+2, \cdots, n$.

In order to get further estimates we need the elementary inequalities of Tschebyscheff, which are valid for any distribution function F with vanishing mean value and standard deviation not greater than B:

$$F(\alpha) \geqq 1 - B/\alpha^2, \qquad F(-\alpha) \leqq B/\alpha^2, \qquad \alpha > 0.$$

From these inequalities it easily follows for two distribution functions F_1, F_2 with vanishing mean values, the standard deviations of which are both not greater than B, and for all x, that

$$F_1(x) - F_2(x+2\alpha) \leqq B/\alpha^2, \qquad \alpha > 0. \tag{12}$$

For, if $x \leqq -\alpha$, one has $F_1(x) \leqq F_1(-\alpha) \leqq B/\alpha^2$; thus (12) follows because $F_2 \geqq 0$. If $x > -\alpha$, one concludes $F_2(x+2\alpha) \geqq F_2(\alpha) \geqq 1 - B/\alpha^2$; therefore

$$F_1(x) - F_2(x+2\alpha) \leqq 1 - F_2(x+2\alpha) \leqq 1 - (1 - B/\alpha^2),$$

that is, (12). On the foundation of (11) and (12) the proof may be carried out. Putting $i=r$ one has, according to the definition of $D_i(x)$ and because both $W_r(x)$ and $U(x, B_r)$ have the mean value 0 and the standard deviation $B_r < 2\eta$, on account of (12),

$$W_r(x) - U(x+2\alpha, B_r) = D_r(x) < 2\eta/\alpha^2 \quad \text{for all } x, \qquad \alpha > 0; \tag{13}$$

whence, denoting by g_i the upper limit of $D_i(x)$, we conclude from (13) $g_r < 2\eta/\alpha^2$ and from (11), for any $i \geqq r+1$, $D_i(x) \leqq g_{i-1} + |R|$; thus

$$g_i \leqq g_{i-1} + [M(\eta)/2]\{b_i^2 + c_i\}$$

and through induction from i to $i+1$

$$g_n \leqq g_r + \frac{M(\eta)}{2} \sum_{i=r+1}^{n} \{b_i^2 + c_i\} < \frac{2\eta}{\alpha^2} + \frac{M(\eta)}{2} \sum_{i=1}^{n} \{b_i^2 + c_i\}. \tag{14}$$

On the other hand one has

$$b_i < \delta, \qquad c_i/b_i < \delta, \qquad\qquad i = 1, 2, \cdots,$$

because of (5) and (4). Therefore

$$\sum_{i=1}^{n} b_i^2 < \delta \sum b_i = \delta B_n = \delta, \qquad \sum_{i=1}^{n} c_i < \delta \sum b_i = \delta.$$

It follows that $g_n < 2\eta/\alpha^2 + \delta M(\eta)$; thus

$$D_n(x) = W_n(x) - \phi\left(\frac{x + 2\alpha}{2^{1/2}}\right) < 2\eta/\alpha^2 + \delta M(\eta).$$

Therefore

$$W_n(x) < \phi(x/2^{1/2}) + 2\alpha/(2\pi)^{1/2} + 2\eta/\alpha^2 + \delta M(\eta). \tag{15}$$

The positive number ϵ being given, one determines first α through $2\alpha/(2\pi)^{1/2} = \epsilon/3$, then η through $2\eta/\alpha^2 = \epsilon/3$, with which $M(\eta)$ is determined, and finally δ through $\delta M(\eta) \leqq \epsilon/3$. It follows that

$$W_n(x) < \phi(x/2^{1/2}) + \epsilon, \tag{16}$$

uniformly in x. Using

$$\phi\left(\frac{x - 2\alpha}{(2B_i)^{1/2}}\right) - W_i(x) = \Delta_i(x)$$

one finds in the same way

$$\Delta_n(x) < 2\eta/\alpha^2 + \delta M(\eta)$$

and

$$W_n(x) > \phi(x/2^{1/2}) - \epsilon. \tag{17}$$

The proof is thus complete.

One should compare this with the proof given by A. Khintchine (*Asymptotische Gesetze der Wahrscheinlichkeitsrechnung*, Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 2, no. 4, Berlin, 1933), which is based on works of I. Petrowsky and A. Kolmogoroff. We have chosen other assumptions in order to make the exposition as simple as possible, while Khintchine strives for wide generality so that nothing is supposed by him as to moments of the third order.

A so-called upper- and lower-function, the introduction of which is denoted by Khintchine as the fundamental idea of the Petrowsky-Kolmogoroffian proof, is not used by us.

We now generalize the investigation and omit the special conditions (2) and (3).

We use the transformation

$$\frac{x_i - a_i}{B_n^{1/2}} = u_i, \qquad i = 1, \cdots, n. \tag{18}$$

If mean value, standard deviation, and absolute moment of the third order of u_i are denoted by $\bar{a}_i$, $\bar{b}_i$, and $\bar{c}_i$ respectively, it becomes

$$\bar{a}_i = 0, \qquad \bar{b}_i = b_i/B_n, \qquad \bar{c}_i = c_i/(B_n)^{3/2} \tag{2'}$$

and accordingly

$$\frac{\bar{c}_i}{\bar{b}_i} = \frac{c_i}{b_i(B_n)^{1/2}},$$

and

$$\bar{B}_n = \text{St. D. } (u_1 + \cdots + u_n) = \bar{b}_1 + \cdots + \bar{b}_n = 1. \tag{3'}$$

The assumptions (4) must be replaced by

$$\frac{c_i}{b_i B_n^{1/2}} < \delta, \qquad i = 1, \cdots, n. \tag{4'}$$

Denoting by $\overline{W}_n$ the distribution function of the sum $u_1 + \cdots + u_n$, we thus have according to (16), (17) uniformly in u

$$\overline{W}_n(u) = \phi(u/2^{1/2}) + \sigma, \qquad |\sigma| < \epsilon. \tag{19}$$

But it easily follows in view of

$$u_1 + \cdots + u_n = (x_1 + \cdots + x_n - A_n)/B_n^{1/2}$$

that the distribution function $W_n(x)$ of $x_1 + \cdots + x_n$ is identical with $\overline{W}_n(u)$ if the relation

$$u = (x - A_n)/B_n^{1/2} \tag{20}$$

between u and x is valid.

According to (19) and $W_n(x) = \overline{W}_n(u)$ we have uniformly in x

$$W_n(x) = \phi\left(\frac{x - A_n}{(2B_n)^{1/2}}\right) + \sigma, \qquad |\sigma| < \epsilon. \tag{21}$$

We can thus say: *To every positive ϵ there is a positive δ, so that* (21) *holds true, if the assumptions* (4′) *are satisfied.*

We call special attention to the case

$$0 < g < b_i; \qquad c_i < H,$$

g and H being independent of i. One has $gn < B_n$ and as for the rest $b_i < c_i^{2/3} < H^{2/3}$ and $B_n < nH^{2/3}$, so that B_n is exactly of the order n. It follows that

$$\frac{c_i}{b_i B_n^{1/2}} < \frac{H}{g g^{1/2} n^{1/2}} \cdot$$

Therefore (4′) can be satisfied, if n is sufficiently large.

In the same way one can discuss sets of chance variables $x_{11}, x_{12}, \cdots, x_{1n_1}, \cdots, x_{k1}, x_{k2}, \cdots, x_{kn_k}, \cdots, n_k$ increasing indefinitely with k. It is supposed that the quotients

$$\frac{c_{k1}}{b_{k1} B_{n_k}^{1/2}}, \cdots, \frac{c_{kn_k}}{b_{kn_k} B_{n_k}^{1/2}}$$

approach 0 as a limit, if k increases indefinitely. That means that if any $\delta > 0$ is given one can choose k sufficiently large so that *all* these quotients are less than δ. Then the distribution function W_{n_k} of $x_{k1} + \cdots + x_{kn_k}$ approaches $\phi((x - A_{n_k})/(2B_{n_k})^{1/2})$ as a limit.

The conditions for the quotients are in particular satisfied if $B_{n_k} > B > 0$ for every k (B being a constant number) and the quotients $c_{k1}/b_{k1}, \cdots, c_{kn_k}/b_{kn_k}$ approach 0, as k increases indefinitely.

NEW YORK, N. Y.

A NEW LOWER BOUND FOR THE EXPONENT IN THE FIRST CASE OF FERMAT'S LAST THEOREM[1]

BARKLEY ROSSER

1. **Introduction.** In this paper is proved the theorem: If p is an odd prime and

$$a^p + b^p + c^p = 0 \tag{1}$$

has a solution in integers prime to p, then $p > 41{,}000{,}000$.

It seems certain that still higher lower bounds for p can be deduced by the methods of this paper. However an argument is given which makes it seem unlikely that an indefinitely high lower bound can be so deduced.

2. **Preliminary results.** Unless otherwise specified, we shall assume that p is an odd prime for which (1) can be satisfied by integers prime to p. Hence[2] $p > 8{,}000{,}000$. Also $x \equiv y$ shall denote $x \equiv y \pmod{p}$. Also any statement regarding factorization of a polynomial is to be understood modulo p.

Morishima has proved[3] that for each odd prime $m \leqq 43$, there is a t ($t \not\equiv 0$, $\not\equiv 1$) such that each of the values

$$t, \quad \frac{1}{t}, \quad 1 - t, \quad \frac{1}{1-t}, \quad \frac{t-1}{t}, \quad \frac{t}{t-1} \tag{2}$$

satisfies each of the following relations when substituted for x:

$$\{(m^{p-1} - 1)/p\}(x^{m-1} - 1) \equiv 0, \tag{3}$$

$$x^4 \not\equiv 1, \tag{4}$$

$$x^6 \not\equiv 1. \tag{5}$$

Morishima further proved that for each odd prime $m \leqq 31$, there is no t such that the values in (2) satisfy (4), (5), and

$$x^{m-1} \equiv 1. \tag{6}$$

Hence for such m's, $(m^{p-1} - 1)/p \equiv 0$. That is

$$m^{p-1} \equiv 1 \pmod{p^2}. \tag{7}$$

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[1] Presented to the Society, September 8, 1939.

[2] Barkley Rosser, *On the first case of Fermat's last theorem*, this Bulletin, vol. 45 (1939), pp. 636–640. This paper will be referred to as I.

[3] Taro Morishima, *Über den Fermatschen Quotienten*, Japanese Journal of Mathematics, vol. 8 (1931), pp. 159–173.

In this paper we shall use the fact that $p>8{,}000{,}000$ to prove that, for $m=37$ or 41, there is no t such that the values in (2) satisfy (4), (5), and (6). Hence we can conclude that (7) holds for each prime $m\leqq 41$ (since it is well known that (7) holds for $m=2$). This fact is then used to prove that $p>41{,}000{,}000$.

Until §6, we assume that m is an odd prime, $n=(m-1)/2$, and that a t exists such that the values of (2) satisfy (4), (5), and (6).

We reduce the consideration of (4), (5), and (6) to a familiar theory[4] by replacing t by $-a$. Then since the polynomials in (4), (5), and (6) are all even functions, we see that each of

$$\begin{array}{c}a,\ \dfrac{1}{a},\ -1-a,\ \dfrac{1}{-1-a},\ \dfrac{1+a}{-a},\ \dfrac{-a}{1+a},\\ -a,\ \dfrac{1}{-a},\ 1+a,\ \dfrac{1}{1+a},\ \dfrac{1+a}{a},\ \dfrac{a}{1+a},\end{array}\tag{8}$$

must satisfy (4), (5), and (6). Put

$$b=\frac{-a^6-3a^5+5a^3-3a-1}{a^2(a+1)^2},$$

and define

$$f(x)=(x-a)(x-1/a)(x+1+a)(x+1/(1+a))(x+(1+a)/a)(x+a/(1+a)).$$

Then $f(x)=x^6+3x^5+bx^4+(2b-5)x^3+bx^2+3x+1$.

We shall prove in §3 that $f(x)f(-x)$ has no multiple factors if $m\leqq 67$. So for $m\leqq 67$, $f(x)f(-x)$ must divide $x^{m-1}-1$. Also note that $f(x)f(-x)$ has no factors in common with either x^4-1 or x^6-1, by (4) and (5).

LEMMA 1. *If $\phi(x)$ is a polynomial in x and $f(x)f(-x)$ has no multiple factors, then a n.a.s.c. that $f(x)f(-x)$ divide $\phi(x)$ is that $f(x)$ divide both $\phi(x)$ and $\phi(x+1)$.*

PROOF. By comparing the values in (8), one quickly sees that $f(-x)=f(x-1)$. So $f(-x)$ divides $\phi(x)$ if and only if $f(x-1)$ divides $\phi(x)$, if and only if $f(x)$ divides $\phi(x+1)$.

LEMMA 2. *If $f(x)f(-x)$ has no multiple factors and divides $x^{4N}-1$, then either*

A. *$f(x)$, $x^{2N}-1$, and $(x+1)^{2N}+1$ have a common factor, or*

B. *$f(x)f(-x)$ divides $x^{2N}-1$.*

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[4] In particular we shall use the results of L. E. Dickson, *On the last theorem of Fermat*, Messenger of Mathematics, vol. 38 (1908), pp. 14–32.

PROOF. Use Lemma 1 and the results of Dickson[4] (loc. cit., §11, pp. 20–21).

3. **$f(x)f(-x)$ has no multiple factors if $m\leqq 67$.** By (4) and (5) one can readily see that the only cases in which $f(x)f(-x)$ can have a multiple root are:

I. $a^2+a-1\equiv 0$,
II. $a^2-a-1\equiv 0$,
III. $a^2+3a+1\equiv 0$.

In case I, $x^{2n}-1$ and x^2-x-1 have the factor $x+a$ in common by (6). In case II, $x^{2n}-1$ and x^2-x-1 have the factor $x-a$ in common by (6). In case III, $x^{2n}-1$ and x^2-x-1 have the factor $x+a+1$ in common by (6). So it suffices to prove that $x^{2n}-1$ and x^2-x-1 have no factors in common.

LEMMA 3. *If $n\leqq 33$, then $x^{2n}-1$ and x^2-x-1 have no common factor modulo p.*

PROOF. Divide $x^n\pm 1$ by x^2-x-1. If $a_1, a_2, \cdots$ are the Fibonacci numbers, that is $a_1=a_2=1$, $a_{j+2}=a_{j+1}+a_j$, and $Q_j(x)=a_1x^j+a_2x^{j-1}+\cdots+a_jx+a_{j+1}$, then $x^n\pm 1=(x^2-x-1)Q_{n-2}(x)+a_nx+(a_{n-1}\pm 1)$. The eliminant of x^2-x-1 and $a_nx+(a_{n-1}\pm 1)$ is $(a_{n-1})^2+a_{n-1}a_n-(a_n)^2\pm(2a_{n-1}+a_n)+1$. However $(a_{j-1})^2+a_ja_{j-1}-(a_j)^2=(-1)^j$. So the eliminant of $x^n\pm 1$ and x^2-x-1 is $1+(-1)^n\pm(2a_{n-1}+a_n)$. However $a_j=(k^j-l^j)/5^{1/2}$, where $k=(1+5^{1/2})/2$, $l=(1-5^{1/2})/2$. So a_j equals the integer nearest to $k^j/5^{1/2}$. Hence $x^n\pm 1$ and x^2-x-1 have no common factor if $n\leqq 33$, since $p>8{,}000{,}000$ and factorization is modulo p. Therefore $x^{2n}-1$ and x^2-x-1 have no common factor if $n\leqq 33$.

4. **Proof that $m\neq 37$.** Assume $m=37$, and then use Lemma 2 with $N=9$.

Case 1. $x-\alpha$ is a common factor of $f(x)$, $x^{18}-1$, and $(x+1)^{18}+1$. By (5), $x-\alpha$ is not a factor of x^6-1. So $x-\alpha$ is a common factor of $(x^6+x^3+1)(x^6-x^3+1)$ and $(x+1)^{18}+1$.

Subcase I. $x-\alpha$ is a common factor of x^6+x^3+1 and $(x+1)^{18}+1$. So $\alpha^9\equiv 1$ and $(\alpha^2+2\alpha+1)^9/\alpha^9+1\equiv 0$. Put $\beta=\alpha+1/\alpha$. Then $A=\beta^3-3\beta+1\equiv 0$, and, since $\beta+2=(\alpha+1)^2/\alpha$,

$$(\beta+2)^9+1=((\beta+2)^3+1)((\beta+2)^6-(\beta+2)^3+1)\equiv 0.$$

If $(\beta+2)^3+1\equiv 0$, then $6\beta^2+15\beta+8\equiv 0$, $23\beta+52\equiv 0$, and[5] $2516=2^2\cdot 17\cdot 37\equiv 0$. So

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[5] Here, and at corresponding places later, a contradiction results from the fact that $p>8{,}000{,}000$.

$$(\beta+2)^6-(\beta+2)^3+1\equiv 0,\qquad B=417\beta^2+699\beta-137\equiv 0,$$
$$B\beta-417A=C=699\beta^2+1096\beta-411\equiv 0,$$
$$3B-C=534\beta^2+1001\beta\equiv 0.$$

However $\beta\equiv 0$ would contradict $A\equiv 0$. So $534\beta+1001\equiv 0$. So $883{,}227=3\cdot 37\cdot 73\cdot 109\equiv 0$.

Subcase II. $x-\alpha$ is a common factor of x^6-x^3+1 and $(x+1)^{18}+1$. So $\alpha^9\equiv -1$ and $(\alpha^2+2\alpha+1)^9/\alpha^9\equiv 1$. Put $\beta=\alpha+1/\alpha$. Then

$$\beta^3-3\beta-1\equiv 0,$$
$$(\beta+2)^9-1\equiv((\beta+2)^3-1)((\beta+2)^6+(\beta+2)^3+1)\equiv 0.$$

If $(\beta+2)^3-1\equiv 0$, then $6\beta^2+15\beta+8=0$, $23\beta+28\equiv 0$, and $724=2^2\cdot 181\equiv 0$. So

$$(\beta+2)^6+(\beta+2)^3+1\equiv 0,\qquad 447\beta^2+861\beta+271\equiv 0,$$
$$37(187\beta+302)\equiv 0.$$

Using $187\beta+302\equiv 0$, we get $1{,}620{,}673=73\cdot 149^2\equiv 0$.

Case 2. $f(x)f(-x)$ divides $x^{18}-1$. Because of (5), $f(x)f(-x)$ divides $x^{12}+x^6+1$. So $f(x)f(-x)\equiv x^{12}+x^6+1$. Comparing the coefficients of x^{10} and x^8, we get $2b-9\equiv 0$ and $b^2-10b+30\equiv 0$. So $21\equiv 0$.

5. **Proof that $m\neq 41$.** Assume $m=41$. Use Lemma 2 with $N=10$.

Case 1. $f(x)$, $x^{20}-1$, and $(x+1)^{20}+1$ have a common factor. The discussion in Dickson (loc. cit., pp. 23–24) will eliminate this case.

Case 2. $f(x)f(-x)$ is a factor of $x^{20}-1$. By Lemma 2 with $N=5$, either $f(x)$, $x^{10}-1$ and $(x+1)^{10}+1$ have a common factor, or else $f(x)f(-x)$ divides $x^{10}-1$. The latter is impossible, so let $x-\alpha$ be a common factor of $f(x)$, $x^{10}-1$, and $(x+1)^{10}+1$. Because of (4), $x-\alpha$ is not a factor of x^2-1.

Subcase I. $x-\alpha$ is a common factor of $x^4+x^3+x^2+x+1$ and $(x+1)^{10}+1$. Then $\alpha^5\equiv 1$. So $(\alpha^2+2\alpha+1)^5/\alpha^5+1\equiv 0$. Put $\beta=\alpha+1/\alpha$. Then $\beta^2\equiv-\beta+1$, $(\beta+2)^5+1\equiv 0$. So $5(11\beta+16)\equiv 0$. So $41\equiv 0$.

Subcase II. $x-\alpha$ is a common factor of $x^4-x^3+x^2-x+1$ and $(x+1)^{10}+1$. Then $\alpha^5\equiv -1$. So put $\beta=\alpha+1/\alpha$. Then $\beta^2\equiv\beta+1$, $(\beta+2)^5-1\equiv 0$. So $275\beta+174\equiv 0$. So $2501=41\cdot 61\equiv 0$.

6. **Proof that $p>41{,}000{,}000$.** We have now proved

$$37^{p-1}\equiv 1 \pmod{p^2},\qquad 41^{p-1}\equiv 1 \pmod{p^2}.$$

So we can argue as in I and conclude that $p>41{,}000{,}000$.

7. **The non-generality of this method.** It seems certain that the

lower bound just obtained is not the best that this method will produce. For instance, now that we know that $p > 41{,}000{,}000$, it is almost certain that we can prove that (7) holds for $m = 43$ (in fact, this probably follows from $p > 8{,}000{,}000$). Also, the fact that $p > 41{,}000{,}000$ will probably make it easier to prove that, for appropriate $m > 43$, the values of (2) must satisfy (3), (4), and (5), and so, for these m's also, (7) undoubtedly holds. Nevertheless the methods of this paper will apparently fail for very large values of p.

To see the difficulty that would arise, let us ignore any difficulties inherent in the proof that the values of (2) must satisfy (3), (4), and (5), and assume that for any m, this result would be easily forthcoming. Then, for any m, (7) would hold except for these p's which are factors of the eliminant of $x^{m-1} - 1$ and $(x+1)^{m-1} - 1$. As $p > 41{,}000{,}000$, we could then prove that (7) holds for all $m \leqq N_1$ (undoubtedly $N_1 > 43$). From this, by the method of I, we could prove that $p > M_1$. From this we could then prove that (7) holds for all $m \leqq N_2$, and so on. Unfortunately the M's and N's would probably not increase indefinitely.

To see this, let us first consider the eliminant of $x^{m-1} - 1$ and $(x+1)^{m-1} - 1$. Let $x - \alpha$ be a common factor. Then $\alpha^{m-1} \equiv 1$ and $(\alpha+1)^{m-1} \equiv 1$. We break this into cases (essentially corresponding to a factorization of the eliminant (see Dickson,[6] p. 28), by considering the eliminants of $(x+1)^u - x^v$ and $x^{m-1} - 1$ for small values of u and v. The case $u = 1$, $v = 2$ was treated in Lemma 3, and two eliminants of the order $k^{m/2}$ were obtained. If m is the rth prime, m is of the order of $r \log r$ (see Landau,[7] pp. 213–215). So the eliminants are of the order of r^{br}.

Now we consider the lower bounds for p which can be obtained by the method of I. Let the Σ's be as in Lemma 5 of I and let r be large. Then $\Sigma_1 = \theta(p_r) - \log 2 < 2r \log r$ (Landau, loc. cit., p. 195). So $\Sigma_j < (\Sigma_1)^j/j! < (2r \log r)^j/j!$. Also, since r is large, $r!(\log 2)\Sigma_{r-1} > (e^{-r}r^r)e^r = r^r$. Substituting these in the formula for $f^r(x)$ in Lemma 5 of I, we get

$$f_r(x) < (x + r \log r)^r/r^r.$$

If $x > (2 \log r)^r$, then $f_r(\log x^2/2) < x/2$. So the method of I will not give a lower bound as great as $(2 \log r)^r$, whereas the eliminant of x^{p_r-1} and $x^2 - x - 1$ is of the order of r^{br}.

To the above argument, one might object that firstly it is not the size of the eliminants but the size of their largest prime factor which is important, and secondly it is not essential to take the m's in order of magnitude. In answer, it should be pointed out that after one passes the limits of factor tables, it becomes impracticable to deal with the factors of the eliminant rather than the eliminant. Therefore, since the eliminant (in one case at least) appears to be an increasing function of m, one is compelled to work with monotone increasing m.

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[6] L. E. Dickson, *On the last theorem of Fermat*, Quarterly Journal of Mathematics, vol. 40 (1908), pp. 27–45.

[7] E. Landau, *Handbuch der Lehre von der Verteilung der Primzahlen.*

CORNELL UNIVERSITY

SOME UNIFORMLY CONVEX SPACES

R. P. BOAS, JR.[1]

1. **Introduction.** A Banach space is said to be uniformly convex if to every ϵ, $0<\epsilon<1$, there is a $\delta(\epsilon)$, $0<\delta(\epsilon)<2$, such that $\|x\|=\|y\|=1$ and $\|x-y\|\geqq\epsilon$ imply $\|x+y\|<2-\delta(\epsilon)$. J. A. Clarkson, who introduced the concept of uniform convexity [5], proved that the spaces L^p and l^p are uniformly convex if $p>1$, basing his proof on the following inequalities[2] among norms of elements of L^p or l^p:

$$(1.1)\qquad \|x+y\|^p+\|x-y\|^p \leqq 2^{p-1}(\|x\|^p+\|y\|^p), \qquad p\geqq 2;$$

$$(1.2)\qquad \|x+y\|^p+\|x-y\|^p \leqq 2(\|x\|^{p'}+\|y\|^{p'})^{p-1}, \qquad p\geqq 2;$$

$$(1.3)\qquad \|x+y\|^{p'}+\|x-y\|^{p'} \leqq 2(\|x\|^p+\|y\|^p)^{p'-1}, \qquad 1<p\leqq 2.$$

The uniform convexity of L^p and l^p follows by inspection from either (1.1) or (1.2) if $p\geqq 2$, and from (1.3) if $1<p\leqq 2$. As Clarkson observed, (1.1) is a consequence of (1.2), since $\{(1/2)(a^r+b^r)\}^{1/r}$ is an increasing function of r for positive a and b [6, p. 26], so that the right side of (1.1) is not less than that of (1.2). However, (1.1) is interesting because it is considerably simpler to prove than (1.2) (see §3), so that the uniform convexity of L^p and l^p can be established more easily for $p\geqq 2$ than for $1<p<2$.

In this note I give a short proof of Clarkson's inequalities (and of a general set of inequalities, which includes them), using M. Riesz's convexity theorem for linear forms. This proof has the advantage that it can be generalized to show that the spaces $L^p\{L^q\}$, $L^p\{l^q\}$, $l^p\{L^q\}$, $l^p\{l^q\}$ are all uniformly convex[3] if $p>1, q>1$. Here $L^p\{E\}$ is the Banach space of functions $x(t)$ from a real interval to the Banach space E, integrable (in the sense of Bochner [3]), and such that $\mathfrak{M}_p[x] = (\int \|x(t)\|^p dt)^{1/p}$ is finite; the norm of x in $L^p\{E\}$ is[4] $\mathfrak{M}_p[x]$. Similarly, $l^p\{E\}$ is the Banach space of sequences $X = \{x_n\}$ of elements of E, such that $\mathfrak{M}_p[X] = (\sum \|x_n\|^p)^{1/p}$ is finite, with $\mathfrak{M}_p[X]$ as the norm of X.

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[1] National Research Fellow.

[2] Here, as throughout this note, $p'=p/(p-1)$; similarly for other letters.

[3] These results suggest the possibility that $L^p\{E\}$ and $l^p\{E\}$ are uniformly convex whenever E is; but I can offer no evidence for or against this conjecture.

I also give a separate proof of (1.1); this proof is strictly elementary, depending only on the inequalities of Hölder, Minkowski, and Jensen.

Clarkson's inequalities can be written in the general form

$$(\|x+y\|^r + \|x-y\|^r)^{1/r} \leqq 2^{1/s'}(\|x\|^s + \|y\|^s)^{1/s}; \tag{1.4}$$

(1.4) reduces to (1.1), (1.2), (1.3) for $r=s=p$; $r=p$, $s=p'$; $r=p'$, $s=p$. In each case, $s \leqq p \leqq r$; this condition will be used in §2.

2. **A preliminary reduction.** We begin by showing that (1.4), for an r and an s with $1 < s \leqq p \leqq r$, is implied by the same inequality for complex x and y (with norms replaced by absolute values). In fact, if we know

$$(|x+y|^r + |x-y|^r)^{1/r} \leqq 2^{1/s'}(|x|^s + |y|^s)^{1/s} \tag{2.1}$$

for all complex x and y, with $1 < s \leqq p \leqq r$, we deduce (1.4) (for L^p) from the inequalities [6, p. 148]

$$\begin{aligned} &\left\{\left(\int |x(t)+y(t)|^p dt\right)^{r/p} + \left(\int |x(t)-y(t)|^p dt\right)^{r/p}\right\}^{1/r} \\ &\leqq \left\{\int \left(|x(t)+y(t)|^r + |x(t)-y(t)|^r\right)^{p/r} dt\right\}^{1/p}, \quad r \geqq p; \end{aligned} \tag{2.2}$$

$$\begin{aligned} &2^{1/s'}\left\{\int \left(|x(t)|^s + |y(t)|^s\right)^{p/s} dt\right\}^{1/p} \\ &\leqq 2^{1/s'}\left\{\left(\int |x(t)|^p dt\right)^{s/p} + \left(\int |y(t)|^p dt\right)^{s/p}\right\}^{1/s}, \quad s \leqq p. \end{aligned} \tag{2.3}$$

For, (2.1) shows that the left side of (2.3) is not less than the right side of (2.2). For l^p, we have the same inequalities with $\int$ replaced by $\sum$ [6, p. 123], and we draw the same conclusions.

3. **Independent proof of** (1.1). According to §2, it is enough to prove

$$(|x+y|^p + |x-y|^p)^{1/p} \leqq 2^{1/p'}(|x|^p + |y|^p)^{1/p}, \quad p \geqq 2. \tag{3.1}$$

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[4] See Bochner and Taylor [4] for a discussion of these spaces.

We have, using first Jensen's inequality [6, p. 28] and then Hölder's inequality with index $p/2$ [6, p. 24],

$$\begin{aligned}(|x+y|^p+|x-y|^p)^{1/p} &\leqq (|x+y|^2+|x-y|^2)^{1/2}\\ &= 2^{1/2}(|x|^2+|y|^2)^{1/2}\\ &\leqq 2^{1/2}\{2^{(p-2)/p}(|x|^p+|y|^p)^{2/p}\}^{1/2}\\ &= 2^{1/p'}(|x|^p+|y|^p)^{1/p},\end{aligned}$$

since $(p/2)'=p/(p-2)$.

This is the simplest proof of the uniform convexity of L^p and l^p, $(p>2)$.

4. **Proof of Clarkson's inequalities.** We shall prove the following general theorem, of which Clarkson's inequalities are special cases: it asserts the truth of (1.2) and (1.3), respectively, when $p\geqq 2$, $r=p$, $s=p'$, and when $1<p\leqq 2$, $r=p'$, $s=p$.

THEOREM 1. *If x and y are elements of L^p or l^p, $(p>1)$, then*

$$(4.1)\qquad (\|x+y\|^r+\|x-y\|^r)^{1/r} \leqq 2^{1/s'}(\|x\|^s+\|y\|^s)^{1/s}$$

whenever the point (r, s) is in the common part of the regions of the (r, s) plane determined by the inequalities

$$(4.2)\qquad r\geqq s\geqq r', \qquad r\geqq p\geqq s>1.$$

As we saw in §2, it is enough (since the second inequality (4.2) is satisfied) to establish

$$(4.3)\qquad (|x+y|^r+|x-y|^r)^{1/r} \leqq 2^{1/s'}(|x|^s+|y|^s)^{1/s}$$

for all complex x and y. We now appeal to M. Riesz's convexity theorem,[5] which, stated for the pair of linear forms $x+y$ and $x-y$, is that

$$M_{a,b} = \log\max\frac{(|x+y|^{1/a}+|x-y|^{1/a})^a}{(|x|^{1/b}+|y|^{1/b})^b}$$

is convex in the triangle[6] $0\leqq a\leqq b\leqq 1$.

Now, we have the identity

$$(|x+y|^2+|x-y|^2)^{1/2} = 2^{1/2}(|x|^2+|y|^2)^{1/2},$$

and the trivial inequalities

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[5] [6, p. 219]; a new and interesting proof (and generalization) has been given by Thorin [8].

[6] The maximum is taken over all x and y. When a or b is zero, the numerator or denominator on the right is to be replaced by its limiting value as a or b approaches zero, namely $\max(|x+y|, |x-y|)$ or $\max(|x|, |y|)$.

$$\max(|x+y|,\ |x-y|) \leqq |x|+|y|,$$
$$\max(|x+y|,\ |x-y|) \leqq 2\max(|x|,\ |y|);$$

it follows easily that

$$(4.4)\qquad M_{1/2,1/2} = (1/2)\log 2, \qquad M_{0,1} = 0, \qquad M_{0,0} = \log 2.$$

The graph of a continuous convex function lies below any chord. Since the linear function of a and b taking the values log 2, 0, (1/2) log 2 at the points (0, 0), (0, 1), and (1/2, 1/2) is $(1-b)\log 2$, we obtain from (4.4) (writing $a=1/r$, $b=1/s$), the inequality (4.3), valid for any r and s such that $(1/r, 1/s)$ is in the triangle whose vertices are (0, 0), (0, 1), and (1/2, 1/2). These values of r and s are those satisfying $r>1$ and the first inequality (4.2).

The uniform convexity of L^p and l^p is a consequence of any case of Theorem 1; Clarkson's inequalities (1.2) and (1.3) are the strongest cases involving p in a simple way.[7] When $r=s=p\geqq 2$, we have the weakest inequality; we should therefore expect to be able to give the simplest proof in this case.

5. **The uniform convexity of $\lambda^p\{\lambda^q\}$.** It is convenient to have a shorthand notation for our L^p and l^p spaces. We shall use λ to stand for either L or l; a statement involving λ is to be interpreted as the set of statements obtained by the reading of L or l for λ in all possible ways.

The space $\lambda^p\{E\}$ is evidently uniformly convex if its elements satisfy any inequality of the form

$$(5.1)\qquad \left\{\mathfrak{M}_p^r[x+y] + \mathfrak{M}_p^r[x-y]\right\}^{1/r} \leqq 2^{1/s'}\left\{\mathfrak{M}_p^s[x] + \mathfrak{M}_p^s[y]\right\}^{1/s}$$

with $r>1$, $s>1$.

The content of (5.1) will perhaps be clearer if we write it out explicitly for one of its cases. For $L^p\{l^q\}$, it states that for any sequences $\{x_n(t)\}$, $\{y_n(t)\}$ of functions belonging to L^p,

$$\left\{\left[\int \left(\sum |x_n(t)+y_n(t)|^q\right)^{p/q} dt\right]^{r/p} + \left[\int \left(\sum |x_n(t)-y_n(t)|^q\right)^{p/q} dt\right]^{r/p}\right\}^{1/r}$$
$$\leqq 2^{1/s'}\left\{\left[\int \left(\sum |x_n(t)|^q\right)^{p/q} dt\right]^{s/p} + \left[\int \left(\sum |y_n(t)|^q\right)^{p/q} dt\right]^{s/p}\right\}^{1/s},$$

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[7] The left side of (2.1) is an increasing function of $1/r$; the right side is a decreasing function of $1/s$. Hence we get a better inequality by increasing $1/r$ or $1/s$.

provided that the right side is finite. For $l^p\{L^q\}$, (5.1) is the same inequality with $\int$ and $\sum$ interchanged.

Even when E is λ^q, (5.1) is not true (naturally enough) for all choices of r and s. Reasonable generalizations of Clarkson's inequalities are the cases of (5.1) described by the following table, when E is λ^q.

q	p	r	s
$1<q\leqq 2$	$p\geqq q'$	p	p'
$1<q\leqq 2$	$q\leqq p\leqq q'$	q'	q
$1<q\leqq 2$	$1<p\leqq q$	p'	p
$q\geqq 2$	$p\geqq q$	p	p'
$q\geqq 2$	$q'\leqq p\leqq q$	q	q'
$q\geqq 2$	$1<p\leqq q'$	p'	p

These inequalities are included in the following theorem.

THEOREM 2. *If x and y are elements of $\lambda^p\{\lambda^q\}$, ($p>1$, $q>1$), then (5.1) is true whenever the point (r, s) is in the common part of the regions determined by the inequalities*

$$r \geqq p \geqq s > 1, \qquad r(q-1) \geqq s \geqq r' \tag{5.2}$$

when $1<q\leqq 2$, or

$$r \geqq p \geqq s > 1, \qquad r(q'-1) \geqq s \geqq r' \tag{5.3}$$

when $q\geqq 2$.

As we have already seen, any case of Theorem 2 implies that the spaces $\lambda^p\{\lambda^q\}$ are uniformly convex for $p>1$, $q>1$.

To prove Theorem 2, we observe that (5.1) is (1.4) with $\mathfrak{M}_p[\cdots]$ for $\|\cdots\|$. If we replace $|\cdots|$ by $\|\cdots\|$ in (2.2) and (2.3), the norm being the norm in λ^q, (2.2) and (2.3) are of course still valid, and we deduce that (5.1) holds if

$$(\|x+y\|^r + \|x-y\|^r)^{1/r} \leqq 2^{1/s'}(\|x\|^s + \|y\|^s)^{1/s} \tag{5.4}$$

with $1<s\leqq p\leqq r$, the norm being a λ^q norm. We shall prove (5.4) in the cases specified by (5.2) and (5.3), and thus establish the corresponding cases of (5.1).

To do this, we generalize the argument of §4; here Clarkson's inequalities will replace the equality

$$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2),$$

which is valid only in euclidean spaces (Jordan and von Neumann [7]).

The form of Riesz's theorem used in §4 is equivalent to another theorem of Riesz on bilinear forms; S. Bochner and the author [2] have observed that the latter remains true, with an appropriate interpretation, when the variables involved are elements of a Banach space. It follows that the convexity theorem for linear forms is also true (see §6). For our special case it states that

$$M_{a,b} = \log \sup \frac{(\|x+y\|^{1/a} + \|x-y\|^{1/a})^a}{(\|x\|^{1/b} + \|y\|^{1/b})^b}$$

is convex in the triangle $0 \leqq a \leqq b \leqq 1$. (The least upper bound is taken as x and y vary over λ^q; the norms, of course, are λ^q norms.)

We may now formulate (5.4) as

$$M_{1/r,1/s} \leqq (1/s') \log 2, \qquad 1 < s \leqq p \leqq r < \infty. \tag{5.5}$$

We have, in λ^q (and in fact in any Banach space),

$$M_{0,0} = \log 2, \qquad M_{0,1} = 0.$$

Clarkson's inequalities (1.2) and (1.3) give us

$$M_{1/q,1/q'} \leqq (1/q) \log 2,\ q \geqq 2; \quad M_{1/q',1/q} \leqq (1/q') \log 2,\ 1 < q \leqq 2.$$

Consequently (5.4) is true whenever $(1/r, 1/s)$ is in the triangle with vertices $(0, 0)$, $(0, 1)$, $(1/q, 1/q')$ if $q \geqq 2$, or in the triangle with vertices $(0, 0)$, $(0, 1)$, $(1/q', 1/q)$ if $1 < q \leqq 2$. These points correspond to points (r, s), with $r > 1$, satisfying the second inequalities (5.2) or (5.3). This completes the proof of Theorem 2.

It should be mentioned that certain cases of (5.2), adequate to establish the uniform convexity of $\lambda^p\{\lambda^q\}$ for some (but not all) p and q, follow directly from Clarkson's inequalities without any convexity theorem. For example, if $q \geqq 2$, (5.4) becomes (1.2) for $r = q$, $s = q'$; hence (5.4) still holds for $r \geqq q$, $s \geqq q'$; therefore, if $p \geqq q'$, we can take $r = s = p$ and thus establish the uniform convexity of $\lambda^p\{\lambda^q\}$ for $p \geqq q'$, $q \geqq 2$.

6. **The abstract convexity theorem.** Let E be a Banach space, with elements x; let its conjugate space have elements f (linear functionals on E). Let $N_{a,b}$ be the least upper bound of

$$\left| \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} f_j(x_i) \right| \tag{6.1}$$

for

$$(6.2)\qquad \sum_{i=1}^{m}\|x_i\|^{1/b} \leqq 1, \qquad \sum_{j=1}^{n}\|f_j\|^{1/c} \leqq 1.$$

It is known[8] that $\log N_{c,b}$ is convex in the triangle $0\leqq b\leqq 1$, $0\leqq c\leqq 1$, $b+c\geqq 1$.

We wish to show that if $M_{a,b}$ is the least upper bound of the expression

$$(6.3)\qquad \left\{\sum_{j=1}^{n}\left\|\sum_{i=1}^{m} a_{ij}x_i\right\|^{1/a}\right\}^{a}$$

for

$$(6.4)\qquad \sum_{i=1}^{m}\|x_i\|^{1/b} \leqq 1,$$

then $\log M_{a,b}$ is convex in the triangle $0\leqq a\leqq b\leqq 1$. This we do by showing that $M_{a,b}=N_{c,b}$ if $c=1-a$.

Let $\{f_j\}$ and $\{x_i\}$ be arbitrary sets satisfying (6.2) with $c=1-a$. Then, by Hölder's inequality,

$$\begin{aligned}\left|\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}f_j(x_i)\right| &= \left|\sum_{j=1}^{n} f_j\left(\sum_{i=1}^{m} a_{ij}x_i\right)\right| \\ &\leqq \left(\sum_{j=1}^{n}\|f_j\|^{1/(1-a)}\right)^{1-a}\left(\sum_{j=1}^{n}\left\|\sum_{i=1}^{m} a_{ij}x_i\right\|^{1/a}\right)^{a} \\ &\leqq M_{a,b}.\end{aligned}$$

Consequently[9]

$$(6.5)\qquad N_{c,b} \leqq M_{a,b}.$$

Now let $\{x_i\}$ be an arbitrary set satisfying (6.4). We can find[10] linear functionals f_j, $(j=1, 2, \cdots, n)$, such that

$$f_j\left(\sum_{i=1}^{m} a_{ij}x_i\right) = \left\|\sum_{i=1}^{m} a_{ij}x_i\right\|\cdot\|f_j\|, \qquad j = 1, 2, \cdots, n;$$

$$\|f_j\|^{1/c} = A\left\|\sum_{i=1}^{m} a_{ij}x_i\right\|^{1/a}, \qquad j = 1, 2, \cdots, n,$$

with constant $A\neq 0$; and $\sum_{j=1}^{n}\|f_j\|^{1/c}=1$. For these f_j we have, using the conditions for equality in Hölder's inequality,

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[8] Boas and Bochner [2, p. 64].

[9] This simple argument was suggested by the referee.

[10] Banach [1, p. 55].

$$\begin{aligned}\sum_{j=1}^{n}\sum_{i=1}^{m} a_{ij}f_j(x_i) &= \sum_{j=1}^{n} f_j\left(\sum_{i=1}^{m} a_{ij}x_i\right)\\ &= \sum_{j=1}^{n} \|f_j\|\cdot\left\|\sum_{i=1}^{m} a_{ij}x_i\right\|\\ &= \left(\sum_{j=1}^{n}\|f_j\|^{1/(1-a)}\right)^{1-a}\left(\sum_{j=1}^{n}\left\|\sum_{i=1}^{m} a_{ij}x_i\right\|^{1/a}\right)^{a}\\ &= \left(\sum_{j=1}^{n}\left\|\sum_{i=1}^{m} a_{ij}x_i\right\|^{1/a}\right)^{a}.\end{aligned}$$

Hence $M_{a,b} \leqq N_{c,b}$ if $c=1-a$; this, with (6.5), completes the proof.

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NOTE ON DEGREE OF TRIGONOMETRIC AND POLYNOMIAL APPROXIMATION TO AN ANALYTIC FUNCTION, IN THE SENSE OF LEAST pTH POWERS

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1. **Introduction.** In a recent note[1] the present writers studied the relation between the continuity properties of a function and the degree of approximation in the sense of Tchebycheff by trigonometric and other polynomials; this approximation in the z-plane is considered either on the unit circle by polynomials in z and $1/z$, or on the segment $-1 \leqq z \leqq 1$ by polynomials in z, or (for functions of period 2π) on the infinite interval $-\infty < z < \infty$ by trigonometric polynomials. In the respective cases, the functions approximated are analytic in an annulus $\rho > |z| > 1/\rho < 1$, in an ellipse whose foci are $+1$ and -1, or in a horizontal strip containing the axis of reals in its interior. It is the purpose of the present note to establish the analogous results when approximation is measured by the integral of the pth power of the error, as in the sense of least pth powers.

The method we employ makes essential use of the specific results concerning Tchebycheff approximation as developed in our previous note, together with certain general methods already developed elsewhere.[2]

2. **Approximation on the unit circle.** Our main result is as follows:

THEOREM 1. *Let the weight function $w(\theta)$ be positive and continuous for all values of θ, and of period 2π. Let the function $f(\theta)$ (not necessarily real) be periodic with period 2π, and suppose the numbers a_{nj} and b_{nj} (not necessarily real) are given so that*

$$s_n(\theta) = \frac{a_{n0}}{2} + \sum_{j=1}^{n} (a_{nj} \cos j\theta + b_{nj} \sin j\theta),$$

with the relation, for $n = 1, 2, \cdots,$

$$\int_{-\pi}^{\pi} w(\theta) \,|\, f(\theta) - s_n(\theta) \,|^p d\theta \leqq \frac{M}{\rho^{np} n^{(k+\alpha+1)p}}, \tag{1}$$

$$0 < \alpha \leqq 1,\ p > 0,\ \rho > 1,$$

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[1] This Bulletin, vol. 44 (1938), pp. 865–873. We shall refer to this note as WS.

[2] Walsh, *Interpolation and Approximation by Rational Functions in the Complex Domain,* American Mathematical Society Colloquium Publications, vol. 20, New York, 1935.

where k is a nonnegative integer and M is a constant independent of n. Then the function

$$F(z) = \lim_{n\to\infty}\left[c_{n0} + \sum_{j=1}^{n}(c_{n,-j}z^{-j} + c_{nj}z^{j})\right], \tag{2}$$

$$2c_{nj} = a_{nj} - ib_{nj}, \qquad 2c_{n,-j} = a_{nj} + ib_{nj},$$

coincides with $f(\theta)$ almost everywhere (everywhere if $f(\theta)$ is continuous) on the circle $|z| = 1$, with $z = \cos\theta + i\sin\theta$, and $F(z)$ is analytic in the annulus $\rho > |z| > 1/\rho$ and continuous in the corresponding closed region. For z_1 and z_2 on $|z| = \rho$ or $1/\rho$ we have[3]

$$|F^{(k)}(z_1) - F^{(k)}(z_2)| \leqq L|z_1 - z_2|^{\alpha}\cdot|\log|z_1 - z_2||^{\beta}, \tag{3}$$

where $\beta = 0$ if $\alpha < 1$ and $\beta = 1$ if $\alpha = 1$, and where L is a constant independent of z_1 and z_2.

As a matter of convenience in the proof of Theorem 1 we establish a preliminary proposition:

LEMMA. *Let C be a rectifiable Jordan curve in the z-plane containing $z = 0$ in its interior; let $P(z, 1/z)$ be a polynomial in z and $1/z$ of degree n:*

$$\begin{aligned} P(z, 1/z) &\equiv P_1(z) + P_2(z), \\ P_1(z) &= a_0 + a_1 z + \cdots + a_n z^n, \\ P_2(z) &= a_{-1}z^{-1} + a_{-2}z^{-2} + \cdots + a_{-n}z^{-n}, \end{aligned}$$

and suppose

$$\int_C |P(z, 1/z)|^p |dz| \leqq N^p, \qquad p > 0. \tag{4}$$

Denote by $w = f_1(z)$ a function which maps the exterior of C onto $|w| > 1$ so that $f_1(\infty) = \infty$, and denote by $w = f_2(z)$ a function which maps the interior of C onto $|w| < 1$ so that $f_2(0) = 0$; denote by C_R the closed annular region bounded by the two Jordan curves $|f_1(z)| = R > 1$, $|f_2(z)| = 1/R$.

Then there exists a constant N' depending on C, p, and R but not on $P(z, 1/z)$ nor on n, such that (4) implies

$$|P(z, 1/z)| \leqq NN'R^n, \quad z \text{ in } C_R.$$

By a known method of proof (Walsh, op. cit., p. 92) we have for suitably chosen N'

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[3] The notation $F^{(k)}(z)$ indicates the kth derivative of $F(z)$ if $k > 0$ and the function $F(z)$ itself for $k = 0$. Here and below such derivatives on the boundaries of regions of analyticity are considered in the one-dimensional sense.

$$|P(z, 1/z)| \leqq NN'R^n, \quad z \text{ on curve } |f_1(z)| = R;$$
$$|P(z, 1/z)| \leqq NN'R^n, \quad z \text{ on curve } |f_2(z)| = 1/R.$$

The lemma follows at once.[4]

We proceed now with the proof of Theorem 1. From (1) by the well known general inequalities

$$(5)\qquad \begin{aligned} &|\chi_1 + \chi_2|^p \leqq 2^{p-1}|\chi_1|^p + 2^{p-1}|\chi_2|^p, && p > 1,\\ &|\chi_1 + \chi_2|^p \leqq |\chi_1|^p + |\chi_2|^p, && 0 < p \leqq 1,\end{aligned}$$

and by virtue of the boundedness of $1/w(\theta)$, we have

$$(6)\qquad \int_{-\pi}^{\pi} |s_{n+1}(\theta) - s_n(\theta)|^p d\theta \leqq \frac{M_1}{\rho^{np} n^{(k+\alpha+1)p}}\cdot$$

On the circle $|z| = 1$ we have $\cos j\theta = (z^j + z^{-j})/2$, $\sin j\theta = (z^j - z^{-j})/2i$, so the first member of (6) can be written

$$\int_{|z|=1} |P_{n+1}(z, 1/z) - P_n(z, 1/z)|^p |dz|,$$

where $P_n(z, 1/z)$ is a polynomial of degree n in z and $1/z$, equal to $s_n(\theta)$ on $|z| = 1$. From inequality (6) we now deduce through the lemma for $\rho \geqq |z| \geqq 1/\rho$,

$$(7)\qquad |P_{n+1}(z, 1/z) - P_n(z, 1/z)| \leqq M_2/n^{k+\alpha+1},$$

where M_2 is independent of n and z. We define $F(z)$ for $\rho \geqq |z| \geqq 1/\rho$ by means of the equation

$$(8)\qquad \begin{aligned} F(z) \equiv P_1(z, 1/z) &+ [P_2(z, 1/z) - P_1(z, 1/z)]\\ &+ [P_3(z, 1/z) - P_2(z, 1/z)] + \cdots ,\end{aligned}$$

whence from (7) we have for $\rho \geqq |z| \geqq 1/\rho$

$$(9)\qquad \begin{aligned} |F(z) - P_n(z, 1/z)| &\leqq |P_{n+1}(z, 1/z) - P_n(z, 1/z)|\\ &\quad + |P_{n+2}(z, 1/z) - P_{n+1}(z, 1/z)| + \cdots\\ &\leqq M_2 \sum_{j=n}^{\infty} \frac{1}{j^{k+\alpha+1}} \leqq \frac{M_3}{n^{k+\alpha}},\end{aligned}$$

where M_3 is independent of n and z.

The uniformity of the convergence in (8) is included in the inequalities (9), so it follows that $F(z)$ is analytic for $\rho > |z| > 1/\rho$, continuous for $\rho \geqq |z| \geqq 1/\rho$; and since $P_n(z, 1/z)$ on $|z| = \rho$ and $|z| = 1/\rho$ is a trigonometric polynomial of order n, it follows from a theorem due to de la Vallée Poussin[5] that $F(z)$ satisfies an inequality *with respect to* θ of form (3), $z = \rho e^{i\theta}$ or $e^{i\theta}/\rho$, from which it follows that inequality (3) itself is fulfilled on $|z| = \rho$ and $|z| = 1/\rho$.

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[4] So far as the writers are aware, this explicit lemma has not been previously formulated in print, but indications of it are given by Walsh, American Journal of Mathematics, vol. 54 (1932), pp. 559–570.

The boundedness of $1/w(z)$ implies from (1)

$$\int_{-\pi}^{\pi} | f(\theta) - s_n(\theta) |^p d\theta \leqq \frac{M_4}{\rho^{np} n^{(k+\alpha+1)p}} \cdot$$

Inequality (9) implies, from the equality of $P_n(z, 1/z)$ and $s_n(\theta)$ on $|z| = 1$,

$$\int_{-\pi}^{\pi} | F(z) - s_n(\theta) |^p d\theta \leqq \frac{M_5}{n^{(k+\alpha)p}},$$

so we now deduce by use of (5) and by allowing n to become infinite

$$\int_{-\pi}^{\pi} | F(z) - f(\theta) |^p d\theta = 0;$$

hence $F(z)$ and $f(\theta)$ are equal almost everywhere on $|z| = 1$, and Theorem 1 is established.

3. **Approximation on the segment** $-1 \leqq z \leqq 1$. We shall indicate rapidly the proof of the next theorem.

THEOREM 2. *Let $f(z)$ be defined on the segment $-1 \leqq z \leqq 1$, and for $n = 1, 2, \cdots$ let a polynomial $P_n(z)$ in z of degree n exist such that*

$$(10) \qquad \int_{-1}^{1} w(z) | f(z) - P_n(z) |^p dz \leqq \frac{M}{\rho^{np} n^{(k+\alpha+1)p}},$$

$$0 < \alpha \leqq 1, \ p > 0, \ \rho > 1,$$

where k is a nonnegative integer, and where $w(z)(1 - z^2)^{1/2}$ is positive and continuous on $-1 \leqq z \leqq 1$. Then the function $f(z)$ is equal to $F(z)$ almost everywhere on the segment $-1 \leqq z \leqq 1$, where

$$(11) \qquad F(z) = \lim_{n \to \infty} P_n(z).$$

Furthermore $F(z)$ is analytic throughout the interior of the ellipse γ whose foci are $+1$ and -1 and the sum of whose semi-axes is ρ; also $F(z)$ is continuous in the corresponding closed region, and satisfies an inequality of type (3) on γ.

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[5] *Leçons sur l'Approximation*, Paris, 1919, chap. 4.

We map the z-plane onto the w-plane by the transformation $z=(w+w^{-1})/2$, which carries the segment $-1\leqq z\leqq 1$ counted twice into the unit circle $|w|=1$, and carries the ellipse γ counted twice into the two circles $|w|=\rho$, $|w|=1/\rho$. The first member of (10) can be written

$$\frac{1}{2}\int_{|w|=1} w(z)\,|f(z)-P_n(z)|^p(1-z^2)^{1/2}|\,dw\,|,$$

so Theorem 1 applies (cf. WS for further details of the transformation and reasoning). A condition of form (3) on $|w|=\rho$ and $|w|=1/\rho$ with respect to w implies condition (3) as stated in Theorem 2, so the conclusion follows.

Another method of studying the situation of Theorem 2 is to avoid the transformation onto the w-plane, but to use the fact that for a polynomial $P(z)$ of degree n the inequality $\int_{-1}^{1}|P(z)|^p dz\leqq N^p$, $p>0$, implies $|P(z)|\leqq NN'R^n$ on and within[6] γ. Here it is natural to assume that $w(z)$ itself is positive and continuous rather than $w(z)(1-z^2)^{1/2}$. We deduce as in the proof of Theorem 1 an inequality analogous to (9) giving the degree of convergence of the sequence $P_n(z)$ to $F(z)$ on γ; this degree of convergence then implies[7] inequality (3) on γ, as we wish to prove. Thus we have proved the following theorem:

THEOREM 2a. *Theorem 2 remains true if the requirement that $w(z)(1-z^2)^{1/2}$ is positive and continuous on $-1\leqq z\leqq 1$ is replaced by the requirement that $w(z)$ be positive and continuous on $-1\leqq z\leqq 1$.*

4. **Approximation to a periodic function on the axis of reals.** A different transformation of the plane will now yield a new result:

THEOREM 3. *Let the function $f(z)$ be periodic with period 2π, and for $n=1, 2, \cdots$ let there exist a trigonometric polynomial $t_n(z)$ of order n such that we have*

$$\int_{-\pi}^{\pi} w(z)\,|f(z)-t_n(z)|^p dz \leqq \frac{M}{\rho^{np}n^{(k+\alpha+1)p}}, \quad 0<\alpha\leqq 1,\ p>0,\ \rho>1,$$

where the weight function $w(z)$ is a positive, continuous function of period 2π, and k is a nonnegative integer. Then if we define $F(z)=\lim_{n\to\infty} t_n(z)$, where $t_n(z)$ is still expressed as a trigonometric polynomial, the two functions $f(z)$ and $F(z)$ are equal almost everywhere

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[6] Walsh, op. cit., Lemma, p. 92.

[7] Walsh and Sewell, this Bulletin, vol. 43 (1937), pp. 557–563. This method of proof of Theorem 2a is used widely in a number of similar situations in a forthcoming paper by the present writers, on "Problem β."

on the axis of reals. Moreover if $z=x+iy$, *the function* $F(z)$ *is analytic for* $|y|<\log \rho$, *continuous in the corresponding closed region, and on the lines* $y=\pm\log \rho$ *satisfies an inequality of type* (3).

Theorem 3 is a direct consequence of Theorem 1, by virtue of the transformation $w=e^{iz}$, in the notation of Theorem 3 (compare WS, proof of Theorem 5).

5. **Convergence properties of sequences.** The methods of proof of Theorems 1, 2, and 3 apply also to the study of the degree of convergence of the sequences involved on various point sets not yet mentioned. Thus under the hypothesis of Theorem 1, the method already used shows that in the annulus $\rho_1 \geqq |z| \geqq 1/\rho_1$, $\rho_1<\rho$, we have instead of (7)

$$| P_{n+1}(z, 1/z) - P_n(z, 1/z) | \leqq \frac{M_2\rho_1^n}{\rho^n n^{k+\alpha+1}}\cdot$$

Then for $\rho_1 \geqq |z| \geqq 1/\rho_1$ we may write

$$\begin{aligned} |F(z) - P_n(z, 1/z)| &\leqq | P_{n+1}(z, 1/z) - P_n(z, 1/z) | \\ &\quad + | P_{n+2}(z, 1/z) - P_{n+1}(z, 1/z) | + \cdots \\ &\leqq M_2\sum_{j=n}^{\infty} \frac{\rho_1^j}{\rho^j j^{k+\alpha+1}} \leqq \frac{M_2}{n^{k+\alpha+1}} \sum_{j=n}^{\infty}\left(\frac{\rho_1}{\rho}\right)^j \leqq \frac{M'\rho_1^n}{\rho^n n^{k+\alpha+1}}. \end{aligned}$$

That is to say, we have established a further result:

COROLLARY TO THEOREM 1. *Under the hypothesis of Theorem* 1 *we have in the annulus* $\rho_1 \geqq |z| \geqq 1/\rho_1$, $\rho_1<\rho$,

$$| F(z) - P_n(z, 1/z) | \leqq \frac{M'\rho_1^n}{\rho^n n^{k+\alpha+1}},$$

where M' *is a suitably chosen constant independent of* n *and* z.

The corollary just established implies corresponding statements connected with Theorems 2 and 3.

COROLLARY TO THEOREM 2 (OR THEOREM 2a). *Under the hypothesis of Theorem* 2 (*or Theorem* 2a) *we have for* z *on and within the ellipse whose foci are* $+1$ *and* -1 *and whose sum of semi-axes is* $\rho_1<\rho$

$$| F(z) - P_n(z) | \leqq \frac{M'\rho_1^n}{\rho^n n^{k+\alpha+1}},$$

where M' *is a suitably chosen constant independent of* n *and* z.

In the proof of the Corollary to Theorem 2a we employ the method of proof of Theorem 2a rather than the Corollary to Theorem 1 itself.

COROLLARY TO THEOREM 3. *Under the hypothesis of Theorem* 3 *we have for z in the region* $|y| < \log \rho_1 < \log \rho$

$$|F(z) - t_n(z)| \leq \frac{M' \rho_1^n}{\rho^n n^{k+\alpha+1}},$$

where M' *is a suitably chosen constant independent of n and z.*

It may be noted that the inequalities representing the conclusions of these corollaries, with the obvious understanding that these functions $P_n(z, 1/z)$, $P_n(z)$, $t_n(z)$ are polynomials of the kind considered of respective degree n, themselves imply the conclusions of Theorems 1, 2 (or 2a), and 3 (compare WS, Theorems 1, 3, and 5).

6. **Reciprocal theorems.** In the direction of a converse of Theorem 1 we indicate the proof of the following result:

THEOREM 4. *Let the function* $F(z)$ *be analytic for* $\rho > |z| > 1/\rho < 1$, *continuous in the corresponding closed region, and let* $F^{(k)}(z)$, *k a nonnegative integer, satisfy a Lipschitz condition*[8] *of order* α *on* $|z| = \rho$ *and* $|z| = 1/\rho$. *Let* $p > 0$ *be given. Let* $w(\theta)$ *be a nonnegative function of* θ, *periodic with period* 2π, *and Lebesgue-integrable for* $-\pi \leqq \theta \leqq \pi$. *Then there exists a sequence of polynomials* $P_n(z, 1/z)$ *in z and* $1/z$ *of degrees* $n = 1, 2, \cdots$ *so that we have*

$$\int_{-\pi}^{\pi} w(\theta) |F(z) - P_n(z, 1/z)|^p d\theta \leqq \frac{M}{\rho^{np} n^{(k+\alpha)p}},$$

where M is a suitably chosen constant independent of n and z.

Theorem 4 is an immediate consequence of the corresponding theorem for Tchebycheff approximation (WS, Theorem 2). We add the remark that the conclusion of Theorem 4, holding for some sequence $P_n(z, 1/z)$, holds *a fortiori* for the sequence $P_n(z, 1/z)$ of best approximation in the sense of least pth powers. The Corollary to Theorem 1 applies to the latter.

Corresponding results in the directions of converses of Theorems 2, 2a, and 3 can be proved with no less ease.

It will be noticed that there is a discrepancy of unity between the exponents of n in Theorems 1 and 4. This discrepancy is not accidental, but is inherent in the nature of the problem, as is shown by examples that the authors plan to publish elsewhere.

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[8] That is to say, let (3) be satisfied with $\beta = 0$.

7. **Generality of weight functions.** The mere casual comparison of Theorems 2 and 2a suggests that the particular restrictions we have placed on the weight functions are artificial rather than essential to the problem. The most general weight function that can be employed seems to be difficult to determine, both here and in the case of polynomial approximation purely in the real domain. Nevertheless we shall prove the following additional proposition:

THEOREM 5. *Theorems* 1, 2, 2a, *and* 3 *persist if the requirement that the respective weight functions* $w(\theta)$, $w(z)(1-z^2)^{1/2}$, $w(z)$, $w(z)$ *be positive and continuous is replaced by the requirement that the weight functions be nonnegative, Lebesgue-integrable, and that some negative power of the corresponding weight functions be Lebesgue-integrable.*[9]

The situation of Theorem 1 is typical. Under the new assumption of Theorem 5 we assume $[w(\theta)]^{-\beta}$ to be integrable in the interval $-\pi \leqq \theta \leqq \pi$, with $\beta > 0$. The Hölder inequality

$$\left| \int F^{\delta} G^{1-\delta} \right| \leqq \left(\int |F| \right)^{\delta} \left(\int |G| \right)^{1-\delta}, \qquad 0 < \delta < 1,$$

with $\delta = 1/(1+\beta)$, gives

$$\int_{-\pi}^{\pi} | f(\theta) - s_n(\theta) |^{p(1-\delta)} d\theta$$
$$\leqq \left(\int_{-\pi}^{\pi} \frac{d\theta}{[w(\theta)]^{(1-\delta)/\delta}} \right)^{\delta} \left(\int_{-\pi}^{\pi} w(\theta) | f(\theta) - s_n(\theta) |^{p} d\theta \right)^{1-\delta},$$

and by virtue of (1) we have

$$\int_{-\pi}^{\pi} | f(\theta) - s_n(\theta) |^{p(1-\delta)} d\theta \leqq \frac{M'}{\rho^{np(1-\delta)} n^{(k+\alpha+1)p(1-\delta)}}.$$

This last inequality leads to an inequality similar to (6) and thence as before to the conclusion of Theorem 1.

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[9] This condition on the weight functions has been previously used by Dunham Jackson in the study of approximation in the real domain, and by Walsh (op. cit., pp. 104–105) in the complex domain.

A RELATION BETWEEN NON-ALTERNATING AND INTERIOR TRANSFORMATIONS[1]

G. T. WHYBURN

Recently in proving certain existence theorems for non-alternating and for interior transformations of a continuum onto a simple arc[2] it was observed that when a transformation of one of these types was set up, usually it satisfied in large measure conditions which brought it also under transformations of the other type. This suggests the existence of common ground shared by these sorts of transformations, and it is the object of this paper to exhibit the nature of such. Our principal conclusion is to the effect that, under certain auxiliary conditions, any interior transformation $f(M)=D$ of a compact continuum M onto a dendrite D can be factored into the form $f=f_2f_1$ where $f_1(M)=M'$ merely shrinks sets of type $f^{-1}(p)$ (p an end point of D) into single points and is topological elsewhere and $f_2(M')=D$ is non-alternating and interior. However, we first prove two theorems on the relation between the non-alternating property of a transformation of a continuum into a dendrite and the connectedness of the inverse sets for the end points of the dendrite.

THEOREM 1. *If M is a compact metric continuum and $f(M)=N$ is non-alternating, then for any end point*[3] *p of N, $f^{-1}(p)$ is connected.*

PROOF. Suppose $f^{-1}(p)$ is not connected. Then there exist points a and b of $f^{-1}(p)$, a closed set $F\subset M-f^{-1}(p)$ and a separation $M-F=M_a+M_b$ where $a\in M_a$, $b\in M_b$. Since $f(F)\cdot p=0$, $f(F)$ is compact, and p is an end point, there exists a neighborhood U of p in N with $\overline{U}\cdot f(F)=0$ and such that the boundary of U is a single point q. Since q separates p and $f(F)$ in N, it follows by a result of the author's[4] that there exists a separation $M-f^{-1}(q)=M_F+M_p$, where $F\subset M_F$, $f^{-1}(p)\subset M_p$. But this gives at once the separation $M-f^{-1}(q)=M_a\cdot M_p+(M_F+M_b\cdot M_p)$ where $M_a\cdot M_p\supset a$, $(M_F+M_b\cdot M_p)\supset b$, contrary to the hypothesis that f is non-alternating.

THEOREM 2. *If $f(M)=D$ is an interior transformation of a compact*

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[1] Presented to the Society, September 8, 1939.

[2] See my papers *The existence of certain transformations*, Duke Mathematical Journal, vol. 5 (1939), pp. 647–655; and *Non-alternating interior retracting transformations*, Annals of Mathematics, (2), vol. 40 (1939), pp. 914–921.

[3] That is, a point of Menger-Urysohn order 1 of N.

[4] See my paper in the American Journal of Mathematics, vol. 56 (1934), pp. 294–302, (1.4).

continuum M onto a dendrite D such that for each end point p of D, $f^{-1}(p)$ is connected, then f is non-alternating.

Proof. Let $x, y \in D$ and let R be the component of $D-x$ containing y. Then R contains at least one end point p of D. Since $f^{-1}(p)$ is connected, there exists a single component Q of $M-f^{-1}(x)$ containing $f^{-1}(p)$. Since,[5] by the interiority of f, any quasi-component of $f^{-1}(R)$ maps onto all of R and hence intersects $f^{-1}(p)$, we have $Q=f^{-1}(R)$, whence $Q \supset f^{-1}(y)$ and thus f is non-alternating.

Corollary. *In order that an interior transformation $f(M)=D$ of a compact continuum M onto a dendrite D be non-alternating it is necessary and sufficient that for each end point p of D, $f^{-1}(p)$ be connected.*

Theorem 3. *Let M be a compact continuum, let D be a dendrite and let H be the set of all end points of D. Any interior transformation $f(M)=D$ such that the collection $[f^{-1}(p)]$, $p \in H$, is semi-closed*[6] *can be factored into the form $f=f_2f_1$, where f_1 is equivalent to f on $f^{-1}(H)$ and topological on $M-f^{-1}(H)$ and f_2 is interior and non-alternating.*

Proof. Let us decompose M into the sets $f^{-1}(p)$, $p \in H$, and individual points of $M-f^{-1}(H)$. Since the collection $[f^{-1}(p)]$, $p \in H$, is semi-closed, it follows by a result of the author's[6] that this decomposition is upper semi-continuous.[7] Let M' be the hyperspace of this decomposition and $f_1(M)=M'$ the associated continuous transformation.

Now for each $x' \in M'$, let us define $f_2(x')=ff_1^{-1}(x')$. Then clearly $f(x)\equiv f_2f_1(x)$ for $x \in M$. The continuity of f_2 results at once from that of f_1 and f. Furthermore, since f is interior it follows[8] that f_2 is interior. Finally, since for any end point p of D we have that $f_2^{-1}(p)=f_1f^{-1}(p)$ is a single point of M', it follows by Theorem 2 that f_2 is non-alternating. This completes the proof of our theorem.

Since any finite collection of disjoint closed sets is semi-closed, we have the following corollary:

Corollary. *Any interior transformation of a compact continuum into a simple arc (or into a dendrite with only a finite number of end points) can be factored into the form described in Theorem* 3.

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[5] See my paper, Duke Mathematical Journal, vol. 3 (1937), pp. 370–381, (1.4).

[6] That is, any convergent sequence of the sets $[f^{-1}(p)]$ whose limit set intersects $M-f^{-1}(H)$ converges to a single point. See my paper in the Duke Mathematical Journal, vol. 2 (1937), pp. 685–690.

[7] See R. L. Moore, *Foundations of Point Set Theory*, American Mathematical Society Colloquium Publications, vol. 13, 1932, chap. 5.

[8] See result (1.6) of the paper cited in footnote 5.

CONCERNING SIMILARITY TRANSFORMATIONS OF LINEARLY ORDERED SETS[1]

BEN DUSHNIK AND E. W. MILLER

1. **Introduction.** As is well known, two linearly ordered sets A and B are said to be similar if there exists a 1-1 correspondence between their elements which preserves order. A function f which defines such a 1-1 correspondence may be called a similarity transformation on A to B. In this note we consider two problems concerning similarity transformations which do not seem to have received attention heretofore. The first problem is the following:

(A) *Is it true that every infinite ordered set is similar to a proper subset of itself?*[2]

Before stating the second problem we recall a classical theorem concerning well-ordered sets.[3]

THEOREM. *If the set A is well-ordered, and if f is any similarity transformation on A to a subset of A, then $f(a) \geqq a$ for every a in A.*

It is natural to inquire whether this theorem characterizes well-ordered sets—and this is our second problem; more explicitly:

(B) *Let A be a linearly ordered set such that if f is any similarity transformation on A to a subset of A then $f(a) \geqq a$ for every a in A. Is it true that any such set A is well-ordered?*

We will demonstrate that if the set A is denumerable, then the answer to both questions is in the affirmative. An example will then be constructed to show that these conclusions need not hold if the set A is nondenumerable.

2. **The denumerable case.** We obtain first the following result:

THEOREM 1. *Every denumerably infinite linearly ordered set A contains a proper subset A' to which it is similar.*

PROOF. For any two elements a and b of A, we will say that a and b are *congruent* if either $a=b$ or if there is only a finite number of elements in A which lie between a and b; we will indicate this by writing $a \equiv b$. The set of all elements of A congruent to a given element a will be designated by $C(a)$ and will be called the ***congruence set*** corresponding to a. It is obvious that if $b \varepsilon C(a)$, then $C(b) = C(a)$, and hence that any two different congruence sets have no elements in common. We now distinguish two cases.

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[1] Presented to the Society, September 8, 1939.

[2] This question is a natural one, in view of the familiar definition of an infinite set as one which is *equivalent* to a proper subset of itself.

[3] For theorems mentioned in this paper one may refer to Hausdorff's *Grundzüge der Mengenlehre*, 1st edition, 1914, or to Sierpiński's *Leçons sur les Nombres Transfinis*, 1928.

Case 1. There is an a in A for which $C(a)$ is infinite.

Since every element of $C(a)$ has either an immediate predecessor or an immediate successor in $C(a)$, it is clear that the order-type of $C(a)$ is either ω or ω^* or $\omega^*+\omega$. Suppose, for example, that the order-type of $C(a)$ is ω. Since every element of A which is not in $C(a)$ either precedes or succeeds *all* the elements of $C(a)$, we will have $A = A_1 + A_2 + A_3$, where $A_2 = C(a)$, A_1 is the set of all elements of A which precede $C(a)$, and A_3 is the set of all the elements of A which follow $C(a)$. We now define a function f on A to a subset of A as follows:

(a) If $\alpha \varepsilon A_1$, or $\alpha \varepsilon A_3$, then $f(\alpha) = \alpha$.

(b) If $\alpha \varepsilon A_2$, then $f(\alpha)$ is the successor of α in $C(a)$.

This function is clearly a similarity transformation; moreover, the set A' into which A is transformed by f does not contain the first element of $C(a)$, and A' is thus a proper subset of A. The case where the order-type of $C(a)$ is ω^* or $\omega^*+\omega$ may be handled in an analogous fashion.

Case 2. All the different congruence sets in A are finite sets.

Since A is denumerable, the set of all congruence sets in A is also denumerable. Let $\bar{A}$ be the ordered subset of A which consists of all first elements of congruence sets in A, and let α and β, $(\alpha < \beta)$, be any two elements of $\bar{A}$. There must exist a γ in $\bar{A}$ for which $\alpha < \gamma < \beta$ (for, if not, then clearly $\alpha \equiv \beta$ in A, which would contradict the fact that α and β belong to two different congruence sets). Thus $\bar{A}$ is a denumerable dense set, and any open interval of it, for example, the set (α, β) of all elements of $\bar{A}$ between α and β, would also be a denumerable dense set. It is well known[4] that such a set contains a subset similar to any given denumerable ordered set. Let A' be a subset of (α, β) which is similar to A. It is clear that A' is a proper subset of A, and Theorem 1 is thus completely demonstrated.

THEOREM 2. *If the denumerably infinite ordered set A is such that any similarity transformation f on A to a subset of A has the property $f(a) \geqq a$ for every a in A, then A is a well-ordered set.*

PROOF. We notice at once that A cannot contain a subset $\bar{A}$ which

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[4] See, for example, Sierpiński, loc. cit., pp. 147–148.

is dense. For, if $\overline{A}$ is such a subset, let $\overline{\overline{A}}$ be the lower segment of $\overline{A}$ which consists of all the elements of $\overline{A}$ which precede some element b in $\overline{A}$. Then $\overline{\overline{A}}$ would be a denumerable dense set, and, as was pointed out in the proof of Theorem 1, $\overline{\overline{A}}$ would contain a subset A' similar to A, that is, there would exist a similarity transformation f on A to A'. Since $f(b) \varepsilon A'$, and all elements of A' precede b, we would obtain $f(b) < b$; but this is impossible, by our hypothesis.

We now redefine "congruent" elements of A as follows: $a \equiv b$ if and only if (1) $a = b$ or (2) the closed interval of A which consists of a and b and all the elements of A between a and b is, as an ordered subset of A, a well-ordered set. In the present sense, the congruence sets have the following properties:

(1) Two different congruence sets have no elements in common.

(2) Any upper segment, or any open or closed interval of a congruence set, is a well-ordered set.

(3) If $a \not\varepsilon C(b)$, then a either precedes all, or succeeds all of the elements of $C(b)$.

It may be emphasized, however, that a lower segment of a congruence set is well-ordered only if the congruence set has a first element. We again distinguish two cases.

Case 1. Every congruence set in A has a first element, that is, every congruence set is well-ordered.

In this case, we will show that there can be only one congruence set altogether; the set A will thus be identical with this unique congruence set, and will therefore be well-ordered. To see that we cannot have more than one congruence set here, suppose the contrary, and let α and β, $(\alpha < \beta)$, be the first elements of any two different congruence sets, which sets may be designated by $C(\alpha)$ and $C(\beta)$. If there were in A no element c which separates $C(\alpha)$ and $C(\beta)$, then the set $D = C(\alpha) + C(\beta)$, considered as an ordered subset of A, would be an interval of A which is well-ordered. This would mean that $\alpha \equiv \beta$; but this cannot be, since α and β belong to different congruence sets. Hence, there must exist a congruence set $C(\gamma)$, whose first element is γ, such that $C(\alpha) < C(\gamma) < C(\beta)$, and we would thus have $\alpha < \gamma < \beta$. This means that the set A' of all the first elements of congruence sets in A will be a dense set—contrary to the fact that A cannot contain such a subset.

Case 2. There is a congruence set C without a first element.

We will show that this supposition leads to a contradiction, so that this case cannot actually arise. The proof of our theorem will then be complete.

We can find in C a sequence of elements $c_1 > c_2 > c_3 > \cdots > c_n > \cdots$

such that for every d in C there is a natural number n for which $c_n \leqq d$. The set of elements $[c_{i+1}, c_i)$ of C between c_i and c_{i+1}, inclusive of c_{i+1}, is a well-ordered set; let β_i be the order-type of this set, and consider the sequence of ordinals $\beta_1, \beta_2, \cdots, \beta_n, \cdots$ thus defined. There can be only a finite number of indices k such that for all $n>k$, $\beta_n<\beta_k$ (otherwise one would obtain an infinite sequence of decreasing ordinals, which is impossible). Hence, there will exist an index r such that for every $p \geqq r$ there will be infinitely many indices $q>p$ for which $\beta_p \leqq \beta_q$. In other words, for $i=r, r+1, r+2, \cdots$ the set $[c_{i+1}, c_i)$ will be similar to a subset of some $[c_{q_i+1}, c_{q_i})$ where $q_i>i$, and where moreover, $q_r<q_{r+1}<q_{r+2}<\cdots$. We now express A as the sum of three sets A_1, A_2 and A_3, where A_2 consists of all the elements of C preceding c_r, A_1 consists of all elements of A preceding every element of A_2, and A_3 consists of all the elements of A following every element of A_2. Define now a function f on A to a subset of A as follows: If $\alpha \varepsilon A_1$ or $\alpha \varepsilon A_3, f(\alpha)=\alpha$. Suppose that $\alpha \varepsilon A_2$. There will be an $i \geqq r$ for which $\alpha \varepsilon [c_{i+1}, c_i)$, and since $[c_{i+1}, c_i)$ is similar to a subset of $[c_{q_i+1}, c_{q_i})$, there will be a γ_α in this last set which corresponds to α under such a similarity. We define, for $\alpha \varepsilon A_2, f(\alpha)=\gamma_\alpha$.

This function is clearly a similarity transformation on A to a subset of itself, and for any α in A_2 we have $f(\alpha)=\gamma_\alpha<\alpha$; but this is impossible, by our hypothesis.

It may be observed that up to this point it has not been necessary to make use of Zermelo's axiom of choice.

3. **The nondenumerable case.** We now prove the following result:

THEOREM 3. *The linear continuum C contains a set E, of power $\mathfrak{c}$, which is not similar to any proper subset of itself.*

PROOF. A similarity transformation of C into a subset of itself is a monotonic increasing function of a real variable, and vice versa. Hence, there are exactly $\mathfrak{c}$ such transformations. Let $\Omega_{\mathfrak{c}}$ denote the first ordinal to correspond to the cardinal number of the continuum, and arrange all of these transformations, with the exception of the identity, in a well-ordered series of type $\Omega_{\mathfrak{c}}$: $T_1, T_2, \cdots, T_\alpha, \cdots, (\alpha<\Omega_{\mathfrak{c}})$.

Now, the fixed points under a monotonic transformation which is not the identity cannot be everywhere dense in C. Hence, if T is such a transformation, there are $\mathfrak{c}$ points p such that $T(p) \neq p$. We may accordingly choose a point p_1 such that $T_1(p_1)=q_1 \neq p_1$. We now assume that p_β and q_β have been defined for all $\beta<\alpha<\Omega_{\mathfrak{c}}$, and choose distinct points p_α and q_α in $C-\sum_{\beta<\alpha}(p_\beta+q_\beta)$ such that $T_\alpha(p_\alpha)=q_\alpha$.

This is possible for every $\alpha < \Omega_{\mathfrak{c}}$, since the power of $\sum_{\beta<\alpha}(p_\beta+q_\beta)$ is less than $\mathfrak{c}$.

We notice first that if C' is any interval on C, then there exists a monotonic increasing function which is the identity on $C-C'$ and is not the identity on C'. Therefore there is an α such that $p_\alpha \varepsilon C'$. Hence the set $E=\sum_{\alpha<\Omega_{\mathfrak{c}}} p_\alpha$ is dense on C. It now follows that there is no similarity transformation of E into a proper subset of itself. In fact, we can show that the only similarity transformation of E into a subset of itself is the identity. For assume that τ is a similarity transformation, other than the identity, of E into a subset of itself. Since τ is not the identity and since E is dense in C, there exists an α such that T_α agrees with τ on E. But this is impossible, since $T_\alpha(p_\alpha)=q_\alpha$, and q_α is not a point of E.

THEOREM 4. *The linear continuum contains a set E which is not well-ordered, but which has the property that if f is any similarity transformation of E into a subset of itself, then $f(e) \geqq e$ for every e in E.*

PROOF. The set E constructed in the proof of Theorem 3 satisfies these requirements.

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ON PERFECT SUMMABILITY OF DOUBLE SEQUENCES

J. D. HILL

1. **Introduction.** The purpose of the present note is to point out that the results of Banach on perfect summability of simple sequences[1] may be extended to certain cases of double sequence summability. The main result obtained is embodied in the theorem of §3.

It will be convenient to begin with the following definitions and notations. We denote by c the class of all real double sequences $x\equiv\{\xi_{kl}\}$ for which the *principal limit* $\lim_{k,l}\xi_{kl}\equiv\xi$ exists finite; by rc, the subclass of c for which the *row limits* $\lim_l \xi_{kl}\equiv\xi_k^r$, $(k=0,1,2,\cdots)$ and the *column limits* $\lim_k\xi_{kl}\equiv\xi_l^c$, $(l=0,1,2,\cdots)$, exist finite; and by $rcrn$, the subclass of rc throughout which $\xi_k^r=\xi_l^c=\xi=0$, $(k,l=0,1,2,\cdots)$. With the conventional definitions of addition and multiplication by a constant the classes rc and $rcrn$ are linear spaces, and they become Banach spaces upon introduction of the norm $\|x\|\equiv\sup_{k,l}|\xi_{kl}|$.

Let $A\equiv(a_{ijkl})$ be a given infinite matrix of real numbers. We shall be concerned with transformations of the form

$$A_{ij}(x)\equiv\sum_{k,l}a_{ijkl}\xi_{kl},\qquad i,j=0,1,2,\cdots,\tag{1.1}$$

on the elements $x\equiv\{\xi_{kl}\}$ of rc. More precisely, if $A_{ij}(x)$, $(i,j=0,1,2,\cdots)$, exists for every $x\ \varepsilon\ rc$ and if the corresponding sequence $\{A_{ij}(x)\}$ belongs to rc $[c]$, one says that rc is transformed into rc $[c]$ by the method of summability corresponding to the matrix A, or simply, by the method A. If $A(x)\equiv\lim_{i,j}A_{ij}(x)=\xi$ for every $x\ \varepsilon\ rc$, the method is called *regular*. A regular transformation of rc into itself is called *completely regular* if it is also *regular by rows and columns*, that is to say, if $A_i^r(x)\equiv\lim_j A_{ij}(x)=\xi_i^r$, $(i=0,1,2,\cdots)$, and $A_j^c(x)$ $\lim_i A_{ij}(x)=\xi_j^c$, $(j=0,1,2,\cdots)$, for every $x\ \varepsilon\ rc$.

For later reference we recall that the condition

$$\sup_{i,j}\sum_{k,l}|a_{ijkl}|\equiv M<\infty\tag{1.2}$$

is necessary[2] in order that A transform $rcrn$ into rc.

If the system of equations

$$\sum_{k,l}a_{ijkl}\xi_{kl}=\eta_{ij},\qquad i,j=0,1,2,\cdots,\tag{1.3}$$

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[1] Banach, *Théorie des Opérations Linéaires*, pp. 90–95.

[2] See Hamilton, *Transformations of multiple sequences*, Duke Mathematical Journal, vol. 2 (1936), pp. 29–60; in particular, p. 42, #5.

has a unique solution $\{\xi_{kl}\}$ (belonging to rc or not) corresponding to each $\{\eta_{ij}\}$ ε rc, the method A will be called *reversible.*[3]

REMARK. If A is reversible, it can be shown[4] that there exist linear functionals $f_{kl}(y)$ defined in rc, such that for each $y \equiv \{\eta_{ij}\}$ ε rc we have $\sum_{k,l} a_{ijkl} f_{kl}(y) = \eta_{ij}$, $(i, j = 0, 1, 2, \cdots)$. This fact is essential to the proof of Lemma 4 in §3.

A will be said to be of *type M* if the conditions[5]

$$\sum_{i,j} |\alpha_{ij}| < \infty, \qquad \sum_{i,j} \alpha_{ij} a_{ijkl} = 0, \qquad k, l = 0, 1, 2, \cdots, \tag{1.4}$$

always imply $\alpha_{ij} = 0$, $(i, j = 0, 1, 2, \cdots)$.

Any method A which is simultaneously reversible, completely regular, and of type M will be called *perfect.*[6]

Frequent use will be made of the following particular double sequences: the sequence X, all of whose terms are 1; the sequence $\overset{r}{X}_k$, $(k = 0, 1, 2, \cdots)$, all of whose terms are 0 except those in the kth row which are 1; the sequences $\overset{c}{X}_l$, $(l = 0, 1, 2, \cdots)$, all of whose terms are 0 except those in the lth column which are 1; and the sequences X_{kl}, $(k, l = 0, 1, 2, \cdots)$, all of whose terms are 0 except for the term common to the kth row and lth column, which is 1.

Corresponding to any given x ε rc we set

$$\begin{aligned} x_i \equiv \xi X &+ \sum_{k=0}^{p_i} (\overset{r}{\xi}_k - \xi)\overset{r}{X}_k + \sum_{l=0}^{p_i} (\overset{c}{\xi}_l - \xi)\overset{c}{X}_l + \sum_{k,l=0}^{p_i} (\xi - \xi_{kl}) X_{kl} \\ &+ \sum_{k,l=0}^{p_i,q_i} (\xi_{kl} - \overset{r}{\xi}_k) X_{kl} + \sum_{k,l=0}^{q_i,p_i} (\xi_{kl} - \overset{c}{\xi}_l) X_{kl}, \quad i = 0, 1, 2, \cdots. \end{aligned} \tag{1.5}$$

Then each x_i ε rc, and it is easy to verify that p_i, q_i, $(q_i \geqq p_i \to \infty)$, can be so chosen that $\lim_i x_i = x$.

2. The general form of linear functionals in *rc*.

The following fact will be needed in the proof of Lemma 1 in §3.

Every linear functional $f(x)$ defined in rc is of the form

$$f(x) = C\xi + \sum_k \overset{r}{C}_k \overset{r}{\xi}_k + \sum_l \overset{c}{C}_l \overset{c}{\xi}_l + \sum_{k,l} C_{kl} \xi_{kl}, \tag{2.1}$$

and

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[3] Compare with Banach, op. cit., p. 90.

[4] See Banach, op. cit., p. 47, Theorem 10. The proof given is easily extended to the system (1.3).

[5] Compare with Banach, op. cit., p. 90.

[6] Compare with Banach, op. cit., p. 90.

$$\|f\| = |C| + \sum_k |C_k^r| + \sum_l |C_l^c| + \sum_{k,l} |C_{kl}|. \tag{2.2}$$

PROOF. Let $f(x)$ be an arbitrary linear functional defined in *rc*, and set $f(X) \equiv A$, $f(X_k^r) \equiv A_k^r$, $f(X_l^c) \equiv A_l^c$, $f(X_{kl}) \equiv A_{kl}$, $(k, l = 0, 1, 2, \cdots)$. We show first that each of the series $\sum_k |A_k^r|$, $\sum_l |A_l^c|$, $\sum_{k,l} |A_{kl}|$ is convergent.

Let $x \in rcrn$ be arbitrary and set $x_{ij} \equiv \sum_{k,l=0}^{i,j} \xi_{kl} X_{kl}$, $(i, j = 0, 1, 2, \cdots)$. Then each $x_{ij} \in rcrn$ and since $x_{ij} \to x$ as $i, j \to \infty$, it follows that $f(x_{ij}) = \sum_{k,l=0}^{i,j} \xi_{kl} A_{kl}$ tends to a limit, namely $f(x)$, as $i, j \to \infty$. In order that this be true for every $x \in rcrn$ it is necessary[7] that $\sum_{k,l} |A_{kl}| < \infty$.

On the other hand let $\{\xi_k\}$ be an arbitrary null sequence, and let $x \equiv \{\xi_{kl}\}$ be such that for each fixed $k = 0, 1, 2, \cdots$ we have $\xi_{kl} = \xi_k$ for $l = 0, 1, 2, \cdots$. If we set $x_i \equiv \sum_{k=0}^{i} \xi_k X_k^r$, $(i = 0, 1, 2, \cdots)$, then each $x_i \in rc$ and $x_i \to x$ as $i \to \infty$. Since $f(x_i) = \sum_{k=0}^{i} \xi_k A_k^r$, it follows that the series $\sum_k \xi_k A_k^r$ is convergent for every null sequence $\{\xi_k\}$. Consequently, the condition $\sum_k |A_k^r| < \infty$ must be satisfied,[8] and it is evident that the symmetrical condition $\sum_l |A_l^c| < \infty$ must also hold.

Now let $x \in rc$ be chosen arbitrarily, and let x_i be defined as in (1.5) so that $x_i \to x$. Then $f(x_i) \to f(x)$, and from (1.5) we find that

$$\begin{aligned} f(x) = \xi A &+ \sum_k (\xi_k^r - \xi) A_k^r + \sum_l (\xi_l^c - \xi) A_l^c \\ &+ \sum_{k,l} (\xi + \xi_{kl} - \xi_k^r - \xi_l^c) A_{kl}. \end{aligned}$$

On rearranging the latter expression and setting

$$\begin{aligned} C &\equiv A - \sum_k A_k^r - \sum_l A_l^c + \sum_{k,l} A_{kl}, & C_k^r &\equiv A_k^r - \sum_l A_{kl}, \\ C_l^c &\equiv A_l^c - \sum_k A_{kl}, & C_{kl} &\equiv A_{kl}, \end{aligned}$$

$f(x)$ reduces to the form (2.1) as was to be shown.

Regarding the evaluation of $\|f\|$, it is clear that $\|f\|$ does not exceed the value given in (2.2). The converse statement can be readily shown to follow from the relation $|f(x_n)| \leqq \|f\|$, wherein $x_n \equiv \{\xi_{kl}^n\}$ for each $n = 0, 1, 2, \cdots$ is defined as follows:[9] $\xi_{kl}^n \equiv \operatorname{sgn} C_{kl}$ for $0 \leqq k, l \leqq n$; $\equiv \operatorname{sgn} C_k^r$ for $0 \leqq k \leqq n$, $n < l < \infty$; $\equiv \operatorname{sgn} C_l^c$ for $0 \leqq l \leqq n$, $n < k < \infty$;

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[7] See Hamilton, loc. cit., p. 41, #1.

[8] See Banach, op. cit., p. 86. Merely replace (c) by (c_0) in the discussion at the top of the page.

[9] To insure $\|x_n\| = 1$ for every n, let sgn $\alpha \equiv 1$ if $\alpha \geqq 0$; $\equiv -1$ if $\alpha < 0$.

$\equiv \operatorname{sgn} C$ for $n<k$, $l<\infty$. Inasmuch as we make no use of (2.2), the remaining details may be omitted.

3. **The analogues of Banach's results.** We come next to the following four lemmas. Since these bear a direct analogy to those of Banach, both in statement and method of proof, we shall give here only the proof of the first lemma. The second one is a straightforward corollary of the first, and the reader who is familiar with the original lemmas will encounter no difficulty in supplying the remaining two proofs.[10]

LEMMA 1. *Let A be a completely regular method and let $y_0 \equiv \{\eta_{ij}^0\}$ be an arbitrary element of rc. If the conditions* (1.4) *always imply $\sum_{i,j}\alpha_{ij}\eta_{ij}^0 = 0$, then corresponding to each $\epsilon>0$ there exists an $x \varepsilon rc$ such that*

$$|A_{ij}(x) - \overset{0}{\eta}_{ij}| < \epsilon, \qquad i, j = 0, 1, 2, \cdots. \tag{3.1}$$

PROOF. Let G denote the subset of rc of all sequences of the form $\{\eta_{ij}\} \equiv \{A_{ij}(x)\}$ for x ranging over rc. Evidently G is a linear set.

Let us assume that for some particular $\epsilon>0$, the condition (3.1) cannot be satisfied. This is equivalent to assuming that y_0 is not a limit element of G. In such a case there must exist a linear functional $F(y)$ defined in rc such that[11] $F(y_0)=1$ and $F(y)=0$ for every $y \varepsilon G$. In view of §2 we are therefore led to relations of the following sort:

$$C\overset{0}{\eta} + \sum_i \overset{r}{C}_i \overset{0r}{\eta}_i + \sum_j \overset{o}{C}_j \overset{0o}{\eta}_j + \sum_{i,j} C_{ij}\overset{0}{\eta}_{ij} = 1, \tag{3.2}$$

$$C\eta + \sum_i \overset{r}{C}_i \overset{r}{\eta}_i + \sum_j \overset{o}{C}_j \overset{o}{\eta}_j + \sum_{i,j} C_{ij}\eta_{ij} = 0 \quad \text{for all} \quad \{\eta_{ij}\} \varepsilon G, \tag{3.3}$$

wherein $\sum_{i,j}|C_{ij}| < \infty$. From the definition of G and the fact that A is completely regular, (3.3) may be written

$$C\xi + \sum_i \overset{r}{C}_i \overset{r}{\xi}_i + \sum_j \overset{c}{C}_j \overset{c}{\xi}_j + \sum_{i,j} C_{ij}A_{ij}(x) = 0 \quad \text{for all} \quad x \varepsilon rc. \tag{3.4}$$

Now by virtue of (1.2) we have $\sum_{i,j,k,l=0}^{n}|C_{ij}a_{ijkl}\xi_{kl}| \leqq M\cdot\|x\| \cdot\sum_{i,j}|C_{ij}| < \infty$ for each $n=0, 1, 2, \cdots$ and each $x \equiv \{\xi_{kl}\} \varepsilon rc$. Consequently, (3.4) may be expressed in the form

$$\Phi(x) \equiv C\xi + \sum_i \overset{r}{C}_i \overset{r}{\xi}_i + \sum_j \overset{c}{C}_j \overset{c}{\xi}_j + \sum_{k,l} \xi_{kl} \sum_{i,j} C_{ij}a_{ijkl} = 0,$$

valid for all $x \varepsilon rc$. Allowing x to take in turn the values X_{kl}, $\overset{r}{X}_k$, $\overset{c}{X}_l$,

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[10] In this connection, however, see the remark in §1.

[11] Banach, op. cit., p. 57, §3, Lemma.

X, $(k, l=0, 1, 2, \cdots)$, in $\Phi(x)$, we find successively the following relations:

$$\sum_{i,j} C_{ij}a_{ijkl} = 0, \qquad k, l = 0, 1, 2, \cdots, \tag{3.5}$$

and $C_i^r = C_j^c = C = 0$, $(i, j=0, 1, 2, \cdots)$. Equation (3.2) then reduces to $\sum_{i,j} C_{ij}\eta_{ij}^0 = 1$, and this in view of (3.5) contradicts the hypothesis. The proof is therefore complete.

LEMMA 2. *If the method A is completely regular and of type M, then corresponding to each $\{\eta_{ij}^0\}$ ε rc and each $\epsilon > 0$, there exists an x ε rc satisfying condition* (3.1).

LEMMA 3. *If x_0 is a bounded sequence transformed into rc by a completely regular method A, then corresponding to each $\epsilon > 0$ there exists an x ε rc such that*

$$|A_{ij}(x) - A_{ij}(x_0)| < \epsilon, \qquad i, j = 0, 1, 2, \cdots. \tag{3.6}$$

LEMMA 4. *Let x_0 be any sequence transformed into rc by a completely regular and reversible method A. If corresponding to each $\epsilon > 0$ there exists an x ε rc satisfying* (3.6), *then the sequence x_0 is summable to the same principal limit by every method B which transforms rc into c regularly and which transforms into c every sequence that A transforms into rc. Moreover, if B is completely regular and transforms into rc every sequence that A does, then the row and column limits assigned by B to x_0 will be respectively equal to those assigned by A.*

Finally, the preceding lemmas may be combined to yield the following counterpart of Banach's theorems.

THEOREM. *Let the method A be perfect [completely regular and reversible], and let the method B transform rc into c regularly and transform into c every sequence that A transforms into rc. Then every sequence [bounded sequence] x_0 transformed into rc by A is transformed into c by B with the same principal limit. Moreover, if B is completely regular and transforms into rc every sequence that A does, then the row and column limits assigned by B to x_0 will be respectively equal to those assigned by A.*

MICHIGAN STATE COLLEGE

NOTE ON THE COMPUTATION OF THE DIFFERENCES OF THE Si(x), Ci(x), Ei(x) AND $-$Ei($-x$) FUNCTIONS

MILTON ABRAMOWITZ

In a recent article,[1] Lowan reached the conclusion that the second differences of the functions $\mathrm{Si}(x)$, $\mathrm{Ci}(x)$ and $\mathrm{Ei}(x)$ for values of x from 0.1 to 2 at intervals of 0.0001 may be approximated to 12 decimal places by certain expressions in closed form, which may be computed with the aid of tabulated functions.

The above functions and also $-\mathrm{Ei}(-x)$ were computed to 12 decimal places by adding an appropriate number of terms in their power series expansion.[2] In order to check the accuracy of the computations, the second differences of the computed values were required to agree to 12 decimal places with the values of the approximate expressions mentioned above.

The functions $\mathrm{Ci}(x)$, $\mathrm{Ei}(x)$, $-\mathrm{Ei}(-x)$ have a logarithmic singularity at the origin. For this reason Lowan's conclusion could not be extended to the interval $0<x<0.1$.

If $\phi(x)$ represents any of the functions $\mathrm{Ci}(x)-\log_e x$, $\mathrm{Ei}(x)-\log_e x$, $-\mathrm{Ei}(-x)+\log_e x$, it will be shown that the second difference of $\phi(x)$ may be approximated by $h^2\phi''(x)$ to 12 decimal places. Let

$$R(x) = [\phi(x+h) - 2\phi(x) + \phi(x-h)] - h^2\phi''(x).$$

Substituting for $\phi(x+h)$ and $\phi(x-h)$ their Taylor expansions, we get

$$R(x) = 2\left[\frac{h^4}{4!}\phi^4(x) + \frac{h^6}{6!}\phi^6(x) + \cdots\right]$$

whence

$$\text{(A)} \qquad |R(x)| < 2\left[\frac{h^4}{4!}\{\phi^4(x)\} + \frac{h^6}{6!}\{\phi^6(x)\} + \cdots\right]$$

where $\{\phi^{2k}(x)\}$ is an upper bound of the modulus of the $2k$th derivative.

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[1] A. N. Lowan, *On the computation of second differences of the* Si(x), Ei(x), *and* Ci(x) *functions*, this Bulletin, vol. 45 (1939), pp. 583–588. This will be referred to as A.N.L.

[2] This work is done by the New York City Work Projects Administration project for the Computation of Mathematical Tables, O. P. #765-97-3-10, under the sponsorship of Dr. Lyman J. Briggs, Director of the National Bureau of Standards. The author wishes to express his appreciation to the W.P.A. and to the sponsor of this project for permission to publish these results.

Case of the function $\mathrm{Ei}(x)-\log_e x$. From

$$\frac{d}{dx}(\mathrm{Ei}\,(x)-\log_e x)=\frac{e^x-1}{x}=\int_0^\infty e^{x(1-t)}dt-\int_0^\infty e^{xt}dt$$

(see A.N.L.) we get

$$\frac{d^{2k}}{dx^{2k}}(\mathrm{Ei}\,(x)-\log_e x)=\int_0^\infty(1-t)^{2k-1}e^{x(1-t)}dt-\int_0^\infty t^{2k-1}e^{xt}dt,$$

and, in view of (9) and (10) of A.N.L.,

$$\left|\frac{d^{2k}}{dx^{2k}}(\mathrm{Ei}\,(x)-\log_e x)\right|<\frac{e^x}{2k}\cdot$$

Making use of this result (A) yields

$$|R(x)|<2e^x\left\{\frac{h^4}{4!}\frac{1}{4}+\frac{h^6}{6!}\frac{1}{6}+\cdots\right\}\sim\frac{e^x}{2}\frac{h^4}{4!}\frac{1}{1-h}\cdot$$

For $h=10^{-4}$ and $0<x<0.1$, the last expression is smaller than 10^{-17}. We thus reach the conclusion that for these values of h and x, the second differences of $\mathrm{Ei}(x)-\log_e x$ may be approximated by

$$h^2\frac{d^2}{dx^2}(\mathrm{Ei}\,(x)-\log_e x).$$

From the series expansion for the function $\mathrm{Ei}(x)-\log_e x$ it is easily seen that the second difference of this function may be approximated to 12 decimal places by

$$h^2\left(\frac{1}{2}+\frac{x}{3}+\frac{1}{4}\frac{x^2}{2!}+\frac{1}{5}\frac{x^3}{3!}\right).$$

In an entirely similar manner it may be shown that $1/(2k)$ is an upper bound of the derivatives of the functions $-\mathrm{Ei}(-x)+\log_e x$ and $\mathrm{Ci}(x)-\log_e x$, and if $h=10^{-4}$ and $0<x<0.1$, the second difference of the latter functions may be approximated to 12 decimal places by the respective expressions

$$h^2\left(-\frac{1}{2}+\frac{x}{3}-\frac{1}{4}\frac{x^2}{2!}+\frac{1}{5}\frac{x^3}{3!}\right),\qquad h^2\left(-\frac{1}{2}+\frac{1}{4}\frac{x^2}{2!}\right).$$

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DIOPHANTINE EQUATIONS OF DEGREE[1] n

A. A. AUCOIN

In a recent issue of the National Mathematics Magazine,[2] W. V. Parker and the author obtained solutions of the Diophantine equation $F(x_1, \cdots, x_p)=G(y_1, \cdots, y_q)$, where F and G are homogeneous polynomials, with integral coefficients, of degree 3, and F is such that for a set of integers $x_i=a_i$ (not all zero), $\partial F/\partial x_i=0$, $(i=1, \cdots, p)$. In this paper the above is extended to functions of degree n. One type, which satisfies the conditions of the main theorem, is also solved by an entirely different method. The solutions obtained are in terms of arbitrary parameters, and they are integral for an integral choice of the parameters.

If $x_i=\alpha_i$, $y_k=\beta_k$ is a solution of the equation $f(x_1, \cdots, x_p)=g(y_1, \cdots, y_q)$, where f and g are homogeneous polynomials, with integral coefficients, of degrees n and m respectively, and there are no integers $s>1$, α_i', β_k' such that $\alpha_i=s^\lambda\alpha_i'$, $\beta_k=s^\mu\beta_k'$ where λ, μ are relatively prime positive integers such that $\lambda n=\mu m$, then $x_i=\alpha_i$, $y_k=\beta_k$ is said to be a primitive solution. If $x_i=\alpha_i$, $y_k=\beta_k$ is a primitive solution of the above equation, then $x_i=\alpha_i t^\lambda$, $y_k=\beta_k t^\mu$ (derived from this primitive solution), where λ, μ are any positive integers such that $\lambda n=\mu m$, is also a solution. Two solutions are said to be equivalent if they are derived from the same primitive solution.

THEOREM 1. *Let $f(x_1, \cdots, x_p)$, $g(y_1, \cdots, y_q)$ be homogeneous polynomials with integral coefficients, of degrees n and m respectively. Let $a_1, \cdots, a_p$ be integers not all zero such that the partial derivatives of f of all orders less than $n-1$ vanish*[3] *when $x_i=a_i$. Then every solution in integers x_i, y_k of*

$$f(x_1, \cdots, x_p) = g(y_1, \cdots, y_q), \tag{1}$$

for which

$$\sum_{j=1}^{p} a_j \frac{\partial f}{\partial x_j} \neq 0, \tag{2}$$

is equivalent to one of the infinitude of solutions given by

$$x_i = a_i s t^{\lambda-1} + \alpha_i t^\lambda, \quad y_k = \beta_k t^\mu, \quad i = 1, 2, \cdots, p; \; k = 1, 2, \cdots, q, \tag{3}$$

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[1] Presented to the Society, December 2, 1939.

[2] *On cubic Diophantine equations*, vol. 13 (1938), pp. 115–117.

[3] It follows from Euler's theorem that the function itself vanishes for this choice of x_i.

where λ, μ *are positive integers such that* $\lambda n=\mu m$, α_i *and* β_k *are arbitrary integers,*

$$(4)\qquad s = g(\beta) - f(\alpha), \qquad t = \sum_{j=1}^{p} a_j \frac{\partial f}{\partial \alpha_j},$$

and $f(\alpha)=f(\alpha_1, \cdots, \alpha_p)$, $g(\beta)=g(\beta_1, \cdots, \beta_q)$.

Proof. By Taylor's theorem, if we set $x_i=a_is+\alpha_it$,

$$f(x_1, \cdots, x_p) = st^{n-1}\sum_{j=1}^{p} a_j \frac{\partial f}{\partial \alpha_j} + t^n f(\alpha_1, \cdots, \alpha_p).$$

Hence if x_i, y_k have the values given by (3), s and t those given by (4), (1) becomes $st^{n\lambda-1}\sum_{j=1}^{p}a_j\partial f/\partial\alpha_j+t^{n\lambda}f(\alpha)=t^{m\mu}g(\beta)$, and is satisfied identically in the α_i and β_k. Hence (3) is a solution of (1) with s and t given by (4).

Suppose $x_i=\rho_i$, $y_k=\sigma_k$ is any solution of (1) and (2). If we choose $\alpha_i=\rho_i$, $\beta_k=\sigma_k$, we have that $s=0$, and (3) becomes $x_i=\rho_it^\lambda$, $y_k=\sigma_kt^\mu$, equivalent to the given solution, since by (2), $t\neq 0$.

If $g\equiv 0$, the theorem still holds, with λ arbitrary.

Corollary. *The equation* $f(x)=\sum_{j=1}^{p}x_jg_j(y)+g(y)$, *where* $g_j(y)=g_j(y_1, \cdots, y_q)$ *and* $g(y)=g(y_1, \cdots, y_q)$ *are homogeneous polynomials with integral coefficients of degrees* $n-1$ *and* n, *respectively, has solutions, and every solution which is not also a solution of* $\sum_{j=1}^{p}a_j[\partial f/\partial x_j-g_j(y)]=0$ *is equivalent to one of the infinitude of solutions given by* $x_i=a_is+\alpha_it$, $y_k=\beta_kt$ *where*

$$s = g(\beta) - f(\alpha) - \sum_{j=1}^{p}\alpha_jg_j(\beta), \qquad t = \sum_{j=1}^{p}a_j\left[\frac{\partial f}{\partial \alpha_j} - g_j(\beta)\right].$$

One function of interest which satisfies the hypothesis of Theorem 1 is the function $D(x)=|a_{ij}x_{ij}|$, a determinant of order n with a_{ij} integral such that not all the a's in any row or column are zero. For this function not all the x_{ij} need be distinct. If there is any x_{ij}, say x_{pq}, which occurs only once in D, we may make the choice $x_{pq}=1$, $x_{ij}=0$ otherwise; then all the partial derivatives of all orders less than $n-1$ vanish.

In the solution the form of the expression is the same for every element except x_{pq}. This fact is illustrated by the equation $D(x)=g(y)$, the solution of which is $x_{pq}=st^{\lambda-1}+\alpha_{pq}t^\lambda$, $x_{ij}=\alpha_{ij}t^\lambda$, $(i\neq p,\ j\neq q)$, $y_k=\beta_kt^\mu$ where $s=g(\beta)-D(\alpha)$, $t=D'(\alpha)$, $D'(\alpha)$ being $D(\alpha)$ with a_{pq} the element in the pth row and qth column, and the other elements in the qth column zero.

It is not necessary, in some cases, that there be a unique element x_{pq}. If $a_{ij}=1$, for example, D may be the circulant. In this case we make the choice $x_{ij}=1$.

Another function of interest which also satisfies the condition of Theorem 1 is $P(x)=\prod_{i=1}^{n}\sum_{j=1}^{n}a_{ij}x_j$, where a_{ij} are integral, and the determinant $A=|a_{ij}|\neq 0$. For we may choose x_j, integral, so that $n-1$ of the above factors vanish and hence for this choice of x_j all partial derivatives of all orders less than $n-1$ vanish.

The next equation satisfies the hypothesis of Theorem 1, but will be solved by an entirely different method. This is given in the following theorem:

THEOREM 2. *The equation*

$$P(x) = g(y), \tag{5}$$

where $P(x)$ is given above and $g(y)$ is given in Theorem 1, *has solutions, and every solution which is not also a solution of*

$$\prod_{i=1}^{n-1}\sum_{j=1}^{n} a_{ij}x_j = 0 \tag{6}$$

is equivalent to one of the infinitude of solutions given by

$$\begin{aligned} x_j &= t^{\lambda}(A_{nn})^{\lambda-1}A_{nn}^{(j)} + st^{\lambda-1}(A_{nn})^{\lambda-1}A_{nj}, \qquad j = 1, \cdots, n-1, \\ x_n &= st^{\lambda-1}(A_{nn})^{\lambda}, \qquad y_k = t^{\mu}(A_{nn})^{\mu}\beta_k, \end{aligned} \tag{7}$$

where A_{ij} is the cofactor of a_{ij} in[4] *A; $A_{nn}^{(j)}$ is the determinant obtained from A_{nn} by replacing the jth column by $\alpha_1, \alpha_2, \cdots, \alpha_{n-1}$; s and t are given by*

$$s = A_{nn}g(\beta) - \prod_{i=1}^{n-1}\alpha_i\sum_{j=1}^{n} a_{nj}A_{nn}^{(j)}, \qquad t = A\prod_{i=1}^{n-1}\alpha_i, \tag{8}$$

λ, μ are relatively prime positive integers such that $\lambda n=\mu m$, and the α's and β's are arbitrary integers.

PROOF. Set

$$\sum_{j=1}^{n} a_{ij}x_j = t^{\lambda}(A_{nn})^{\lambda}\alpha_i, \qquad i = 1, \cdots, n-1. \tag{9}$$

If we let $x_n=st^{\lambda-1}(A_{nn})^{\lambda}$, we may write equation (9) in the form

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[4] Since $A\neq 0$, there is a minor of order $n-1$ which does not vanish. Without loss of generality, we may choose the notation so that $A_{nn}\neq 0$.

$\sum_{j=1}^{n-1} a_{ij}x_j = t^{\lambda}(A_{nn})^{\lambda}\alpha_i - st^{\lambda-1}a_{in}(A_{nn})^{\lambda}$. Solving this system of equations we get

$$(10)\qquad x_j = t^{\lambda}(A_{nn})^{\lambda-1}A_{nn}^{(j)} + st^{\lambda-1}(A_{nn})^{\lambda-1}A_{nj}, \qquad j = 1, \cdots, n-1.$$

It follows then that

$$(11)\qquad \sum_{j=1}^{n} a_{nj}x_j = t^{\lambda-1}(A_{nn})^{\lambda-1}\left[t\sum_{j=1}^{n-1} a_{nj}A_{nn}^{(j)} + sA\right].$$

If we let $y_k = t^{\mu}(A_{nn})^{\mu}\beta_k$, then by (9) and (11), (5) becomes

$$(12)\quad t^{n\lambda-1}(A_{nn})^{n\lambda-1}\prod_{i=1}^{n-1}\alpha_i\left[t\sum_{j=1}^{n-1} a_{nj}A_{nn}^{(j)} + sA\right] = t^{m\mu}(A_{nn})^{m\mu}g(\beta),$$

and since $\lambda n = \mu m$, (12) is identically satisfied in the α's and β's if s and t are given by (8). Hence (7) forms a solution of (5) with s and t given by (8).

Suppose now that $x_j = \rho_j$, $y_k = \sigma_k$ is any solution of (5). If we choose $\alpha_i = \sum_{j=1}^{n} a_{ij}\rho_j$, $\beta_k = \sigma_k$, we have[5]

$$t = A\prod_{i=1}^{n-1}\sum_{j=1}^{n} a_{ij}\rho_j,$$

$$\begin{aligned} s &= A_{nn}g(\sigma) - \prod_{i=1}^{n-1}\sum_{j=1}^{n} a_{ij}\rho_j\sum_{k=1}^{n-1} a_{nk}[\rho_k A_{nn} - \rho_n A_{nk}] \\ &= A_{nn}\prod_{i=1}^{n}\sum_{j=1}^{n} a_{ij}\rho_j - \prod_{i=1}^{n-1}\sum_{j=1}^{n} a_{ij}\rho_j\sum_{k=1}^{n-1} a_{nk}[\rho_k A_{nn} - \rho_n A_{nk}] \\ &= \prod_{i=1}^{n-1}\sum_{j=1}^{n} a_{ij}\rho_j\left[A_{nn}\sum_{j=1}^{n} a_{nj}\rho_j - A_{nn}\sum_{j=1}^{n-1} a_{nj}\rho_j + \rho_n\sum_{j=1}^{n-1} a_{nj}A_{nj}\right] \\ &= \rho_n A\prod_{i=1}^{n-1}\sum_{j=1}^{n} a_{ij}\rho_j = \rho_n t. \end{aligned}$$

Hence

$$x_j = t^{\lambda}(A_{nn})^{\lambda-1}[\rho_j A_{nn} - \rho_n A_{nj}] + (A_{nn})^{\lambda-1}t^{\lambda}\rho_n A_{nj} = (tA_{nn})^{\lambda}\rho_j,$$

$$x_n = (tA_{nn})^{\lambda}\rho_n, \qquad\qquad y_k = (tA_{nn})^{\mu}\sigma_k,$$

which is equivalent to the given solution provided $x_j = \rho_j$, $y_k = \sigma_k$ is not a solution of (6). We may find, however, values of x_j which satisfy (6) and these values, together with $y_k = 0$, afford additional solutions of (5).

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[5] $A_{nn}^{(j)}$ becomes $\rho_j A_{nn} - \rho_n A_{nj}$ when α_i is replaced by $\sum_{j=1}^{n} a_{ij}\rho_j$.

By the above method we may show the following consequence.

COROLLARY. *The equation* $P(x)=\sum_{j=1}^{n} x_j g_j(y)+g(y)$, *where* g_j *and* g *are the functions of the corollary to Theorem* 1, *has solutions, and every solution not also a solution of* $\sum_{j=1}^{n} A_{nj} g_j(y) - A\prod_{i=1}^{n-1}\sum_{j=1}^{n} a_{ij}x_j = 0$ *is given by* $x_j = A_{nn}^{(j)}t + A_{nj}s$, $(j=1, \cdots, n-1)$, $x_n = A_{nn}s$, $y_k = A_{nn}\beta_k t$, *where*

$$s = \prod_{i=1}^{n-1} \alpha_i \sum_{j=1}^{n} a_{nj} A_{nn}^{(j)} - \sum_{j=1}^{n-1} A_{nn}^{(j)} g_j(\beta) - A_{nn} g(\beta),$$

$$t = \sum_{j=1}^{n} A_{nj} g_j(\beta) - A \prod_{i=1}^{n-1} \alpha_i.$$

The final theorem treats an equation which satisfies the hypothesis of Theorem 1, but is reduced to an equivalent problem and then solved.

THEOREM 3. *The equation*

$$f(x) \sum_{j=1}^{p} d_j x_j = R(y), \tag{13}$$

where $f(x)$ *satisfies the conditions of Theorem* 1, *and* $R(y)=R(y_1, \cdots, y_q)$ *is a homogeneous polynomial with integral coefficients of degree* $n-1$, *has solutions; and every solution which is not also a solution of*

$$f(x) \sum_{j=1}^{p} a_j \frac{\partial f}{\partial x_j}\left[\sum_{j=1}^{p} a_j \frac{\partial f}{\partial x_j} \sum_{k=1}^{p} d_k x_k - f(x) \sum_{j=1}^{p} d_j a_j \right] = 0 \tag{14}$$

is equivalent to one of the infinitude of solutions given by

$$x_i = a_i s + \alpha_i t, \qquad y_k = \beta_k t \tag{15}$$

where

$$\begin{aligned} s &= A^{n-1}[\lambda(AD - BC)]^{n-2}[D\lambda^2 - BR(\mu)], \\ t &= A^{n-1}[\lambda(AD - BC)]^{n-2}[AR(\mu) - C\lambda^2], \\ \beta_k &= A^2[\lambda(AD - BC)]^2 \mu_k, \end{aligned} \tag{16}$$

and $A=\sum_{j=1}^{p} a_j \partial f/\partial \alpha_j$, $B=f(\alpha)$, $C=\sum_{j=1}^{p} d_j a_j$, $D=\sum_{j=1}^{p} d_j \alpha_j$, *the* α_j, λ, μ_k *being arbitrary integers.*

PROOF. If we let x_i, y_k have the values given by (15), (13) becomes, after dividing out the factor[6] t^{n-1},

$$(As + Bt)(Cs + Dt) = R(\beta). \tag{17}$$

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[6] It will be shown later that $t \neq 0$.

By Theorem 2, the solutions of the equation (17) are given by (16).

If $x_i=\rho_i$, $y_k=\sigma_k$ is any solution of (13) and we choose $\alpha_i=\rho_i$, $\mu_k=\sigma_k$, $\lambda=f(\rho)$, we have that $s=0$ and the solution becomes $x_i=\rho_iK^{n-1}$, $y_k=\sigma_kK^{n+1}$, where $K=A\lambda(AD-BC)$, which is equivalent to the given solution provided $K\neq 0$; that is, provided $x_i=\rho_i$, $y_k=\sigma_k$ is not a solution of (14). It will be noted that if $K\neq 0$, then $t\neq 0$.

LOUISIANA STATE UNIVERSITY

A MULTIPLE NULL-CORRESPONDENCE AND A SPACE CREMONA INVOLUTION OF ORDER $2n-1$[1]

EDWIN J. PURCELL

PART I. A NULL-SYSTEM $(1, mn, m+n)$ BETWEEN THE PLANES AND POINTS OF SPACE $(m, n=1, 2, 3, \cdots)$

1. **Introduction.** Consider a curve δ_m of order m having $m-1$ points in common with a straight line d, and a curve δ_n' of order n having $n-1$ points in common with a straight line d', $(m, n=1, 2, 3, \cdots)$. It is assumed for the present that neither δ_m nor d intersects either δ_n' or d'.

In general, through any point P of space there passes one ray ρ which intersects δ_m once and d once, and one ray ρ' which intersects δ_n' once and d' once; ρ and ρ' determine a plane π, the null-plane of P. Conversely, a plane π determines m rays ρ_i and n rays ρ_j' lying in it which intersect, a ray ρ with a ray ρ', in mn points, the null-points of the plane π.

Any point α in general position determines a ray ρ. As α describes a line l, the plane π of ρ and l contains n rays ρ', which intersect l in n points β; conversely, any point β on l determines a ray ρ' which determines with l the plane π, and π contains m rays ρ which intersect l in m points α—one being the original α. Thus an (m, n) correspondence is set up among the points of l with valence zero; there are $m+n$ coincidences and therefore $m+n$ points on any line l whose null-planes contain l.

2. **Planes whose null-points behave peculiarly.** We can obtain the last result by another method; this will yield additional information about planes whose null-points behave peculiarly.

Let a plane π turn about a line l as axis. A ruled surface will be generated by the m rays ρ_i lying in π. This surface is of order $m+1$; δ_m is a onefold curve on the surface and d is an m-fold line. Another ruled surface will be generated in this manner by the rays ρ_j' lying in π; its order is $n+1$, δ_n' is a onefold curve and d' is an n-fold line on this surface. The curve of intersection of these two surfaces is of order $(m+1)(n+1)$ and consists of l and a twisted curve k_{mn+m+n} of order $(m+1)(n+1)-1=mn+m+n$. *This* k_{mn+m+n} *is the locus of the null-points of all planes* π *through* l.

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[1] Presented to the Society, December 2, 1939.

Since a plane π meets this in mn points outside l, k_{mn+m+n} must intersect l in $m+n$ points through each of which a ray ρ and a ray ρ' pass which are coplanar with l. Call such a point on l, P. The plane $\rho\rho'$ is the null-plane of P and has $mn-1$ null-points outside l, and it follows that plane $\rho\rho'$ is tangent to k_{mn+m+n} at P. *The null-planes of the* $m+n$ *points of intersection of* k_{mn+m+n} *with* l *are tangent planes of* k_{mn+m+n} *at these points.*

The line d, an m-fold line on the first of the two surfaces described above, intersects the second surface in $n+1$ points, which are m-fold points on the first surface. The line d' intersects the first of the two surfaces in $m+1$ points which are n-fold points on the second surface. These points all lie on k_{mn+m+n} and the $m+1$ are n-fold points of it and $n+1$ are m-fold points of it. k_{mn+m+n} *has* $m+1$ n*-fold points on* d' *and* $n+1$ m*-fold points on* d.

δ_m has no actual double points or other multiple points. It is, however, rational and has $(m-1)(m-2)/2$ apparent double points and its rank is $r=m(m-1)-(m-1)(m-2)=2(m-1)$; that is, the order of its developable surface is $2(m-1)$. Similarly, the order of the developable surface of δ_n' is $2(n-1)$. The line l will intersect $2(m-1)$ tangents of δ_m and $2(n-1)$ tangents of δ_n'. In the plane π through l and a tangent line t of the first group, two rays ρ coincide in the line which joins the point of tangency of t with the intersection of d and π. Of the mn null-points in the plane π, n lie on each of the other $m-2$ rays ρ, and $2n$ fall two and two together on the coinciding rays; *in these points* k_{mn+m+n} *is tangent to the plane of* l *and* t *and the number of these planes is* $2(m+n-2)$.

From the discussion of this section we have the following conclusions:

(1) *The planes,* m *of whose null points coincide with a point of* d, *envelope a surface of class* $n+1$; *and the planes,* n *of whose null points coincide with a point of* d, *envelope a surface of class* $m+1$.

(2) *The planes,* $2n$ *of whose null-points coincide two and two on a ray* ρ, *envelope a surface of class* $2(m-1)$, n *of the remaining null-points lying on each of the other* $m-2$ *rays* ρ; *the planes,* $2m$ *of whose null-points coincide two and two on a ray* ρ', *envelope a surface of class*

$2(n-1)$, *m of the remaining null-points lying on each of the other $n-2$ rays ρ'.*

Consider a plane π through l, whose intersection with d is also an intersection with δ_m. Call this common point of d and δ_m, Δ. Then the rays ρ_i lying in π will be the $m-1$ lines joining Δ to the $m-1$ points of intersection of δ_m and π, not lying on d, and the line λ joining Δ to the intersection of l and the plane of d and the tangent line to δ_m at Δ. This line λ will be the limiting position of a ray ρ as a plane revolves about l into the position of π.

In the osculating planes of δ_m and δ_n', three rays coincide. Therefore, *in the osculating planes of δ_m, $3n$ of the null-points coincide three and three on the triple ray; in the osculating planes of δ_n', $3m$ of the null-points coincide three and three on the triple ray.*

3. **Points whose null-planes behave peculiarly.** Consider a point P on d. The point P determines one ρ'. Any plane π through ρ' determines m rays ρ through P. Therefore π counts m times as null-plane of P. Conversely, for every plane through ρ' there fall m null-points together at P. The surface of class $n+1$ mentioned in §2 must have the planes π as tangent planes. *This surface is a ruled surface consisting of rays ρ' which intersect d, and conversely.* Call this surface Σ.

The surface formed by rays ρ' which intersect a general straight line l is (§2) of order $n+1$, and d intersects this surface in $n+1$ points. Thus there are n rays ρ' which intersect d and also an arbitrary line l. Therefore the surface Σ is of degree $n+1$. The line d is a onefold directrix on Σ_{n+1} and d' is an n-fold directrix; for, the n-ic cone of δ_n' projected from a point of d' will intersect d in n points. *The locus of points whose null-planes have m null-points coinciding is Σ_{n+1}.*

Similarly, the ruled surface Σ_{m+1}' of order $m+1$, consisting of rays ρ that intersect d', is the *locus of points whose null-planes have n null-points coinciding.*

Now Σ_{n+1} and Σ_{m+1}' have $mn+1$ generators in common. For the congruence of rays ρ has the characteristic $(1, m)$ and the congruence of rays ρ' has the characteristic $(1, n)$ so that, from Halphen's theorem,[2] there are $1\cdot 1+m\cdot n=mn+1$ common rays.

Since both rays ρ and ρ' through any point on one of these $mn+1$ common rays coincide, any plane through the ray can be taken as null-plane of the point. *Every plane of the pencil through any one of the $mn+1$ common rays has m null-points coinciding on d and n null-points coinciding on d'.*

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[2] C. M. Jessop, *A Treatise on the Line Complex*, 1903, p. 259.

The intersection of Σ_{n+1} and Σ'_{m+1} is of degree $(n+1)(m+1)$. Since d' was shown to be an n-fold line on Σ_{n+1} and is clearly a onefold line on Σ'_{m+1}, d' therefore counts n times in the intersection of these two surfaces. Similarly d counts m times in the intersection. Each of the $mn+1$ common rays of the two congruences counts once in the intersection. The parts just enumerated have total degree $n+m+mn+1 = (n+1)(m+1)$. Therefore, *the locus of points whose null-planes have m null-points coinciding in one point and n null-points coinciding in another consists of the lines d and d' and the $mn+1$ common rays of the two congruences.*

Now consider a plane containing d; let it intersect d' in D' and δ'_n in n points N_i. *Every point of the n lines $D'N_i$ is a null-point of this plane*—similarly for planes through d'.

Let point P be on δ_m but not on d. One ρ' is determined but every line from P to d will be a ρ. Therefore, *any point of δ_m or δ'_n not also a point of d or d' has the pencil of planes through the ray of the opposite congruence as null-planes.*

PART II. A SPACE CREMONA INVOLUTION OF ORDER $2n-1$ (n ANY INTEGER)

4. **Definition.** Not every skew curve of order n has a secant meeting it in $n-1$ points, and some have only one such secant, but there are also skew curves of order n that have two $(n-1)$-secant lines. In such case they lie on a quadric surface and have a singly infinite system of such secants. The two selected must be two generators of the same regulus.

Consider a fixed twisted curve δ_n of order n having $n-1$ points in common with a fixed line d and $n-1$ points in common with another fixed line d'. This construction occurs when the two twisted curves δ'_n and δ_m in Part I are identical but lines d and d' remain skew to each other.

A general point P determines a unique line intersecting δ_n once, at A, and d once, at D, and a unique line intersecting δ_n once, at B, and d' once, at D'. We define P', the correspondent of P, to be the intersection of lines AD' and BD. It is an involution.

5. **Equations.** Let d be $x_1=0$, $x_2=0$, and d' be $x_3=0$, $x_4=0$, and the parametric equations of δ_n be

$$x_1 = (as + bt)\prod_{1}^{n-1}(t_is - s_it), \qquad x_2 = (cs + dt)\prod_{1}^{n-1}(t_is - s_it),$$

$$x_3 = (es + ft)\prod_{n}^{2n-2}(t_is - s_it), \qquad x_4 = (gs + ht)\prod_{n}^{2n-2}(t_is - s_it),$$

where (s_i, t_i), $(i=1, 2, \cdots, n-1)$, are values of the parameter at the $n-1$ points of δ_n on d, and for $i=n, n+1, \cdots, 2n-2$ are values of the parameter at the $n-1$ points of δ_n on d'. Then the equations of the involution are

$$x_1' = (ad - bc)\{(ah - bg)x_3 - (af - be)x_4\} \prod_1^{n-1} \alpha_i \prod_1^{n-1} \beta_i,$$

$$x_2' = (ad - bc)\{(ch - dg)x_3 - (cf - de)x_4\} \prod_1^{n-1} \alpha_i \prod_1^{n-1} \beta_i,$$

$$x_3' = (fg - eh)\{(cf - de)x_1 - (af - be)x_2\} \prod_n^{2n-2} \alpha_i \prod_n^{2n-2} \beta_i,$$

$$x_4' = (fg - eh)\{(ch - dg)x_1 - (ah - bg)x_2\} \prod_n^{2n-2} \alpha_i \prod_n^{2n-2} \beta_i,$$

where $\alpha_i \equiv (t_i d + s_i c)x_1 - (t_i b + s_i a)x_2$ and $\beta_i \equiv (t_i h + s_i g)x_3 - (t_i b + s_i e)x_4$. It is of order $2n-1$, n any integer.

6. **The fundamental system.** Line d is an $(n-1)$-fold fundamental line of simple contact. The $n-1$ fixed tangent planes through d are $\alpha_i = 0$, $(i=1, 2, \cdots, n-1)$. The line d is an F-line of the first species whose principal surface consists in the $n-1$ planes $\beta_i = 0$, $(i=1, 2, \cdots, n-1)$.

Line d' is an $(n-1)$-fold F-line of simple contact. The $n-1$ fixed tangent planes through d' are $\beta_i = 0$, $(i=n, n+1, \cdots, 2n-2)$. d' is an F-line of the first species whose P-surface is $\prod_n^{2n-2} \alpha_i = 0$.

Points Δ_i, $(i=1, 2, \cdots, n-1)$, intersections of d with δ_n whose parameters on δ_n are (s_i, t_i), and points Δ_i', $(i=n, n+1, \cdots, 2n-2)$, intersections of d' with δ_n, are isolated n-fold F-points whose P-surfaces are, respectively, the above mentioned fixed tangent planes $\alpha_i = 0$, $(i=1, 2, \cdots, n-1)$, and $\beta_i = 0$, $(i=n, n+1, \cdots, 2n-2)$.

The $(n-1)^2$ lines, each joining a Δ_i to a Δ_i', are simple F-lines without contact. They are F-lines of the second species.

The $(n-1)^2$ lines of intersection of the fixed tangent planes through d with the fixed tangent planes through d' are simple F-lines without contact. They are F-lines of the second species.

7. **Invariant locus.** Every point of the curve δ_n is invariant. Every line that intersects d, d', and δ_n, each once, goes over into itself although it is not pointwise invariant. The locus of these lines is the quadric surface on which d, d', and δ_n lie.

8. **Intersection of two homaloids.** Since they are surfaces of order

$2n-1$, two homaloids intersect in a space curve of order $(2n-1)^2$.

The fixed part of this curve consists in the lines d and d', each counting $n(n-1)$ times, the $(n-1)^2$ lines joining the isolated n-fold F-points of d with those of d', each counting once, and the $(n-1)^2$ lines of intersection of the fixed tangent planes through d with those through d', each counting once. The order of this fixed part is $2n(n-1)+2(n-1)^2$.

The variable part of the curve of intersection is of order $2n-1$ and corresponds to the line of intersection of the two general planes which go over into the pair of homaloids.

UNIVERSITY OF ARIZONA

SYSTEMS OF ORTHOGONAL POLYNOMIALS ON CERTAIN ALGEBRAIC CURVES[1]

FULTON KOEHLER

1. Introduction. A system of orthogonal polynomials in two real variables x and y can be defined with respect to any domain of integration in the xy plane and any nonnegative weight function which has a positive integral over that domain. Although a general study of the formal properties of such systems has been made,[2] the corresponding problem of convergence has been adequately dealt with only for cases which reduce almost immediately to familiar problems in a single variable. The purpose of this paper is to give more substantial examples of systems for which it is possible, by special methods, to present an account of convergence which is fully comparable, at least in some of its main features, with such highly developed theories as those of Fourier and Legendre series.

The proofs, as in the case of series of orthogonal polynomials in one variable with a more or less general weight function, can be made to depend on properties of boundedness. The method used here for obtaining these properties is to establish relationships between the systems in two variables considered and systems in one variable whose properties are well known.

The domain of integration to which primary consideration is to be given is the perimeter of the square whose sides are segments of the lines $x=\pm 1$, $y=\pm 1$. Two other domains will be dealt with briefly in a concluding section.

The square contour will be denoted by the letter C. We shall take the following sequence as the basis for the construction of a set of orthogonal polynomials on C: 1, $x+y$, $x-y$, xy, x^2+y^2, x^2-y^2, x^2y+xy^2, x^2y-xy^2, x^3+y^3, x^3-y^3, $\cdots$, $x^{n-1}y+xy^{n-1}$, $x^{n-1}y-xy^{n-1}$, x^n+y^n, x^n-y^n, $\cdots$. The terms of this sequence have the property that any finite number of them are linearly independent on C; and also, by means of the identity $x^2y^2-x^2-y^2+1\equiv 0$ which holds everywhere on C, any polynomial in x and y can be expressed on C as a linear combination of them. If $\rho(x, y)$ is a function which is positive almost everywhere on C, we can multiply each member of the sequence by $\rho^{1/2}$ and apply the Schmidt process of orthogonalization, thus obtaining an orthogonal system of polynomials with respect to the weight function $\rho(x, y)$. Let these polynomials be normalized and given positive leading coefficients, where the leading coefficient of a polynomial is defined as being the coefficient of the last member of the above sequence which appears in it. We shall denote the polynomials thus uniquely defined by $p_{nk}(x, y)$, where n is the degree of the polynomial, and k is 1, 2, 3, or 4 according as the leading term is the first, second, third, or fourth term of the nth degree in the above sequence.

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[1] Presented to the Society, April 14, 1939.

[2] D. Jackson, (1) *Formal properties of orthogonal polynomials in two variables*, Duke Mathematical Journal, vol. 2 (1936), pp. 423–434. (2) *Orthogonal polynomials on a plane curve*, Duke Mathematical Journal, vol. 3 (1937), pp. 228–236.

2. **Boundedness of the normalized polynomials for some weight functions.** Let $P_n[x; f(x)]$ be the polynomial of the nth degree in the normalized orthogonal system on the interval $(-1, 1)$ corresponding to the weight function $f(x)$.

THEOREM I. *If* $\rho(x, y) = \rho(y, x) = \rho(-x, -y)$ *for all points* (x, y) *of* C, *and if the polynomials* $P_n[x;\ \rho(x, 1)]$ *are uniformly bounded for all values of* n *on every closed interval interior to* $(-1, 1)$, *then the polynomials* $p_{nk}(x, y)$ *are uniformly bounded for all values of* n *and* k *on every closed point set of* C *not including one of the corner points.*

To prove the theorem it is first necessary to point out certain facts regarding integration over the contour C. The integral of any function $g(x, y)$ over C is given by the formula

$$\int_C g(x, y)ds = \int_{-1}^{1} g(x, 1)dx + \int_{-1}^{1} g(x, -1)dx + \int_{-1}^{1} g(1, y)dy + \int_{-1}^{1} g(-1, y)dy.$$

Hence, if $g(x, y) = -g(y, x)$ at all points of C or if $g(x, y) = -g(-x, -y)$ at all points of C, $\int_C g(x, y)ds = 0$. From this it is possible to prove that each polynomial $p_{nk}(x, y)$ is either symmetric or skew-symmetric and that each is either even or odd; that is, $p_{nk}(x, y) \equiv \pm p_{nk}(y, x)$ and $p_{nk}(x, y) \equiv \pm p_{nk}(-x, -y)$.

The polynomial $p_{2n,1}(x, y)$ is therefore a linear combination of the terms $x^{2n-1}y + xy^{2n-1}$, $x^{2n-2} + y^{2n-2}$, $x^{2n-3}y + xy^{2n-3}$, $\cdots$, $x^2 + y^2$, xy, 1. It has the following properties of orthogonality:

$$\int_C \rho(x, y)p_{2n,1}(x, y)(x^{2k} + y^{2k})ds = 0, \quad k = 0, 1, \cdots, n-1,$$

$$\int_C \rho(x, y)p_{2n,1}(x, y)(x^{2k-1}y + xy^{2k-1})ds = 0, \quad k = 1, 2, \cdots, n-1.$$

The integrands in these two equations are even and symmetric and the integral over C of such a function is equal to four times the integral of that function over any one side of C. Hence,

$$\int_{-1}^{1} \rho(x, 1)p_{2n,1}(x, 1)(x^{2k} + 1)dx = 0, \qquad k = 0, 1, \cdots, n - 1,$$

$$\int_{-1}^{1} \rho(x, 1)p_{2n,1}(x, 1)(x^{2k-1} + x)dx = 0, \qquad k = 1, 2, \cdots, n - 1.$$

That is, $p_{2n,1}(x, 1)$ is a polynomial in x of degree $2n-1$ which is orthogonal over $(-1, 1)$ with $\rho(x, 1)$ as weight function to $1, x, x^2, \cdots, x^{2n-2}$. Hence,

$$p_{2n,1}(x, 1) = (1/2)P_{2n-1}[x; \rho(x, 1)].$$

By similar methods it can be shown that $p_{2n,2}(x, 1)/(x^2-1)$ is a polynomial of degree $2n-3$ which is orthogonal over $(-1, 1)$ with $(1-x^2)^2\rho(x, 1)$ as weight function to $1, x, x^2, \cdots, x^{2n-4}$; and so

$$p_{2n,2}(x, 1) = (1/2)(x^2 - 1)P_{2n-3}[x; (1 - x^2)^2\rho(x, 1)].$$

The corresponding formulas for the other two polynomials of even degree and the four of odd degree are

$$\begin{aligned}
p_{2n,3}(x, 1) &= (1/2)P_{2n}[x; \rho(x, 1)],\\
p_{2n,4}(x, 1) &= (1/2)(x^2 - 1)P_{2n-2}[x; (1 - x^2)^2\rho(x, 1)],\\
p_{2n+1,1}(x, 1) &= (1/2)(x + 1)P_{2n-1}[x; (1 + x)^2\rho(x, 1)],\\
p_{2n+1,2}(x, 1) &= (1/2)(x - 1)P_{2n-1}[x; (1 - x)^2\rho(x, 1)],\\
p_{2n+1,3}(x, 1) &= (1/2)(x + 1)P_{2n}[x; (1 + x)^2\rho(x, 1)],\\
p_{2n+1,4}(x, 1) &= (1/2)(x - 1)P_{2n}[x; (1 - x)^2\rho(x, 1)].
\end{aligned}$$

It is well known that if the polynomials $P_n[x; f(x)]$ are uniformly bounded on every closed interval interior to $(-1, 1)$, then the polynomials $P_n[x; \pi(x)f(x)]$ are likewise bounded provided $\pi(x)$ is a polynomial which does not vanish on the interior of the interval $(-1, 1)$. Hence, the above formulas, together with the fact that each polynomial $p_{nk}(x, y)$ is symmetric or skew-symmetric and even or odd, establish the theorem.

It is of interest to note that if $\rho(x, y) = |y+x|^{\alpha}|y-x|^{\beta}$ where $\alpha > -1$, $\beta > -1$, then the polynomials $p_{nk}(x, 1)$ are expressible in terms of Jacobi polynomials.

3. **Convergence theorems.** Let $f(x, y)$ be any function of x and y

which is integrable in the sense of Lebesgue over C and let us consider the series

$$(1)\qquad \sum_{n=0}^{\infty}\sum_{k=1}^{4} a_{nk}p_{nk}(x, y)$$

where $a_{nk}=\int_C\rho(u, v)f(u, v)p_{nk}(u, v)ds$ and where (x, y) is any point of C. (For convenience of notation we define p_{02}, p_{03}, p_{04}, p_{13}, p_{14}, and p_{24} to be identically zero.) Let $K_n(x, y, u, v)=\sum_{m=0}^{n}\sum_{k=1}^{4}p_{mk}(x, y)p_{mk}(u, v)$, and let $S_n(x, y)$ be the partial sum of (1) through terms of the nth degree. Then

$$(2)\quad S_n(x, y) - f(x, y) = \int_C \rho(u, v)[f(u, v) - f(x, y)]K_n(x, y, u, v)ds.$$

(In formulas where x, y, u, and v appear under the integral sign u and v are to be taken as the variables of integration.) By using a formula[3] for $K_n(x, y, u, v)$ analogous to the Christoffel-Darboux identity, the difference $S_n(x, y)-f(x, y)$ can be expressed as a sum of at most thirty-two terms of the form

$$(3)\qquad kp_{jh}(x, y)\int_C \rho(u, v)\frac{f(u, v) - f(x, y)}{(Au + Bv) - (Ax + By)}p_{mk}(u, v)ds$$

where A and B are any arbitrary constants, where k is a constant numerically less that $|A|+|B|$, and where m and j take on the values n and $n+1$. The problem of convergence at a specified point thus reduces to proving that an expression of the form (3) approaches zero as n becomes infinite.

LEMMA 1. *If the weight function $\rho(x, y)$ is of such a nature that the corresponding polynomials are uniformly bounded on every closed point set of C not containing a corner point and if $\phi(x, y)$ is a function such that $\rho\phi$ is integrable over C and $\rho\phi^2$ is integrable over some neighborhood on C of each corner point, then*

$$\lim_{n\to\infty}\int_C \rho(u, v)\phi(u, v)p_{nk}(u, v)ds = 0, \qquad k = 1, 2, 3, 4.$$

The proof of this lemma is omitted since it can be easily constructed from proofs already known.[4] On the basis of this lemma we can now prove the following assertion:

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[3] D. Jackson, (1), p. 433.

[4] See, for example, D. Jackson, *Series of orthogonal polynomials*, Annals of Mathematics, (2), vol. 34 (1933), pp. 536–537.

THEOREM II. *Assume*

(a) *the weight function* $\rho(x, y)$ *is such that the corresponding polynomials* $p_{nk}(x, y)$ *are uniformly bounded on every closed point set of* C *not containing a corner point;*

(b) (x, y) *is a point of* C, *not a corner;*

(c) *the function* ρf *is integrable over* C *and* ρf^2 *is integrable over a neighborhood on* C *of each corner point;*

(d) *the function*

$$\rho(u, v)\,\frac{f(u, v) - f(x, y)}{(u - x) + (v - y)}$$

is integrable over some neighborhood on C *of the point* (x, y).

Then the series (1) *converges at the point* (x, y) *to the value* $f(x, y)$.

Let us first make the additional hypothesis that there is a point (x', y') on C on the interior of a side different from the one containing (x, y) such that the function

$$\rho(u, v)\,\frac{f(u, v) - f(x, y)}{(u - x') + (v - y')}$$

is integrable over some neighborhood on C of the point (x', y'). Determine A and B so that the straight line $A(u-x)+B(v-y)=0$ in the uv plane passes through the point (x', y'). Then in some neighborhood on C of (x, y) the expression $A(u-x)+B(v-y)$ is a non-vanishing constant multiple of $u-x+v-y$ and in some neighborhood of (x', y'), of $u-x'+v-y'$. If we let

$$\phi(u, v) = \frac{f(u, v) - f(x, y)}{A(u - x) + B(v - y)},$$

then $\rho\phi$ is integrable over C and $\rho\phi^2$ is integrable over a neighborhood of each corner point; hence, by Lemma 1, the entire expression (3) approaches zero as n becomes infinite, which insures the convergence of (1) to the value $f(x, y)$.

To prove the theorem in the general case let us choose two distinct points (x', y') and (x'', y''), each on the interior of a side of C different from the side containing (x, y), and let us define a function $g(u, v)$ to be equal to $f(u, v)$ throughout neighborhoods of (x, y) and (x', y'), equal to $g(x, y) = f(x, y)$ throughout a neighborhood of (x'', y''), and equal to zero elsewhere on C. Then the series of form (1) corresponding to the functions $f-g$ and g will converge at the point (x, y) to the values $f(x, y)-g(x, y)$ and $g(x, y)$ respectively. Hence, the series for f will converge at the point (x, y) to the value $f(x, y)$.

COROLLARY. *If the function f is such that* ρf *is integrable over C and* ρf^2 *is integrable over a neighborhood of each corner point, the behavior of the series* (1) *at a point* (x, y) *on the interior of a side of C depends only on the behavior of the function* $f(u, v)$ *in the neighborhood of that point.*

A study of the uniform convergence of (1) to a continuous function $f(x, y)$ can be made with the same methods that have been used in a study of Legendre series.[5] In applying these methods it is first necessary to consider the degree of polynomial approximation obtainable over the contour C to a function which is continuous on C. By adaptation of existing theorems[6] with regard to polynomial approximation on an interval, the following conclusion is obtained:

THEOREM III. *If the polynomials* $p_{nk}(x, y)$ *are uniformly bounded and the weight function* $\rho(x, y)$ *bounded on every closed point set of C not including one of the corner points and if* $f(x, y)$ *is a function such that* $f(x, \pm 1)$ *and* $f(\pm 1, x)$ *have continuous pth derivatives* $(p \geqq 0)$ *on the interval* $(-1, 1)$ *with moduli of continuity at most equal to* $\omega(\delta)$; *then, for every* $\eta > 0$,

$$|S_n(x, y) - f(x, y)| \leqq \frac{K \log n}{n^p} \omega(1/n)$$

for all (x, y) *on C not within a distance* η *of any corner point and for all* $n > 1$, *the constant K being independent of n and of* (x, y) *but dependent on* η.

4. **Other contours.** The methods used in the convergence proofs of the preceding section are applicable to orthogonal systems over a highly general contour provided the normalized polynomials of the system possess the requisite property of boundedness on that contour. There has been as yet, however, no general method devised for establishing the latter property. Two other ranges of integration for which the result can be established by reasoning similar to that already used are a pair of bisecting line segments and a pair of concentric circles.

An orthogonal system over the first of these domains will have two polynomials of the nth degree for $n \geqq 1$. If the range is taken as consisting of the portions of the x and y axes contained in the intervals $-1 \leqq x \leqq 1$, $-1 \leqq y \leqq 1$, the two polynomials of nth degree of a normalized, orthogonal system corresponding to a symmetric weight function $\rho(x, y)$ are given by the formulas

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[5] See, for example, D. Jackson, *The Theory of Approximation*, American Mathematical Society Colloquium Publications, vol. 11, New York, 1930, pp. 25–32.

[6] D. Jackson, loc. cit., pp. 13–18.

$$p_{n1}(x, 0) = 2^{-1/2}P_n[x; \rho(x, 0)], \qquad p_{n2}(x, 0) = 2^{-1/2}xP_{n-1}[x; x^2\rho(x, 0)],$$
$$p_{nk}(x, y) = (-1)^{k-1}p_{nk}(y, x).$$

The convergence theorems for this system are similar to those of the last section with the points $(\pm 1, 0), (0, \pm 1), (0, 0)$ appearing as exceptional points analogous to the corner points of the square.

The contour made up of two concentric circles is an example of a contour which has no singular points. An orthogonal system over this contour will have four polynomials of the nth degree for $n \geqq 3$. If the two circles are taken with center at the origin and radii a and b, the polynomials of a normalized system corresponding to the weight function 1 are represented by the formulas

$$p_{nk}(a \cos \theta, a \sin \theta) = \alpha_{nk1} \cos n\theta + \alpha_{nk2} \cos (n-2)\theta,$$
$$p_{nk}(b \cos \theta, b \sin \theta) = \beta_{nk1} \cos n\theta + \beta_{nk2} \cos (n-2)\theta, \quad k = 1, 3.$$

The formulas for p_{n2} and p_{n4} are of the same form with the cosines replaced by sines. The constants α_{nkm} and β_{nkm} involved in the formulas are bounded for all values of n, k, and m; so the polynomials themselves are uniformly bounded over the entire contour. The convergence theorems for this system of polynomials can thus be formulated without reference to any exceptional points.

THE UNIVERSITY OF MINNESOTA

ON THE CONVERSE OF THE TRANSITIVITY OF MODULARITY

Y. K. WONG

E. H. Moore's theorem on the transitivity of modularity is as follows: Consider the basis[1] $\mathfrak{A}$, $\mathfrak{P}$, ϵ; if a positive hermitian matrix ϵ_0 is modular as to $\epsilon\ \epsilon$, then every vector which is modular as to ϵ_0 is modular[2] as to ϵ (that is, $\mathfrak{M}_{\epsilon_0} \subset \mathfrak{M}_{\epsilon}$).

In his doctoral thesis, the author establishes the converse of the preceding theorem as a consequence of the Hellinger-Toeplitz theorem.[3] In this note, we give a new proof for the converse of the transitivity of modularity, and then deduce the generalized Hellinger-Toeplitz theorem as a corollary. The converse of the transitivity of modularity is, therefore, equivalent to the Hellinger-Toeplitz theorem. We also establish the converse of the transitivity of modularity for matrices, and a theorem on the transitivity of accordance and finiteness.

THEOREM I. *Consider the basis* $\mathfrak{A}$, $\mathfrak{P}$, ϵ; *and let* ϵ_0 *be a positive hermitian matrix. Then the following assertions are equivalent*:

(1) *every vector* μ_0 *modular as to* ϵ_0 *is modular as to* ϵ;

(2) ϵ_0 *is modular as to* $\epsilon\ \epsilon_0$;

(3) ϵ_0 *is modular as to* $\epsilon\ \epsilon$.

If one of the preceding conditions is satisfied, the modulus of ϵ_0 *as to* $\epsilon\ \epsilon$ *is equal to the norm of* ϵ_0 *as to* $\epsilon\ \epsilon_0$.

In the course of demonstration, we let $\mathfrak{M}_0$ denote the space of vectors μ_0 modular as to ϵ_0; J_0, the integration process based on ϵ_0; and M_0, the modulus as to ϵ_0. Similar interpretations are given to the symbols $\mathfrak{M}$, J, M, for the base matrix ϵ. A vector which is finite as to ϵ is denoted by β.

If every μ_0 is modular as to ϵ, the matrix ϵ_0 is of type $\mathfrak{M}_0\overline{\mathfrak{M}}$. Then $J\epsilon_0\beta$ is in $\mathfrak{M}_0$ for every β, and $J_0(J\bar{\beta}\epsilon_0)\mu_0 = J\bar{\beta}J_0\epsilon_0\mu_0 = J\bar{\beta}\mu_0$ for every pair β, μ_0. Consequently, for every β, $M_0J\epsilon_0\beta$ is equal to the least upper bound of $|J\bar{\beta}\mu_0|$ for all μ_0 such that $M_0\mu_0 \leqq 1$, by part (2) of Theorem (41.9) in G.A. Similarly, for every μ_0, which is modular as to ϵ by hypothesis, $M\mu_0$ is equal to the least upper bound of $|J\bar{\beta}\mu_0|$

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[1] E. H. Moore, *General Analysis* (G.A. for abbreviation), Part I, p. 4, and Part II, p. 84.

[2] Theorem (46.4), part (1) in G.A., II, p. 137.

[3] *Spaces associated with non-modular matrices with applications to reciprocals*, Chicago thesis, 1931, pp. 3–9. The same proof is given in G.A., II, p. 193.

for all β such that $M\beta \leqq 1$. If the class $\mathfrak{L}$ is identified as the class of vectors $\bar{\beta}$ such that $M\beta \leqq 1$, and F_l, on $\mathfrak{M}_0$ to $\mathfrak{N}$, is defined to be $(|J\bar{\beta}\mu_0| \, |\mu_0)$ for every $l = \bar{\beta}$, then by Theorem (53.55) in G.A.,[4] the upper bound of $M_0 J \epsilon_0 \beta$, for all β finite as to ϵ such that $M\beta \leqq 1$, is finite. By Theorem (46.85) in G.A., ϵ_0 is modular as to $\epsilon_0 \ \epsilon$. Since ϵ_0 is hermitian, ϵ_0 is also modular[5] as to $\epsilon \ \epsilon_0$.

When condition (2) is true, then condition (3) is secured by a simple application[6] of the composition of modularity[7] to $\epsilon_0 = J_0 \epsilon_0 \epsilon_0$. That the last condition implies the first is proved in Theorem (46.4) of Moore's G.A.

From $\epsilon_0 = J_0 \epsilon_0 \epsilon_0$ and part (2) of Theorem (46.9) in G.A., we have $N_{\epsilon_0 \epsilon} \epsilon_0 = M_{\epsilon\epsilon} J_0 \epsilon_0 \epsilon_0 = M_{\epsilon\epsilon} \epsilon_0$. This completes the proof.

The hypothesis of the preceding theorem is assumed for the following corollary:

COROLLARY. *Let $\mathfrak{M}_{0*}$ consist of all μ_0 whose moduli as to ϵ_0 are bounded by a fixed constant. If $\mathfrak{M}_{0*}$ is a subset of $\mathfrak{M}$, then the moduli as to ϵ of all vectors in $\mathfrak{M}_{0*}$ are also bounded.*

We may assume, without losing generality, that the moduli of all vectors in $\mathfrak{M}_{0*}$ are at most unity. Since the spaces $\mathfrak{M}_0$ and $\mathfrak{M}$ are linear, the condition that every μ_0 for which $M_0 \mu_0 \leqq 1$ is modular as to ϵ is equivalent to condition (1) in the preceding theorem. Consequently ϵ_0 is modular as to $\epsilon \ \epsilon_0$. The equation $\mu_0 = J_0 \epsilon_0 \mu_0$ gives, by Theorem (46.7) in G.A., that $M\mu_0 \leqq M_{\epsilon\epsilon_0} \epsilon_0$ whenever $M_0 \mu_0 \leqq 1$.

THEOREM II. *Consider the basis $\mathfrak{A}$, $\mathfrak{P}^1$, $\mathfrak{P}^2$, ϵ^1, ϵ^2; and let ϵ_0^1, ϵ_0^2 be two positive hermitian matrices. Then the following assertions are equivalent:*

(1) *every matrix κ^{12} modular as to $\epsilon_0^1 \ \epsilon_0^2$ is of type $\mathfrak{M}^1 \overline{\mathfrak{M}}^2$;*
(2) *ϵ_0^1 is modular as to $\epsilon^1 \ \epsilon^1$, and ϵ_0^2 is modular as to $\epsilon^2 \ \epsilon^2$;*
(3) *every matrix κ^{12} modular as to $\epsilon_0^1 \ \epsilon_0^2$ is modular as to $\epsilon^1 \ \epsilon^2$.*

For the demonstration of the theorem, we shall show that (1)→(2)→(3)→(1). The second implication is proved in part (2) of Theorem (46.4) in G.A. The last implication follows from the fact that every matrix κ^{12} modular as to $\epsilon^1 \ \epsilon^2$ is of type $\mathfrak{M}^1 \overline{\mathfrak{M}}^2$. To show the first implication, consider any μ_0^1, $\mu_0^2 \neq 0^2$ which are modular as to ϵ_0^1, ϵ_0^2 respectively, Theorem (47.2) in G.A. shows that $\mu_0^1\bar{\mu}_0^2$ is modular as to $\epsilon^1\ \epsilon^2$, and hence by hypothesis, $\mu_0^1\bar{\mu}_0^2$ is of type $\mathfrak{M}^1\overline{\mathfrak{M}}^2$. Since $\mu^2 \neq 0^2$, let $a \equiv \mu^2(p^2) \neq 0$. Then $\mu_0^1 \cdot a$, and hence μ_0^1, is modular as to ϵ^1. This proves that every μ_0^1 modular as to ϵ_0^1 is modular as to ϵ^1. By Theorem I, ϵ_0^1 is modular as to $\epsilon^1\ \epsilon^1$. Similarly, we prove that ϵ_0^2 is modular as to $\epsilon^2\ \epsilon^2$. The proof was suggested by Dr. Coral.

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[4] See also Hildebrandt, *On uniform limitedness of sets of functional operations*, this Bulletin, vol. 29 (1923), pp. 309–315; Fréchet, *Sur les fonctionelles bilinéaires*, Transactions of this Society, vol. 16 (1915), pp. 217–218.

[5] By a similar reasoning, we may, of course, deduce the Hellinger-Toeplitz theorem as a consequence of Theorem (53.55).

[6] See the author's thesis, loc. cit., p. 8, or Moore, G.A., II, p. 193.

[7] Moore, G.A., II, p. 144.

THEOREM III. (Generalized Hellinger-Toeplitz theorem.) *Consider the basis* $\mathfrak{A}$, $\mathfrak{P}^1$, $\mathfrak{P}^2$, ϵ^1, ϵ^2. *A matrix* κ^{12} *is modular as to* $\epsilon^1\ \epsilon^2$ *if and only if* κ^{12} *is by rows of* $\overline{\mathfrak{M}}^2$ *and* $J^2\kappa^{12}\mu^2$ *is modular as to* ϵ^1 *for every* μ^2.

To prove the theorem, we make use of the fact that κ^{12} is modular as to $\epsilon^1\ \epsilon^2$ if and only if the following condition holds:

(M) κ^{12} *is by rows of* $\overline{\mathfrak{M}}^2$, *and* $J^2\kappa^{12}\kappa^{*21}$ *is modular as to* $\epsilon^1\ \epsilon^1$.

This is Theorem (46.9) in G.A., with the omission of the redundant condition that κ^{12} is by columns accordant as to ϵ^1. (For when κ^{12} satisfies the conditions (M), κ^{12} is by columns A^1. To prove this, we note that $J^2\kappa^{12}\kappa^{*21}$ is A^{11} by Theorem (46.65) in G.A. Consequently, when $S_\sigma^1\epsilon^1\alpha^1 = 0^1$, then $J^2(S_\sigma^1\bar{\alpha}^1\kappa^{12},\ S_\sigma^1\kappa^{*21}\alpha^1) = S_\sigma^1 S_\sigma^1\bar{\alpha}^1 J^2\kappa^{12}\kappa^{*21}\alpha^1 = 0$, which implies that $S_\sigma^1\bar{\alpha}^1\kappa^{12} = 0^2$, since J^2 is proper. Thus κ^{12} is by columns A^1.) Consequently, it suffices to prove the following statement: *When* κ^{12} *is by rows of* $\overline{\mathfrak{M}}^2$, *the matrix* $J^2\kappa^{12}\kappa^{*21}$ *is modular as to* $\epsilon^1\ \epsilon^1$ *if and only if* $J^2\kappa^{21}\mu^2$ *is modular as to* ϵ^1 *for every* μ^2.

Using the notation introduced by E. H. Moore in his study of generalized Fourier theory, we denote the positive hermitian matrix $J^2\kappa^{21}\kappa^{*21}$ by $\epsilon_{\kappa *}^1$. It was shown by Moore that the space of vectors modular as to $\epsilon_{\kappa *}^1$ is equal[8] to the space of vectors $J^2\kappa^{12}\mu^2$ for all μ^2 in $\mathfrak{M}^2$. When κ^{12} is assumed to be by rows of $\overline{\mathfrak{M}}^2$, the assertion that $J^2\kappa^{12}\mu^2$ is modular as to ϵ^1 for every μ^2 is equivalent to the assertion that every vector modular as to $\epsilon_{\kappa *}^1$ is modular as to ϵ^1. By Theorem I, the latter assertion is valid if and only if $\epsilon_{\kappa *}^1$ is modular as to $\epsilon^1\ \epsilon^1$. This proves the theorem.

The basis stated in the preceding theorem is assumed for the following corollary:

COROLLARY. *Suppose that* κ^{12} *is by rows of* $\overline{\mathfrak{M}}^2$. *Then* κ^{12} *is modular as to* $\epsilon^1\ \epsilon^2$ *if and only if every vector modular as to* $J^2\kappa^{12}\kappa^{*21}$ *is modular as to* ϵ^1.

The transitivity for accordance and finiteness is stated in the following theorem:

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[8] Moore, G. A., I, p. 22.

THEOREM IV. *Consider the basis* $\mathfrak{A}$, $\mathfrak{P}$, ϵ *and let* ϵ_0 *be a positive hermitian matrix. Then*

(a) *every vector accordant as to* ϵ_0 *is accordant as to* ϵ *if and only if* ϵ_0 *is accordant as to* ϵ ϵ;

(b) *every vector finite as to* ϵ_0 *is finite as to* ϵ *if and only if* ϵ_0 *is of type* $F\overline{F}$.

In part (a), if every vector accordant as to ϵ_0 is accordant as to ϵ, then ϵ_0, being of type $A_0\overline{A}_0$, is of type $A\overline{A}$. By Theorem (46.5) in G.A., ϵ_0 is accordant as to ϵ ϵ. Conversely every vector ξ accordant as to ϵ_0 satisfies the relation $\xi = J_0\epsilon_0\xi = L_\sigma J_0\epsilon_0\xi_\sigma$. Now $J_0\epsilon_0\xi_\sigma$, being a finite (right) linear combination of the columns of ϵ_0, is a vector accordant as to ϵ for every σ. By Theorem (48.2) in G.A., ξ is accordant as to ϵ. Part (b) is an immediate consequence of the definition of finiteness.

THE UNIVERSITY OF CHICAGO

ON THE EXTENSION OF A FUNCTIONAL INEQUALITY OF S. BERNSTEIN TO NON-ANALYTIC FUNCTIONS[1]

R. J. DUFFIN AND A. C. SCHAEFFER

We wish to demonstrate here the following elementary inequality of the differential calculus. If a function[2] satisfies the conditions $\{f(x)\}^2 \leq 1$ and $\{f^{(n)}(x)\}^2 + \{f^{(n-1)}(x)\}^2 \leq 1$ for all x and for some positive integer n, then the latter inequality is valid also when n is replaced by any smaller positive integer.

That such an inequality might be true is suggested by the validity of a similar but more specialized inequality concerning trigonometric polynomials.[3] Thus, if $P(x) = \sum_0^N \{a_\nu \cos(\nu x/N) + b_\nu \sin(\nu x/N)\}$ and if $\{P(x)\}^2 \leq 1$ for all x, it has been proved that $\{P^{(k)}(x)\}^2 + \{P^{(k-1)}(x)\}^2 \leq 1$, $(k = 1, 2, 3, \cdots)$. This theorem, a refinement of a theorem of S. Bernstein, has been proved by several different methods,[4] and generalizations have been given which prove that the inequality is true for a wider class of analytic functions. It will be shown here that this theorem is a rapid deduction from the elementary inequality given above. Moreover, this method of proof serves to distinguish those features of Bernstein's theorem arising from the characteristic properties of trigonometric polynomials from those which are merely properties of the differential coefficient.

The second part of this paper is concerned with finding the functions which cause the inequality to become an equality at some point. For example, if $\{f^{(3)}(x)\}^2 + \{f^{(2)}(x)\}^2 \leq 1$ and $\{f(x)\}^2 \leq 1$, we find that the equality $f'(0) = 1$ necessitates that $f(x) \equiv \sin x$. It is to be noted that the behavior of this system of differential inequalities is similar to the behavior of a differential equation because the assignment of certain initial values gives rise to a unique solution. It is possible to construct other examples of differential inequalities with this stability property.

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[1] Presented to the Society, September 6, 1938, under the title *Extension of a theorem of S. Bernstein to non-analytic functions.*

[2] In stating that an inequality involving $f^{(n)}(x)$ is satisfied in an interval we imply that the derivatives of all orders up to and including the nth exist in the same interval. All functions and constants appearing in this paper are considered real unless the contrary is explicitly stated.

[3] In formulating such a comparison, we have taken a hint from a paper by Oystein Ore, *On functions with bounded derivatives*, Transactions of this Society, vol. 43 (1938), pp. 321–326. Ore introduces an "extension principle" which indicates how certain theorems concerning ordinary algebraic polynomials may be modified in order to be applicable to arbitrary functions.

[4] G. Szegö, Schriften der Königsberger gelehrten Gesellschaft, 5th year, no. 4 (1928), pp. 59–70. J. van der Corput and G. Schaake, Compositio Mathematica, vol. 2 (1935), pp. 321–361; vol. 3 (1936), p. 128. R. Boas, Transactions of this Society, vol. 40 (1936), pp. 287–308. R. Duffin and A. Schaeffer, this Bulletin, vol. 43 (1937), pp. 554–556; vol. 44 (1938), pp. 236–240.

The method of proof employs only fundamental notions. Essentially all that is needed is the mean value theorem and Weierstrass' theorem on the maximum of a continuous function.

THEOREM A. *If $f(x)$ is a function which for some integer n and for all x satisfies the conditions*

$$\{f(x)\}^2 \leqq 1, \tag{a}$$

$$\{f^{(n)}(x)\}^2 + \{f^{(n-1)}(x)\}^2 \leqq 1, \tag{b}$$

then for $k=1, 2, 3, \cdots, n$ and for all x

$$\{f^{(k)}(x)\}^2 + \{f^{(k-1)}(x)\}^2 \leqq 1. \tag{c}$$

PROOF. Considering first the case $n=2$, we have by hypothesis $\{f(x)\}^2 \leqq 1$ and $\{f'(x)\}^2 + \{f''(x)\}^2 \leqq 1$. It is to be shown that the function $\phi(x) \equiv \{f(x)\}^2 + \{f'(x)\}^2$ is not greater than 1. Suppose, on the contrary, that at some point x_1, $\phi(x_1) = 1 + 2\delta$, $\delta > 0$. If x is any point to the right of x_1, then by the mean value theorem a point x_2 exists such that

$$f'(x_2) = \{f(x) - f(x_1)\}/(x - x_1), \qquad x_1 < x_2 < x.$$

Taking $x - x_1 = 2/(\delta)^{1/2}$, we see that $|f'(x_2)| \leqq (\delta)^{1/2}$. Hence $\phi(x_2) \leqq 1 + \{f'(x_2)\}^2 \leqq 1 + \delta$. The same method is available to show that there is a point to the left of x_1 where $\phi(x) \leqq 1 + \delta$. Thus, $\phi(x)$ is greater at x_1 than it is at certain points to the left and to the right; it must, therefore, have a maximum in the interior of the interval defined by these points. At this maximum point, say x_0, we have $\phi(x_0) \geqq 1 + 2\delta$; and the derivative is equal to zero:

$$2f'(x_0)\{f(x_0) + f''(x_0)\} = 0.$$

But, if the first factor is zero, then $\phi(x_0) = \{f(x_0)\}^2 \leqq 1$, while if the second factor is zero, then

$$\phi(x_0) = \{-f''(x_0)\}^2 + \{f'(x_0)\}^2 \leqq 1.$$

This contradiction proves the theorem for $n=2$.

For n greater than 2 we proceed by induction. Thus, suppose the theorem is true for $n=m-1$ and consider the case $n=m$. First, in order to show that $f'(x)$ is bounded uniformly in x, we obtain the crude estimate $|f'(x_0)| \leqq 5$ for an arbitrary point x_0. By the mean value theorem

$$f(x_0 + 1) - f(x_0) = f'(x_0 + h), \qquad 0 < h < 1.$$

Because f is bounded by 1, this gives, for some h between 0 and 1,

$$|f'(x_0 + h)| \leqq 2. \tag{1}$$

Again, by the mean value theorem, if x is any point and h is the same constant appearing in (1), then for some point x_1

$$f^{(m-2)}(x + h) - f^{(m-2)}(x) = hf^{(m-1)}(x_1). \tag{2}$$

But, by hypothesis $f^{(m-1)}$ is bounded by 1; so, using (2), we obtain

$$\{f^{(m-2)}(x + h) - f^{(m-2)}(x)\}^2 + \{f^{(m-1)}(x + h) - f^{(m-1)}(x)\}^2 \leqq 9.$$

Note that $\{f(x+h)-f(x)\}^2<9$ follows from hypothesis (a), so the theorem is applicable to the function $\{f(x+h)-f(x)\}/3$ in the case $n=m-1$ for which it has been supposed true. As a consequence

$$\left\{\frac{f'(x + h) - f'(x)}{3}\right\}^2 + \left\{\frac{f(x + h) - f(x)}{3}\right\}^2 \leqq 1.$$

In particular the last relation gives for the point x_0 the inequality

$$|f'(x_0 + h) - f'(x_0)| \leqq 3. \tag{3}$$

Combining (1) and (3) we obtain the desired bound $|f'(x_0)| \leqq 5$.

Let β be the least upper bound of $|f'(x)|$ for all values of x; we have just shown that β is finite. If $\beta > 1$, the function $\beta^{-1}f'(x)$ satisfies the conditions of the theorem in the case $n=m-1$ and consequently

$$\{\beta^{-1}f''(x)\}^2 + \{\beta^{-1}f'(x)\}^2 \leqq 1. \tag{4}$$

Let $F(x)=\beta^{-1}f(x)+\gamma(1-\beta^{-1})$ where $\gamma=+1$ or -1 and is to be determined later. Because $\{f(x)\}^2 \leqq 1$, clearly $\{F(x)\}^2 \leqq 1$; and by (4), $\{F''(x)\}^2+\{F'(x)\}^2 \leqq 1$. Applying the theorem to the function $F(x)$ in the case $n=2$, we have $\{F'(x)\}^2+\{F(x)\}^2 \leqq 1$ or

$$\{\beta^{-1}f'(x)\}^2 + \{\beta^{-1}f(x)\}^2 + 2\gamma\beta^{-1}(1 - \beta^{-1})f(x) + (1 - \beta^{-1})^2 \leqq 1.$$

By the definition of β there exist points for which the first term on the left is arbitrarily close to 1. And for fixed x the term containing γ can be made nonnegative since γ is either $+1$ or -1. Thus, the least upper bound of the left-hand side is not smaller than $1+(1-\beta^{-1})^2$; and because this can be no larger than 1, it follows that $\beta \leqq 1$.

The problem is by the above considerations reduced to the case $\{f'(x)\}^2 \leqq 1$. Applying the theorem to the function $f'(x)$ with $n = m-1$, we have for $k = 2, 3, 4, \cdots, m-1, m$

$$\text{(c)} \qquad \{f^{(k)}(x)\}^2 + \{f^{(k-1)}(x)\}^2 \leqq 1.$$

But, now we may apply the theorem to $f(x)$ with $n = 2$, and this shows that (c) is true for $k = 1$ also.[5]

We wish now to determine the class of functions which make relation (c) of Theorem A an equality at some point and for some value of k. The following lemma is of central importance for this investigation.

LEMMA. *If the function $f(x)$ satisfies in the interval $0 \leqq \alpha \leqq x \leqq \pi/2$ the conditions*

$$\text{(a)} \qquad \{f''(x)\}^2 + \{f'(x)\}^2 \leqq 1,$$

$$\text{(b)} \qquad f(\alpha) = \sin\alpha, \qquad f'(\alpha) = \cos\alpha,$$

$$\text{(c)} \qquad f(\pi/2) \leqq 1,$$

then in the same interval

$$\text{(d)} \qquad f(x) \equiv \sin x.$$

PROOF. It is convenient to introduce two auxiliary functions $\omega(x)$ and $\psi(x)$, defined by the relations

$$\omega(x) = f(x) - \sin x,$$

$$2\psi(x) = 1 + \{\omega''(x)\}^2 + \{\omega'(x)\}^2 - \{f''(x)\}^2 - \{f'(x)\}^2.$$

By condition (b) $\omega(\alpha) = \omega'(\alpha) = 0$, and by (a) $\psi(x)$ is nonnegative in the interval under consideration. Writing $f(x) = \omega(x) + \sin x$ and substituting in the expression for $2\psi(x)$ gives

$$\psi(x) = \sin x\, \omega''(x) - \cos x\, \omega'(x);$$

and if $x \neq 0$, this relation may be put in the form

$$\text{(1)} \qquad \frac{d}{dx}(\omega'(x)/\sin x) = \psi(x)/\sin^2 x.$$

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[5] The following three modifications of Theorem A are of some interest: (1) The assumption that $f^{(n)}(x)$ exists may be replaced by weaker conditions. For example, in condition (b), $\{f^{(n)}(x)\}^2$ may be interpreted as $\limsup_{h\to\pm 0}\{(f^{(n-1)}(x+h) - f^{(n-1)}(x))/h\}^2$. (2) The theorem may be stated for a finite interval provided the inequalities (c) are given to be satisfied at the end points of the interval for every value of k. (3) The theorem is valid for complex functions $f(x)$ provided $|f(x)| \leqq 1$ and relations of the form $\{f^{(k)}(x)\}^2 + \{f^{(k-1)}(x)\}^2 \leqq 1$ are replaced by relations of the form $|f^{(k)}(x)\cos\alpha + f^{(k-1)}(x)\sin\alpha| \leqq 1$ for all real x and α.

Considering first the case $\alpha > 0$, we have by the mean value theorem if $0 < \alpha < x < \pi/2$

$$\omega'(x)/\sin x - \omega'(\alpha)/\sin \alpha = (x - \alpha)\psi(x_1)/\sin^2 x_1, \qquad \alpha < x_1 < x.$$

But, $\omega'(\alpha) = 0$ and $\psi(x) \geqq 0$; so

$$\omega'(x) \geqq 0, \qquad \alpha \leqq x \leqq \pi/2, \tag{2}$$

which implies, by the mean value theorem, that $\omega(x)$ is non-decreasing in this interval. Because ω vanishes at $x = \alpha$ and because, by (c), ω cannot be greater than zero at $x = \pi/2$, it follows from (2) that $\omega(x)$ vanishes identically in the interval $(\alpha, \pi/2)$.

Special attention must be given the case $\alpha = 0$. If $x_0 < x$ are any two points in the interior of the interval $(0, \pi/2)$, by the mean value theorem the relation (1) gives

$$\omega'(x)/\sin x - \omega'(x_0)/\sin x_0 = (x - x_0)\psi(x_1)/\sin^2 x_1,$$
$$0 < x_0 < x_1 < x < \pi/2.$$

Consider the limiting form of this relation as x_0 approaches zero for fixed x. The second term on the left approaches $\omega''(0)$, and the expression on the right, being always nonnegative, must have a nonnegative limit. By conditions (a) and (b) it is clear that $f''(\alpha) = 0$ if $\alpha = 0$, and this implies that $\omega''(0) = 0$. Thus relation (2) is valid also for the case $\alpha = 0$, and $\omega(x)$ must vanish identically in the interval $(0, \pi/2)$; this completes the proof.

THEOREM B. *If $f(x)$ is a function which for all x satisfies the conditions*

$$\{f(x)\}^2 \leqq 1, \tag{a}$$

$$\{f''(x)\}^2 + \{f'(x)\}^2 \leqq 1, \tag{b}$$

and if at some point x_1

$$\{f'(x_1)\}^2 + \{f(x_1)\}^2 = 1, \tag{c}$$

then for some constant γ

$$f(x) \equiv \sin(x + \gamma) \tag{d}$$

in an interval from x_1 to the nearest point where $|\sin(x+\gamma)| = 1$, and in case $f(x_1) = 0$ the interval extends a distance $\pi/2$ on each side of x_1.

PROOF. There is no loss of generality in supposing $f(x_1) \geqq 0$, $f'(x_1) \geqq 0$ because the general case may be reduced to this by considering the function $\pm f(\pm x)$ for some choice of signs. (If $f(x_1) = 0$, the signs can be chosen in two ways.) Thus let $f(x_1)=\sin\alpha$ and $f'(x_1)=\cos\alpha$, $0\leqq\alpha\leqq\pi/2$. Again there is no loss of generality if we assume $x_1=\alpha$ since this situation may be realized by a suitable translation of the origin. Application of the preceding lemma completes the proof of Theorem B.

THEOREM C. *If $f(x)$ is a function which for all x and for some integer n greater than 2 satisfies the conditions*

$$\text{(a)}\qquad \{f(x)\}^2\leqq 1,$$

$$\text{(b)}\qquad \{f^{(n)}(x)\}^2+\{f^{(n-1)}(x)\}^2\leqq 1,$$

and if at some point x_1 and for some integer k in the range $1\leqq k\leqq n-1$,

$$\text{(c)}\qquad \{f^{(k)}(x_1)\}^2+\{f^{(k-1)}(x_1)\}^2=1,$$

then there is a constant γ such that for all x

$$\text{(d)}\qquad f(x)\equiv\sin(x+\gamma)$$

except possibly if $k=1$ and $\{f(x_1)\}^2=1$.

PROOF. We shall tacitly employ Theorem A throughout. First consider the case $k=1$; suppose

$$\{f'(x_1)\}^2+\{f(x_1)\}^2=1,\qquad \{f(x_1)\}^2<1.$$

By Theorem B, $f(x)\equiv\sin(x+\gamma)$ throughout an interval at one end of which $f(x)=\pm 1$. At this end point, say x_2, it follows by continuity that $\{f''(x_2)\}^2+\{f'(x_2)\}^2=1$ and $f'(x_2)=0$. Applying Theorem B to the function $f'(x)$, we obtain $f'(x)\equiv\cos(x+\gamma)$ over an interval extending a distance $\pi/2$ on each side of x_2. At the end points of this interval $\{f'(x)\}^2+\{f(x)\}^2=1$ and $f(x)=0$; so applying Theorem B to the function $f(x)$, we see that $f(x)\equiv\sin(x+\gamma)$ in an interval of length 2π. Clearly this process may be repeated to give a continuation of $f(x)$ for all values of x, and this proves the Theorem for $k=1$.

The problem for higher values of k is easily reduced to the case $k=1$ just discussed. Suppose for some x_1 and k

$$\text{(1)}\qquad \{f^{(k)}(x_1)\}^2+\{f^{(k-1)}(x_1)\}^2=1,\qquad 1<k<n.$$

By application of Theorem B to the function $f^{(k-1)}(x)$ we see that there is a point where $f^{(k-1)}(x)=\pm 1$. At this point we have

$$\text{(2)}\qquad \{f^{(k-1)}(x)\}^2+\{f^{(k-2)}(x)\}^2=1,\qquad f^{(k-2)}(x)=0.$$

Now, note that (2) is the same as (1) with k replaced by $k-1$ and with the added condition that $f^{(k-2)}(x)$ is zero. By repeating this process a sufficient number of times we shall find that at some point $\{f'(x)\}^2+\{f(x)\}^2=1$ and $f(x)=0$. This is an instance of the case already discussed, and the theorem is proved.

We are now able to give a proof of S. Bernstein's theorem on trigonometric polynomials, which may be stated in the following form:[6]

If $\phi(x)$ *is a trigonometric polynomial of degree* N *or less,* $\phi(x)=\sum_0^N(a_\nu \cos \nu x+b_\nu \sin \nu x)$; *and if* $|\phi(x)|\leqq 1$ *for all* x, *then* $|\phi'(x)|<N$ *for all* x, *unless* $\phi(x)\equiv\sin(Nx+\alpha)$ *for some constant* α.

Proof. Let $f(x)=\phi(x/\lambda)$ where λ is some constant greater than N. Then as $n\to\infty$, $f^{(n)}(x)\to 0$ uniformly in $(-\infty,\infty)$; hence for sufficiently large n, $\{f^{(n)}(x)\}^2+\{f^{(n-1)}(x)\}^2\leqq 1$. Thus, by Theorem A we have $\{f^{(k)}(x)\}^2+\{f^{(k-1)}(x)\}^2\leqq 1$, or

$$\left\{\frac{1}{\lambda^k}\phi^{(k)}(x)\right\}^2+\left\{\frac{1}{\lambda^{k-1}}\phi^{(k-1)}(x)\right\}^2\leqq 1, \qquad k=1,2,3,\cdots.$$

Considering any fixed k and letting $\lambda\to N+0$, we obtain

$$\left\{\frac{1}{N^k}\phi^{(k)}(x)\right\}^2+\left\{\frac{1}{N^{k-1}}\phi^{(k-1)}(x)\right\}^2\leqq 1, \qquad k=1,2,3,\cdots.$$

To obtain conditions of equality Theorem C is directly applicable.

Finally we give another simple inference from Theorem A.

If $f_1(x)$ *and* $f_2(x)$ *satisfy the conditions of Theorem* A *for some integer* n, *then*

$$\left|\frac{d^k}{dx^k}f_1(x)f_2(x)\right|\leqq 2^{k-1}, \qquad k=1,2,3,\cdots,n.$$

Proof. By the Leibniz rule

$$\frac{d^p}{dx^p}(f_1'+if_1)(f_2'+if_2)=\sum_0^p C_{p,\nu}(f_1^{(\nu+1)}+if_1^{(\nu)})(f_2^{(p+1-\nu)}+if_2^{(p-\nu)})$$

where the constants $C_{p,\nu}$ are the binomial coefficients. Then by Theorem A

$$\sum_0^p C_{p,\nu}\left|f_1^{(\nu+1)}+if_1^{(\nu)}\right|\cdot\left|f_2^{(p+1-\nu)}+if_2^{(p-\nu)}\right|\leqq\sum_0^p C_{p,\nu}=2^p.$$

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[6] M. Riesz, Jahresbericht der Deutschen Mathematiker-Vereinigung, vol. 23 (1914), pp. 354–368. C. J. de la Vallée Poussin, Comptes Rendus de l'Académie des Sciences, vol. 166 (1918), pp. 843–846. G. Pólya and G. Szegö, *Aufgaben und Lehrsätze aus der Analysis*, vol. 2, pp. 35, 90.

To separate real and imaginary parts we may write

$$(f_1' + if_1)(f_2' + if_2) = f_1'f_2' - f_1f_2 + i\frac{d}{dx}(f_1f_2).$$

Differentiating this last expression p times and taking absolute magnitudes, we obtain

$$\left\{\frac{d^{p+1}}{dx^{p+1}}(f_1f_2)\right\}^2 + \left\{\frac{d^p}{dx^p}(f_1'f_2' - f_1f_2)\right\}^2 \leqq (2^p)^2,$$

which is more than we set out to prove. The functions $f_1(x) = \sin(x+\alpha_1)$ and $f_2(x) = \sin(x+\alpha_2)$ show that our constant is the "best possible."

PURDUE UNIVERSITY AND
STANFORD UNIVERSITY

ON THE CARATHÉODORY CONDITION FOR UNILATERAL VARIATIONS[1]

JULIAN D. MANCILL

The two formulations and proofs of the Carathéodory condition in the calculus of variations given by Graves[2] do not necessarily apply to the case when the minimizing curve may have arcs in common with the boundary of the region of admissible variations.[3] The purpose of this note is to show how his first formulation and proof can be modified so as to be applicable to unilateral (one-sided) variations in the plane.

An admissible curve

$$E_0:\quad x^\alpha = x^\alpha(t), \qquad t_0 \leqq t \leqq T,\ \alpha = 1, 2, \cdots, n,$$

which minimizes the integral

$$J = \int_{t_0}^{T} F(x_1, \cdots, x_n, x_1', \cdots, x_n')dt \equiv \int_{t_0}^{T} F(x, x')dt$$

in the class of all admissible curves joining two fixed points x_0 and X in space of n dimensions ($n > 1$), must satisfy certain well known conditions.[4] Suppose that E_0 is an extremaloid joining the points x_0 and X and having corners at $x_1, \cdots, x_m$, and suppose that the function F_1 is different from zero along E_0 including both sides of corners. If the function

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[1] Presented to the Society, December 29, 1939.

[2] *Discontinuous solutions in space problems of the calculus of variations*, American Journal of Mathematics, vol. 52 (1930), pp. 13–19.

[3] Cf. Mancill, *The minimum of a definite integral with respect to unilateral variations*, Contributions to the Calculus of Variations, 1933–37, University of Chicago, 1937, p. 121, condition C_{3C}.

$$\Omega_0(x, p^-, p^+) = (\partial F^+/\partial x^\alpha)p^{\alpha-} - (\partial F^-/\partial x^\alpha)p^{\alpha+},$$

where $p^{\alpha-}$ and $p^{\alpha+}$ represent the direction cosines of the tangents to E_0 at the corners, is different from zero at each corner of E_0, there exists an $(n-1)$-parameter family

$$x^\alpha = \phi^\alpha(t, a) \tag{1}$$

of extremaloids defined for $t_0 - \delta \leqq t \leqq T + \delta$, passing through the point x_0 for $t = t_0$, and containing[5] E_0 for $a^i = a_0^i$, $t_0 \leqq t \leqq T$. We shall now prove the following form of the Carathéodory condition:

THEOREM. *Let E_0 be an extremaloid joining the points x_0 and X and minimizing the integral J, satisfying the following conditions*:

(1) *E_0 is positively strong*;[6]
(2) *$\Omega_0 \neq 0$ at the corners on E_0*;
(3) *E_0 is uniquely determined by each of its elements (x, x')*;[7]
(4) *the determinant*

$$D(t, a_0) \equiv \left| \frac{\partial \phi^\alpha}{\partial a^i} \quad \frac{\partial \phi^\alpha}{\partial t} \right|_{a=a_0}$$

does not vanish at the corners on E_0.

Then the determinant $D(t, a_0)$ does not change sign at the corners on E_0.

Suppose that the determinant $D(t, a_0)$ changes sign at a corner x_k where $t = t_k$. Also, suppose that the corners on all extremaloids of the family occur for fixed values of the parameter[8] t. Consider the one-parameter family of curves E_u constructed by Graves.[9] The value of the integral J taken along E_u is a function $J(u)$ whose derivative near $u = t_k$ is

$$J'(u) = - \mathcal{E}(\phi^{\alpha-}, \phi'^{\alpha-}, x'^{\alpha+}) \equiv - \mathcal{E}[\bar{t}(u), \bar{a}(u), x'^{\alpha+}(u)]. \tag{2}$$

The second derivative of $J(u)$ *at* $u = t_k$ is

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[4] For a full statement of the problem and the classical necessary conditions, see Graves, loc. cit., pp. 2–3.

[5] Graves, loc. cit., p. 6.

[6] Cf. Graves, loc. cit., p. 9.

[7] Ibid., p. 11.

[8] Ibid., p. 7.

[9] Ibid., p. 14.

$$(3)\qquad J''(t_k) = -\ [(\partial F^+/\partial x^\alpha)x'^{\alpha-} - (\partial F^-/\partial x^\alpha)x'^{\alpha+}]\bar{t}'(t_k),$$

on account of the extremal property of E_0, and since the derivative of the function $\mathcal{E}[t_k, \bar{a}(u), x'^{\alpha+}(u)]$ is zero at $u=t_k$. This last statement follows from assumptions (1) and (3) of the theorem and the fact that we have assumed that the corners on all the extremaloids of the family occur for fixed values of the parameter t, and therefore the function $\mathcal{E}[t_k, \bar{a}(u), x'^{\alpha+}(u)] \geqq 0$ near $u=t_k$ and equals zero at t_k. By making use of the homogeneity property of the function F and the expression for $\bar{t}'(t_k)$ given by Graves,[10] the derivative (3) may be reduced to

$$(4)\qquad \begin{aligned} J''(t_k) = -\ &[(x'^{\alpha-}x'^{\alpha-})(x'^{\alpha+}x'^{\alpha+})]^{1/2} \\ &\cdot \Omega_0(x, p^-, p^+)D(t_k + 0, a_0)/D(t_k - 0, a_0) < 0, \end{aligned}$$

where $p^{\alpha-}=x'^{\alpha-}/(x'^{\alpha-}x'^{\alpha-})^{1/2}$ and $p^{\alpha+}=x'^{\alpha+}/(x'^{\alpha+}x'^{\alpha+})^{1/2}$. Consequently, $J(t_k)$ *is a maximum of the function* $J(u)$ *for* u *in a neighborhood of* t_k, since $J'(t_k)=0$ and $J''(t_k)<0$. Thus, E_0 could not minimize the integral J.

Let us now consider the problem of unilateral variations in the plane. In this case it can be shown that a Carathéodory condition as stated here applies to every arc of the minimizing curve *which is an arc of an extremaloid.* The condition may be stated in terms of the one-parameter family of extremaloids through any fixed point of such an arc. For such arcs of the minimizing curve as E_{23} immediately following a non-extremal arc E_{12} of the boundary of the region of admissible curves whose direction at the point 2 where it meets E_{12} is not directed towards the exterior side of E_{12} but is as shown in Fig. 1, the Carathéodory condition may be stated in terms of the one-parameter family of extremaloids whose members, at least those members on the admissible side of the minimizing curve, are tangent to the arc E_{12} of the boundary.[11] In order to insure the existence of the families of extremaloids just described, it is necessary to assume that the continuity and homogeneity properties of the integrand F hold in an extended region containing in its interior those arcs of the minimizing curve which are extremaloids in common with the boundary. It should also be pointed out that in the construction of the family of extremaloids tangent to E_{12}, it is necessary to assume that the Weierstrass $\mathcal{E}$-function is greater than or equal to zero along

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[10] Loc. cit., p. 15.

[11] Mancill, loc. cit., pp. 105–107. For a discussion of the case when the extremaloid is directed towards the exterior side of E_{12} see pages 138–141. In this case the minimizing curve need not satisfy the corner conditions at the point 2.

E_{12} preceding 2.[13] The family of extremaloids is then composed of the family of extremals tangent to E_{12} in a neighborhood of and preceding the point 2, and the family of extremaloids containing the extremaloid E_{23} which is the *continuation* family of the extremals tangent to E_{12} near the point 2.

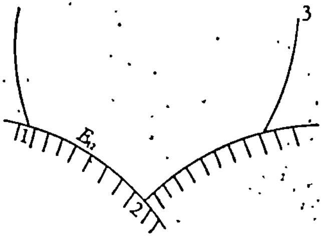

FIG. 1

In order to show how the proof of the theorem applies in the way just described, let (1) represent the one-parameter family of extremaloids through a fixed point of such an arc of the boundary. Let x_k be the *first corner beyond this fixed point.* Then the family of curves E_u used in the proof of the theorem exists for u near t_k and its members are admissible curves for u on one side of t_k. Therefore, the relation (2) holds here for u on the admissible side of t_k. The function $-\mathcal{E}[\bar{t}(u), \bar{a}(u), x'^{\alpha+}(u)]$ is defined locally and in a neighborhood of t_k. Its derivative at $u=t_k$ is $J''(t_k)$ as given in (4). Consequently, it follows from (2) that $J(t_k)$ is greater than $J(u)$ for all values of u on the admissible side of t_k and sufficiently near t_k. The same argument can now be made at each succeeding corner in turn.

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[13] Mancill, ibid., p. 107. For a proof of the Weierstrass condition along E_{12}, see Graves, American Mathematical Monthly, vol. 41 (1934), pp. 502–504.

THE SOCIETY'S PRIZES

The American Mathematical Society offers three prizes in fields as noted below, and awards them at five-year intervals in accordance with the following schedule.

(1) The Bôcher Prize in Analysis, to be awarded at the end of 1943 for papers in analysis published during the years 1938–42 inclusive (five previous awards).

(2) The Cole Prize in Algebra, to be awarded at the end of 1944 for papers published during the years 1939–43 inclusive (two previous awards).

(3) The Cole Prize in the Theory of Numbers, to be awarded at the end of 1941 for papers published during the years 1936–40 inclusive (one previous award).

If the committee on awards finds no memoir of outstanding merit at the time appointed for a given award, it will report this fact to the Council; the Council concurring, no award shall be made at that time.

The individual amount of each prize is set by the Council; ordinarily it has been one hundred dollars. The following conditions for eligibility have been adopted:

1. A memoir to be eligible for consideration must have been published in the appropriate interval of time in a recognized journal published in the United States or Canada.
2. The author at the time of publication of the memoir must have been a member of the American Mathematical Society.
3. The author at the time of publication of the memoir must have been not more than fifty years of age.
4. No dissertation shall be eligible for any one of the prizes.
5. No author who has once received a particular prize shall again be awarded the same prize.
6. No paper shall be eligible for more than one prize.

R. G. D. RICHARDSON,
Secretary

THE FEBRUARY MEETING IN NEW YORK

The three hundred sixty-sixth meeting of the American Mathematical Society was held at Columbia University on Saturday, February 24, 1940. The attendance included the following two hundred forty-two members of the Society:

R. P. Agnew, C. B. Allendoerfer, Warren Ambrose, R. L. Anderson, R. G. Archibald, Herbert E. Arnold, L. A. Aroian, K. J. Arrow, F. E. Baker, M. F. Becker, D. L. Bernstein, A. C. Berry, E. E. Betz, Gertrude Blanch, J. H. Blau, Salomon Bochner, Frank Boehm, W. M. Bond, Samuel Borofsky, C. B. Boyer, A. D. Bradley, A. T. Brauer, H. W. Brinkmann, A. A. F. Brown, A. B. Brown, K. A. Bush, J. H. Bushey, Jewell H. Bushey, S. S. Cairns, W. B. Campbell, F. E. Carner, Claude Chevalley, Alonzo Church, T. F. Cope, Richard Courant, H. B. Curry, M. D. Darkow, Norman Davids, M. M. Day, J. J. De Cicco, M. E. A. Dever, R. P. Dilworth, C. H. Dix, W. J. Dixon, Jesse Douglas, T. L. Downs, Arnold Dresden, F. G. Dressel, W. D. Duthie, J. E. Eaton, W. F. Eberlein, M. L. Elveback, J. M. Feld, W. K. Feller, Aaron Fialkow, W. B. Fite, W. W. Flexner, G. E. Forsythe, Tomlinson Fort, R. M. Foster, K. O. Friedrichs, Orrin Frink, Guido Fubini, G. N. Garrison, Hilda Geiringer, Abe Gelbart, B. P. Gill, M. A. Girshick, Saul Gorn, P. H. Graham, A. M. C. Grant, H. S. Grant, C. H. Graves, C. C. Grove, D. W. Hall, F. C. Hall, N. A. Hall, P. R. Halmos, Hans-Karl Hammer, O. G. Harrold, Philip Hartman, G. G. Harvey, C. M. Hebbert, G. A. Hedlund, Edward Helly, J. G. Herriot, Maximilian Herzberger, Fritz Herzog, E. H. C. Hildebrandt, L. S. Hill, Einar Hille, Banesh Hoffmann, Lulu Hofmann, T. R. Hollcroft, E. M. Hull, W. H. Ingram, S. A. Jennings, R. A. Johnson, B. W. Jones, H. A. Jordan, Mark Kac, I. N. Kagno, E. R. van Kampen, Edward Kasner, L. S. Kennison, S. C. Kleene, J. R. Kline, Morris Kline, E. R. Kolchin, B. O. Koopman, Arthur Korn, Jack Laderman, M. E. Ladue, A. W. Landers, M. K. Landers, Solomon Lefschetz, A. M. M. Lehr, B. A. Lengyel, Howard Levi, Madeline Levin, D. P. Ling, E. R. Lorch, A. N. Lowan, P. H. McGrath, I. C. McLaughlin, Brockway McMillan, L. A. MacColl, J. K. L. MacDonald, H. F. MacNeish, W. G. Madow, P. T. Maker, A. J. Maria, D. H. Maria, A. V. Martin, Walther Mayer, A. E. Meder, A. K. Mitchell, Don Mittleman, E. C. Molina, Deane Montgomery, J. C. Montgomery, L. T. Moore, T. W. Moore, Marston Morse, G. W. Mullins, D. C. Murdoch, F. J. Murray, C. A. Nelson, E. P. Northrop, C. O. Oakley, P. S. Olmstead, J. M. H. Olmsted, Oystein Ore, F. W. Owens, J. C. Oxtoby, C. K. Payne, R. S. Phillips, Everett Pitcher, E. L. Post, Walter Prenowitz, H. A. Rademacher, H. W. Raudenbush, H. W. Reddick, M. S. Rees, F. G. Reynolds, H. J. Riblet, Moses Richardson, R. G. D. Richardson, J. F. Ritt, H. E. Robbins, J. H. Roberts, H. P. Robertson, M. S. Robertson, S. L. Robinson, Benjamin Rosenbaum, J. E. Rosenthal, Barkley Rosser, Arthur Sard, Max Sasuly, Rubin Schatten, E. R. Schneckenburger, Abraham Schwartz, C. H. W. Sedgewick, I. E. Segal, Wladimir Seidel, Stephan Serghiesco, L. W. Sheridan, Jacob Sherman, Seymour Sherman, Max Shiffman, L. P. Siceloff, L. G. Simons, James Singer, L. L. Smail, M. F. Smiley, P. A. Smith, C. M. Sparrow, H. W. Steinhaus, R. C. Stephens, Wolfgang Sternberg, R. E. Street, W. C. Strodt, M. M. Sullivan, F. X. Sutton, J. D. Tamarkin, J. H. Taylor, B. J. Tepping, G. B. Thomas, J. M. Thomas, Arthur Tilley, C. B. Tompkins, W. R. Transue, A. W. Tucker, J. W. Tukey, Annita Tuller, Oswald Veblen, Abraham Wald, Henry Wallman, R. M. Walter, J. V. Wehausen, Louis Weisner, M. E. Wells, William Wernick, F. J. Weyl, Hermann Weyl, A. P. Wheeler,

O. L. Wheeler, A. L. Whiteman, Hassler Whitney, V. A. Widder, E. P. Wiggin, S. S. Wilks, John Williamson, Aurel Wintner, H. P. Wirth, W. H. Wise, Jack Wolfe, Jacob Wolfowitz, H. A. Wood, Oscar Zariski, Leo Zippin, O. J. Zobel.

The meeting opened Saturday morning with two sessions: Analysis, Professor Einar Hille presiding; and Algebra and Theory of Numbers, Professor B. W. Jones presiding.

At the opening of the afternoon general session, at which Professor Marston Morse presided, the Society voted, on recommendation of the Council, to adopt the following amendment to the by-laws as Article 6, Section 9:

> Any member who has been retired from active service on account of age and has been a member of the Society for twenty years or more may, upon notification to the Secretary of such retirement, have his dues remitted, on the understanding that he will thereafter receive the programs of the meetings, but not the Bulletin.

On recommendation of the Council, the Society also approved sending the letter quoted below to sister scientific organizations in other parts of the world:

> The American Mathematical Society, gathered at New York City for its February Meeting, marks this occasion by giving collective expression to the deep sympathy felt by its members for their many scientific colleagues who today are suffering the deprivations and the agonies brought upon them by the spreading cancer of war; and by giving collective expression also to their grave, even desperate, concern for the preservation of the cultural values and the effective organs of scientific research throughout the world during these days of destruction. In the cause of simple humanity and in the interest of the unborn generations for whom science can prepare benefits as yet but dimly descried, the American Mathematical Society now appeals to its sister societies in every land—and most particularly to those in lands which are at war with one another—to exert all possible effort toward the conservation of the scientific resources of the world against the day when peace shall reign once more. The Society calls upon its sister societies to concern themselves especially for those scientists who, by the fortunes of war, may fall prisoner or may come under new flags, to the end that their individual sufferings may be mitigated and their scientific activity continued to the benefit of all men; and it declares its readiness to lend its good offices and its assistance, both moral and material, to those of its sister societies which may respond to this urgent appeal.

After the short business meeting, Professor B. O. Koopman gave an address entitled *The bases of probability.* The general session was followed by two sectional sessions: Geometry and Topology, Dr. T. W. Moore presiding; Logic, Statistics and Probability, Professor Barkley Rosser presiding.

Titles and cross references to the abstracts of the papers read follow below. Papers whose abstract numbers are followed by the letter *t* were read by title. The papers numbered 1 to 10 were read before the section for Analysis; those numbered 11 to 18 before the section for Algebra and Theory of Numbers; those numbered 19 to 26 before the section for Geometry and Topology; those numbered 27 to 33 before the section for Logic, Statistics and Probability; papers numbered 34 to 65 were read by title. Paper 5 was read by Mr. Segal, paper 19 by Mr. Ling, paper 21 by Mr. Martin, paper 22 by Professor Kasner, paper 26 by Professor Courant, and paper 29 by Dr. Kac. Sister M. Patricia Callaghan was introduced by Professor Francis Regan, Mr. Henry Hurwitz by Professor G. D. Birkhoff, and Miss Harlan C. Miller by Professor R. L. Moore.

1. Seymour Sherman: *A comparison of linear measures of point sets in the plane.* Preliminary report. (Abstract 46-3-192.)

2. Jenny E. Rosenthal: *Generating functions and properties of certain orthogonal polynomials.* Preliminary report. (Abstract 46-3-187.)

3. Fritz Herzog: *Uniqueness theorems for rational functions.* (Abstract 46-3-174.)

4. M. M. Day: *Linear methods of summability.* (Abstract 46-3-164.)

5. Einar Hille and I. E. Segal: *Blaschke products as Laplace-Stieltjes integrals.* (Abstract 46-3-176.)

6. Tomlinson Fort: *Summability and the definition of a limit.* (Abstract 46-3-170.)

7. R. P. Agnew: *Tauberian conditions.* (Abstract 46-3-152.)

8. R. S. Phillips: *On the space of completely additive set functions.* (Abstract 46-3-183.)

9. J. Hobart Bushey: *Expansions of products of certain symmetric orthogonal polynomials.* (Abstract 46-3-160.)

10. F. G. Dressel: *The fundamental solution of the parabolic equation.* (Abstract 46-3-168.)

11. G. B. Thomas: *Regular positive ternary quadratic forms.* Preliminary report. (Abstract 46-3-195.)

12. H. J. Riblet: *Certain theorems for algebraic differential fields.* (Abstract 46-3-186.)

13. R. P. Dilworth: *A characterization of lattices of ideals.* (Abstract 46-3-166.)

14. Hermann Weyl: *Theory of reduction for arithmetical equivalence.* (Abstract 46-3-201.)

15. A. L. Whiteman: *Additive prime number theory in real quadratic fields.* (Abstract 46-3-151.)

16. E. R. Kolchin: *On the exponents of differential ideals.* (Abstract 46-3-179.)

17. S. A. Jennings: *Nilpotent groups and nilpotent rings.* (Abstract 46-3-177.)

18. John Williamson: *An algebraic problem involving the involutory integrals of linear dynamical systems.* (Abstract 46-3-202.)

19. D. P. Ling and Leon Recht: *Geodesics on a paraboloid of revolution.* Preliminary report. (Abstract 46-3-180.)

20. J. J. De Cicco: *The horn angle in the geometry of element-series.* (Abstract 46-1-58.)

21. A. V. Martin and J. H. Roberts: *Two-to-one transformations of 2-manifolds.* Preliminary report. (Abstract 46-3-148.)

22. Edward Kasner and J. J. De Cicco: *The conformal near-Moebius transformations.* (Abstract 46-1-92.)

23. G. A. Hedlund: *Surfaces of negative curvature and metric transitivity.* (Abstract 46-3-173.)

24. Everett Pitcher: *Homology groups under continuous maps.* I. (Abstract 46-5-216.)

25. I. N. Kagno: *On a certain non-separating graph on an orientable surface.* (Abstract 46-3-178.)

26. Richard Courant and N. Davids: *Minimal surfaces spanning closed manifolds.* (Abstract 46-3-163.)

27. William Wernick: *Complete sets of logical functions.* (Abstract 46-3-200.)

28. P. R. Halmos: *Statistics, set functions, and spectra.* (Abstract 46-3-144.)

29. R. P. Agnew and Mark Kac: *Translated functions and statistical independence.* (Abstract 46-3-153.)

30. A. C. Berry: *The accuracy of the Gaussian approximation to a sum of independent variates.* (Abstract 46-3-156.)

31. Wolfgang Sternberg: *The general limit theorem for probability densities.* (Abstract 46-5-218.)

32. W. K. Feller: *On the integrodifferential equations of purely discontinuous Markoff processes.* (Abstract 46-3-169.)

33. Abraham Wald: *Asymptotically most powerful tests of statistical hypotheses.* (Abstract 46-3-199.)

34. R. P. Agnew: *On Tauberian theorems for double series.* (Abstract 46-1-43-*t*.)

35. Reinhold Baer: *Abelian groups that are direct summands of every containing abelian group.* (Abstract 46-3-155-*t*.)

36. Garrett Birkhoff: *On a class of positive matrices.* (Abstract 46-3-139-*t*.)

37. W. Z. Birnbaum and H. S. Zuckerman: *On the properties of a collective.* (Abstract 46-3-157-*t*.)

38. A. B. Brown: *On transformation of multiple integrals.* (Abstract 46-3-158-*t*.)

39. L. E. Bush: *An asymptotic formula for the average sum of the digits of integers.* (Abstract 46-3-159-*t*.)

40. Sister M. Patricia Callaghan: *Generalized Frégier curves.* (Abstract 46-3-161-*t*.)

41. R. H. Cameron and W. T. Martin: *An unsymmetric Fubini theorem.* (Abstract 46-3-162-*t*.)

42. Nathaniel Coburn: *A characterization of Schouten's and Hayden's deformation methods.* (Abstract 46-3-140-*t*.)

43. Nathaniel Coburn: *Conformal unitary spaces.* (Abstract 46-3-141-*t*.)

44. M. M. Day: *The spaces* L^p *with* $0<p<1$. (Abstract 46-3-165-*t*.)

45. J. J. De Cicco: *The magnilong near-Laguerre transformations.* (Abstract 46-1-59-*t*.)

46. F. G. Dressel: *A Stieltjes integral equation.* (Abstract 46-3-167-*t*.)

47. G. A. Hedlund: *A new proof for a metrically transitive system.* (Abstract 46-3-172-*t*.)

48. E. V. Huntington: *Congressional reapportionment by the method of smallest divisors.* (Abstract 46-5-208-*t*.)

49. Henry Hurwitz: *Total regularity of infinite matrix transformations.* (Abstract 46-3-146-*t*.)

50. Edward Kasner and J. J. De Cicco: *Transformation theory of integrable double-series of lineal elements.* (Abstract 46-1-93-*t*.)

51. A. N. Lowan: *Note on upper bounds of the derivatives of certain transcendental functions.* (Abstract 46-1-98-*t*.)

52. A. N. Lowan and Gertrude Blanch: *Analysis of computing errors in the process of analytic continuation.* (Abstract 46-3-147-*t*.)

53. J. K. L. MacDonald: *Existence theorems for node and amplitude properties near singularities of linear differential equations.* (Abstract 46-3-181-*t*.)

54. Saunders MacLane: *Note on the relative structure of p-adic fields.* (Abstract 46-3-182-*t*.)

55. Harlan C. Miller: *A theorem concerning closed and compact point sets which lie in connected domains.* (Abstract 46-1-106-*t*.)

56. Eric Reissner: *On Green's function of* $\nabla^2\nabla^2 w=0$ *for the half plane.* (Abstract 46-3-185-*t*.)

57. L. B. Robinson: *The solution of a functional equation on the contour of the unit circle.* (Abstract 46-5-217-*t*.)

58. Peter Scherk: *Estimations with integers in the demesne of the* α- β- *hypothesis.* (Abstract 46-3-188-*t*.)

59. F. K. Schmidt and Saunders MacLane: *On inseparable fields.* (Abstract 46-3-189-*t*.)

60. W. T. Scott: *Approximation to real irrationals by certain classes of rational fractions.* (Abstract 46-3-191-*t*.)

61. M. F. Smiley: *Measurability and modularity in the theory of lattices.* (Abstract 46-3-193-*t*.)

62. Gabor Szegö: *Remarks on a note of R. Wilson and related subjects.* (Abstract 46-3-194-*t*.)

63. H. S. Vandiver: *Elements of a theory of abstract discrete semigroups.* (Abstract 46-3-197-*t*.)

64. H. S. Vandiver: *On general methods for obtaining congruences involving Bernoulli numbers.* (Abstract 46-3-198-*t*.)

65. H. S. Wall: *Continued fractions and totally monotone sequences.* (Abstract 46-1-130-*t*.)

T. R. HOLLCROFT,
Associate Secretary

CORRECTION

In the report of the meeting of the Society in New York on October 28, 1939, which appears in the current volume of this journal, pages 1–5, the following corrections should be made.

P. 3, paper 21. Abstract 46-1-17 instead of Abstract 46–1–456.
P. 4, paper 28. Abstract 45-9-315 instead of Abstract 45-11-315.
P. 4, paper 30. Abstract 46-1-2-*t* instead of Abstract 46-1-441-*t*.
P. 4, paper 43. Abstract 46-1-32-*t* instead of Abstract 46-1-471-*t*.

JOHN HENRY TANNER—IN MEMORIAM

On Monday, March 11, 1940, the telephone announced, throughout the city of Ithaca, New York, the death of Professor Tanner. He had lived very quietly during the last several years, and the end came peacefully as the fitting close of a long and useful life.

John Henry Tanner was born on March 1, 1861, in Fort Plain, New York, where he spent most of his early life. He taught in the Clinton Liberal Institute from 1883 to 1887, then entered Cornell University, graduating in 1891. He was immediately appointed instructor in mathematics, promoted to an assistant professorship in 1894, but he did not assume this office at once, since he spent the next two years studying at Göttingen, under Klein and Hilbert. The second of these years was considerably interrupted. In 1894, through the efforts of Mr. Tanner, Dr. Ernst Ritter was appointed assistant professor of mathematics at Cornell, and died on the way to Ithaca. Then Mr. Tanner visited many of the German universities in the attempt to find a worthy successor to Ritter; he did not make a final selection. During the years 1897–1903 Tanner served as secretary of the Faculty of Arts and Sciences at Cornell and served on various committees during most of that period. It was in this interval that he wrote most of his successful textbooks. In 1904 he was promoted to a full professorship, which position he held until 1926, when he retired with the rank of professor emeritus.

In 1905 Tanner was elected to the Council of the American Mathematical Society, in which office he was active in its financial affairs. In 1908 he was elected treasurer of the Society, which office he held until 1920. In addition to the routine duties of the office, he performed valuable service in formulating its financial policy.

In 1920 Professor and Mrs. Tanner established the Tanner Foundation to promote mathematical research and instruction at Cornell with a gift of $100,000. The income of this sum is to be added to the principal for a considerable period before the sum is to be used for its final purpose.

Tanner was an excellent teacher and was well liked both by associates and students. The Tanners always had a welcome both for university and townspeople and were particularly generous to students and others who needed help of any kind.

VIRGIL SNYDER

BOOK REVIEWS

The Probability Integral. By W. F. Sheppard. Cambridge, University Press, 1939. 11+34 pp.

This book contains six tables of values of functions related to the function $(2\pi)^{-1/2}\int_x^\infty e^{-t^2/2}dt$, and was published for the British Association for the Advancement of Science. It is the seventh volume of the series of mathematical tables published by the British Association. The tables may be described as follows:

Table I. The ratio of the tail area of the normal curve to its bounding ordinate, with reduced derivatives, at intervals of one-hundredth of the standard deviation, to twelve decimal places.

Table II. The ratio of the tail area of the normal curve to its bounding ordinate, with reduced derivatives, at intervals of one-tenth of the standard deviation, to twenty-four decimal places.

Table III. The negative natural logarithm of the tail area of the normal curve, for integral multiples of the standard deviation, to twenty-four decimal places.

Table IV. The negative natural logarithm of the tail area of the normal curve, with reduced derivatives, at intervals of one-tenth of the standard deviation, to sixteen decimal places.

Table V. The common logarithm of the tail area of the normal curve, with reduced derivatives, at intervals of one-tenth of the standard deviation, to twelve decimal places.

Table VI. The common logarithm of the tail area of the normal curve, with second central differences, at intervals of one-hundredth of the standard deviation, to eight decimal places.

By the nth reduced derivative of a function $f(x)$ is meant the expression $h^nf^{(n)}(x)/n!$, and in the tabulations the value chosen for h is the argument interval. Thus accurate interpolation is made possible by the use of the Taylor expansion. The number of reduced derivatives given for each entry varies from 3 to 16.

There is an introduction, by J. O. Irwin, which explains the use of the tables clearly. The Introduction also states that the idea of preparing such a volume of tables was originally conceived by the late Dr. Sheppard, who had in mind a set of tables which would form the basis of the computation of the probability integral "to as many decimal places as would ever be required." The work was not completely finished by Sheppard; the reduced derivatives in Table II, and all of Table VI, were computed by individuals appointed by the Committee for the Calculation of Mathematical Tables of the British Association. All the tables have been checked.

The book is attractively and very legibly printed on paper of good quality.

J. H. Curtiss

Le Origini del Calcolo Infinitesimale nell' Era Moderna. By G. Castelnuovo. Bologna, Zanichelli, 1938. 164 pp.

This book is number 12 of the series Per la Storia e la Filosofia della Matematiche, a collection published under the auspices of the Italian Istituto Nazionale per la Storia delle Scienze and which was founded and directed by F. Enriques, whose name no longer appears. After an introductory chapter on Archimedes it reports on the growth of the infinitesimal calculus in the sixteenth and seventeenth centuries, beginning with Maurolycus and Commandinus and ending with Newton and Leibniz. An appendix contains Newton's "Tractatus de quadratura curvarum" (1704) and Leibniz' "Nova methodus pro maximis et minimis" (1684) in an Italian translation with notes by Professor E. Carruccio. Professor Vacca has given valuable advice in the composition of the book.

The distinguished geometer of the University of Rome has given an unusually clear picture of the different steps which the mathematics of the infinitesimal took in its formative years. "I have tried to show in the clearest possible way how the new science originated when the antique conceptions of Archimedes' genius were fertilized with the new doctrines of algebra and analytical geometry on one side, and of dynamics (or better cinematics) on the other," he writes in the preface. Though he gives a brief sketch of the lives of each of the great contributors, he intentionally avoids discussion of the many personal and priority squabbles which characterized some of their work. Concentrating on the major figures, he has been able to reconstruct before our eyes the important results which gradually completed the structure of the differential and integral calculus. He uses a number of more recent papers, especially some written by Italian historians, and adds some interesting information in the concise bibliographies which end each chapter.

The reader will be wise to compare this book with the recent book of C. B. Boyer, *The Concepts of the Calculus* (Columbia University Press, New York, 1939), which discusses a larger period and also penetrates deeper into the philosophical arguments which formed an important aspect of the study of the infinitesimals from the time of Zeno to the time of Cantor, but has not Castelnuovo's wealth of mathematical details. A sociological study of the development of the calculus, which can only explain its rise and growth in the sixteenth and seventeenth centuries, still has to be written.

D. J. STRUIK

Algebren. By M. Deuring. (Ergebnisse der Mathematik, vol. 4, no. 1.) *Gruppen von linearen Transformationen*. By B. L. van der Waerden (Ergebnisse der Mathematik, vol. 4, no. 2.) Berlin, Springer, 1935. 5+143 and 3+91 pp., respectively.

The theory of algebras, now about to enter the second century of its existence, constitutes today an integrating part of algebra and arithmetics. The most fundamental step in its development seems to have been the introduction of general reference fields, essentially due to Wedderburn. In order to describe approximately the degree of generality we may say that Wedderburn's theory holds at least for those fields which obey the theory of Galois. We quote from Dickson's *Linear Algebras* (1914): "Any linear associative algebra over a field F is the sum of a semisimple algebra and a nilpotent invariant subalgebra (the radical) each over F. A semisimple algebra is either simple or the direct sum of algebras over F. Any simple algebra over F is the direct product of a division algebra and a simple matrix algebra each over F."

Other results are found in Dickson's book *Algebras and their Arithmetics* (1923), which concludes with an instructive list of unsolved problems: (I) the determination of all division algebras, (II) the classification of nilpotent algebras, the discovery of relations between an algebra and its maximal nilpotent invariant subalgebra (the radical), (III) theory of non-associative algebras, and (IV) theory of ideals in the arithmetic of a division algebra and the extension to algebras of the whole theory of algebraic numbers.

Progress in the study of problems II and III has been moderate, in the sense that we have many beautiful special results but no general theory.

As to problems I and IV, our knowledge has advanced considerably, to say the least; this advance is reported in Deuring's report *Algebren* and Albert's Colloquium lectures *Structure of Algebras*. We quote both authors in saying that R. Brauer, H. Hasse, E. Noether and A. A. Albert have given the solution of problem I for algebraic number fields. From the authors who contributed to the solution of problem IV we single out (at the expense of others) H. Brandt who provided the fundamental idea that a left ideal in one maximal domain of integrity is a right ideal in another, and his pupil Eichler who did much (1936–1938) towards the solution of problem IV with respect to ideal classes and units. The reader will find an independent account of the ideal theory in the seventh section of Deuring's book.

Deuring's report opens in the spirit of modern abstract algebra with a streamlined introduction into the Wedderburn theory; not only algebras but rings are considered wherever possible. He then proceeds to the theory of matrix representations; following E. Noether he shows us what representations by matrices mean to the structure of the algebra and vice versa. This part of noncommutative algebra leads to the study of simple algebras (section VI); the mathematical concepts which permit an adequate treatment go back to Dickson and R. Brauer. Dickson invented the "cross products" as a tool of constructing normal simple algebras and in particular division algebras from extension fields Z of F.

Brauer defines the "Brauer group" of a field F in the following way: First of all, normal simple algebras are called equivalent if their division algebras are the same; that is, if they are complete matrix algebras over the same division algebra. He proves that these classes yield an abelian group if direct multiplication is chosen as composition.

The relations between cross products and Brauer group form the object of section V which deals with factor sets. Factor sets are sets of numbers in commutative fields which enter into the process of constructing non-commutative algebras by cross multiplication. We find their analogues in various places, as far distant as addition of one-digit numbers and generation of rings of operators in a Hilbert space. In the special case of cross multiplication which leads to the so-called Dickson algebras, they may be reduced to single numbers in the reference field F itself; the theory of Brauer classes is brought into the closest connection with the theory of norms of the extension Z over F. This link between algebras and arithmetic algebraic numbers, foreshadowed in papers of Dickson and Wedderburn, proved to be the key to the fundamental theorem on algebras over algebraic number fields, which is developed in section VII, on the basis of Hasse's theory of p-adic algebras. The p-adic fields are a generalization of the real and complex numbers, in the sense that they permit an analysis on the basis of a limit concept; real and complex numbers are more related to geometry, whereas the other p-adic fields (Hensel's invention) are related to the arithmetic of the congruence concept. Algebraic number fields may be imbedded in several ways into the complex or real numbers (infinite prime spots, corresponding to their conjugates) and the p-adic fields of Hensel, corresponding to their prime ideals. The division algebras over infinite p-adic fields (real or complex) are to be commutative or else quaternion systems, the Brauer group having one or two elements. Hasse determined the division algebras over the "finite" p-adic fields; they are all of the Dickson type and simple explicit constructions are given. Their Brauer groups are mapped by an explicit isomorphism on the rational numbers modulo 1. Normal simple algebras over an algebraic number field are described by their behavior under imbedding of the reference field into p-adic fields (fundamental theorem): (a) All except a finite number of p-adic imbeddings yield a full matrix algebra; therefore the rational number modulo 1 (Hasse symbol), which belongs to the prime spot and indicates the p-adic Brauer class, is 0 except for a finite number of cases. (b) If and only if all these numbers are 0 we have a complete matrix algebra; two algebras are equivalent if and only if their p-adic Brauer classes and therefore the Hasse symbols are the same. (c) The sum of the Hasse symbols is always zero (law of reciprocity). (d) If Hasse symbols are assigned with the only condition that (c) be satisfied, then there exists an algebra with these Hasse symbols. (e) Every division algebra is of the Dickson type and may be represented as a cross product belonging to a cyclotomic field.

This theorem is certainly one of which every mathematician would be proud *de porter le nom*; an inquiry into the priority situation is therefore interesting (and a little dangerous, too). The reviewer ventures the statement that the full theorem has been found by Hasse, Brauer, Noether, that the algebraic part including the "cyclic" Brauer group was developed independently by Albert and that the arithmetical side was also contained essentially in K. Hey's dissertation (1927–1929).

From the development after the appearance of Deuring's *Algebren* we shall only cite the subjects Riemann matrices, p-algebras and hypercomplex ideal theory (Albert, Eichler, Weyl, Witt and others).

The hypercomplex numbers have been in close touch with groups of linear substitutions; in fact, the preface to Dickson's *Linear Algebras* starts with this sentence: "The theory of linear associative algebras is essentially the theory of pairs of reciprocal linear groups or the theory of certain sets of matrices." It is therefore natural that B. L. van der Waerden's report on groups of linear transformations should contain also the theory of hypercomplex systems and their representations. We find there even the most general form of Brauer's factor sets, which Deuring has omitted in his presentation. This (entirely justified) omission is due to Deuring's tendency to present complete theories rather than individual theorems, the state of affairs in the theory of algebras permitting this luxury. Such an attitude is not possible with respect to groups of linear transformations; the essential reason is perhaps that algebras have been studied as representatives of a few large types, whereas groups are in many cases individuals or representatives of small families. Correspondingly, the essentials of the theory which Deuring covers are contained in comparatively few papers, whereas van der Waerden had to incorporate explicitly a host of individual investigations. If one should examine the Fortschritte der Mathematik, say since 1900, one would hardly find a paper related to the subject not mentioned in his book (excepting papers on continuous groups which were reserved for another report in the same collection).

Whatever the abstract algebraic method can do in the line of unification has been done, as in the theory of the classical linear groups (like the orthogonal groups, and so on), or the theory of representations of general classes like hypercomplex systems, finite groups, bounded representations of arbitrary groups; but in fields like groups of given degree (2, 3, 4), or the representation of individual groups (like modular groups), the special character of the problem necessitates a special treatment.

MAX ZORN

Mécanique Statistique Quantique. By Francis Perrin. (Traité du Calcul des Probabilités et de ses Applications, vol. 2, no. 5.) Paris, Gauthier-Villars, 1939. 224 pp.

This book forms a complete introductory outline of modern statistical mechanics. The first part (five chapters) deals with the rudimentary notions of classical statistical mechanics of Hamiltonian systems, ergodic theory, canonical ensembles, equipartition, coupled systems, thermostats, the thermodynamic quantities and laws, with applications to perfect gases and radiation. The second part (four chapters) introduces the rudiments of the quantum theory and extends to systems obeying its laws many of the considerations of the first part. The third part (entitled *Statistique Quantique des Systèmes Indiscernables*, eight chapters) forms the main body of the work, to which the earlier parts form a sort of introduction. The principles of indistinguishability and exclusivity are introduced and the Bose-Einstein and Fermi-Dirac schemata established. The three laws of thermodynamics are then derived from the statistical theory, and applications are made to the chemical constants, gaseous degenerescence, magnetism, neutron stars, and so on. There follows a chapter on statistical kinetics in which the increase of entropy and tendency towards statistical equilibrium are treated. The closing chapter deals with indistinguishability and exclusivity from the viewpoint of symmetric and antisymmetric wave functions.

The author appears to have imposed two restrictions upon himself: *Firstly*, the mathematics used is elementary (advanced calculus, elementary combinatorial analysis, and Stirling's formula). The result is not merely that he exerts to the full the physicists' privilege of lack of rigor, but rather that, owing to the lack of the universal language and framework of conceptions with which modern mathematics would have supplied him, certain subjects are not placed in the clearest light. Thus, to give only one example, in the ergodic theory the author assumes all limiting functions to be continuous and hence loses sight of the nature of the present-day problem, which is one of measurable functions. *Secondly*, the quantum mechanics used is quite rudimentary, consisting solely of the simplest notions of the wave equation, quantized orbits and the principles of indistinguishability and exclusivity. Thus the great and fundamental questions (such as are treated in the last chapters of J. von Neumann's *Mathematische Grundlagen der Quantenmechanik*, and other modern works) which are suggested by the very title of the present book are left out of consideration. For the most part the method is the enumeration of possible distributions among states, the approximation by Stirling's formula and the maximizing of the resulting expression for the number of complexions, it being assumed that the course of the system is through conditions of maximum number of complexions. The justification of this in the chapter on statistical equilibrium is logically rather meagre. Granted these limitations, we feel that the author has been most successful.

One point deserves especial commendation: The author defines in the following precise mathematical form the idea of two ergodic systems (that is, closed metrically transitive hamiltonian systems) "in feeble interaction": (1) for all calculations for finite time, the combined equation of motion is to be taken the same as the combination of the two separate equations of motion (that is, without mutual potential term); (2) on the other hand, in limits ($t\rightarrow\infty$), the combined system is to be regarded as metrically transitive (that is, $H_1+H_2=E$ is to be regarded as the only integral, not $H_1=E_1$ and $H_2=E_2$). The intuitive notion of feeble interaction is familiar but, to our knowledge, it has not been given in such a clean-cut mathematical form.

We are of the opinion that the very fact which forms the strength of this book is also the source of its weakness, namely, that it is in last analysis only an introductory outline. It is written with the traditional French clarity, and will be a useful addition to the library of the mathematician taking an interest in modern physics.

B. O. KOOPMAN

Topological Groups. By L. Pontrjagin. Translated by Emma Lehmer. Princeton, University Press, 1939. 9+299 pp.

The topological group is a combination of two fundamental mathematical concepts—group and topological space. A topological group G is a group and at the same time a topological space in which the group operations in G are continuous. Historically, the concept arose from the study of groups of continuous transformations. Pontrjagin gives an axiomatic treatment of topological groups. Later he points out their connections with continuous transformations as well as with other older concepts. In the language of the author: "This book is intended for the reader with rather modest mathematical preparation." This is accomplished very successfully by both the included material and its organization. All material needed is precisely formulated, and in most cases proofs are given. The understanding of the text is enhanced by the inclusion of seventy-five examples, which deal largely with real numbers, matrices, and vector spaces. The author points out questions left unanswered in most of the general problems discussed.

The first three chapters give an excellent introduction to topological groups. Chapter I discusses the usual topics in elementary abstract group theory. These include normal subgroups, factor groups, homomorphisms, the center of a group, direct products, and commutative groups. In Chapter II a topological space is defined by means of axioms in terms of the closure of a set. An equivalent neighborhood definition is set up and is used extensively. Among the concepts studied are connectedness, regularity, second axiom of countability, compactness, and topological products. Continuous mappings are introduced early and have a prominent place in the chapter. Chapter III contains the first steps in the theory of topological groups. The fundamental relations holding for abstract groups and topological spaces are adapted to topological groups. Additional concepts introduced include interior (open) mappings and local isomorphsims.

After the introductory material in the first three chapters, the reader may proceed to any one of Chapters IV, VI, or VIII. Chapter IV proves that any compact group satisfying the second axiom of countability admits a complete system of representations. The establishment of an invariant measure on the group is the first step in the proof. Haar gave the first construction of such a measure, but the author gives von Neumann's construction of invariant integration because it is simpler. The proof also contains an exposition of the results of Peter and Weyl concerning the completeness of the system of functions arising from irreducible representations.

The fundamental results of Chapter V, which depend on the theory of representations of Peter and Weyl, are due to the author. Locally compact commutative groups satisfying the second axiom of countability are investigated principally through the construction of a character group. It is shown that to such a group G there corresponds a group X of the same kind, which is called the character group of G. Since the correspondence between G and X is symmetric, any question concerning one of the groups reduces to the corresponding question about the other. When G is compact its character group X is shown to be discrete, and conversely. Hence the study of a compact G reduces to the study of an abstract group.

Some of the fundamental problems of Lie groups (which may be discrete) are solved in Chapter VI. The results of the chapter are intended primarily as preparatory material for Chapter VII, in which it is shown that the study of compact groups satisfying the second axiom of countability can be reduced to the study of Lie groups and their limits. The last named chapter contains a positive solution of Hilbert's fifth problem (is every parameter group a Lie group?) for compact groups. A positive solution was given in Chapter V for locally compact commutative groups.

Chapter VIII develops the following results (organized and formulated by Schreier) concerning the connection in the large between locally isomorphic groups: Let Δ be the aggregate of all connected, locally connected, and locally simply connected groups which are locally isomorphic with one such group G. There exists one and only one (up to an isomorphism) simply connected group G^* in Δ, which is called the universal covering group of G. Moreover, any group G' of Δ can be written in the form G^*/N, where N is a discrete central normal subgroup isomorphic with the fundamental group of G'. The chapter contains a satisfying treatment of the fundamental group and universal covering space of a topological space.

The main object of Chapter IX is to show that the local study of Lie groups can be reduced to the study of infinitesimal groups. While the deeper results of Killing, Cartan, and Weyl in the theory of infinitesimal groups are not taken up here, some are stated without proof. A complete classification of semi-simple Lie groups is given in this way. Among the other topics discussed are compact Lie groups and groups of continuous transformations.

This book should prove of invaluable aid both to the beginner in the field of topological groups and to the more advanced student. While a number of typographical errors were found, they should not prove confusing to an alert reader.

W. T. PUCKETT, JR.

Statistical Methods. By Paul R. Rider. New York, Wiley, 1939. 9+220 pp.

This book has two aims: first, to serve as a textbook for an elementary course in statistics, and second, to help students with some previous knowledge of statistics to gain an insight into the more modern methods. It proceeds from some preliminary development of the classical theory, through such topics as "Student's" distribution, to the various significance tests associated with the χ^2 and Fisher z distributions. The notation of the calculus is used in a number of the formulas. However, so much emphasis is placed on the practical applications of the theory that the statistical worker who uses the book as a laboratory manual will probably not find the mathematical notation disconcerting, no matter what his previous mathematical training may have been.

The classical theory is presented in Chapters I–IV and the first part of Chapter V. Chapters I and II are concerned with the elementary theory of frequency distributions, and with averages and moments. Chapter III contains a discussion of regression, with an exposition of Fisher's method of handling the normal equations in the case of multiple regression. Chapter IV is on simple and multiple correlation, and deals entirely with the observational theory. In Chapter V we find brief descriptive treatments of such topics as the continuous approximation to the binomial distribution, the normal and Gram-Charlier type A distributions, the significance of the difference between two means and two proportions, the significance of correlation coefficients (tested by means of Fisher's logarithmic transformation); and there is also an introduction to the theory of confidence limits. In Chapter VI, we are introduced to "Student's" t distribution which is then applied to appropriate problems, such as testing the significance of regression coefficients. At the beginning of Chapter VII the χ^2 distribution is described, and there follow the usual applications to homogeneity tests, tests of goodness of fit, and contingency tables. Yate's correction for continuity is explained. Chapter VIII, on analysis of variance and covariance, is perhaps the climax of the book. The Fisher z distribution is presented, and is then applied to significance tests in the theory of linear, curvilinear, and multiple regression, and to the analysis of variance within and among classes of single and paired variates. One of the sections of this chapter contains some interesting and hitherto unpublished results by Churchill Eisenhart on absolute criteria in the theory of regression. The last chapter is concerned with Fisher's theory of experimental design, and deals with such topics as randomized blocks, factorial design, and confounding. The book contains tables of the normal distribution and of t, χ^2, and z, and concludes with a good index.

Proofs are given for most of the formulas until the end of Chapter IV. From there on, except for one or two trivial algebraic manipulations, there are no proofs at all. But it is obviously not the purpose of the author to go into such matters, and the references to the literature in this connection are usually adequate. A much more important mathematical criticism of the book is that it is seriously infected by the tradition of vagueness which vitiates so much of the original work on which it is based. For example, although the book is supposed to be suitable for a first course, definitions throughout are given in extremely sketchy form, usually by means of examples. Then too, in a number of cases, which occur both in the early parts of the book and in the later parts, technical statistical terms are used without any explanation or definition at all, it being the apparent intention of the author that the appropriate explanations should be supplied by a teacher. But there is no consistency in these cases; for instance, the terms "mean" and "standard deviation" are defined for observed and continuous distributions, and are then used in a later section without further explanation in connection with the binomial distribution. Certain less elementary concepts, such as statistical independence and degrees of freedom, are never defined, although used repeatedly; and of course no explanation is ever given as to what is meant by the words "population" or "universe." There are a number of loose statements, such as the following one on page 104: "It may be noted that the denominator of (11) is the expected value of the population variance as estimated from the sample." (Expected value had been used in the technical sense previously in the book; and if this statement happens to mean that the population variance is the expected value of the denominator, then it is simply false.) It is also worth mentioning that in the applications of the z-test to the analysis of variance, nothing is ever said as to whether or how the hypothesis of the z distribution as to independence of the estimates of the variance is satisfied.

But from a less rigorous point of view, there are many features in the book which are commendable. Some of these have already been indicated in our description of the contents; but special mention should be made of the fact that every significant piece of theory is illustrated with an illuminating numerical example, and there is a generous number of good exercises at the end of each chapter.

It is the reviewer's personal opinion that the book would be successful in its first aim (as a classroom text for a first course in statistics) only if it were used to supplement a set of lectures by a skillful teacher. But for the more advanced student of statistics who is not entirely familiar with the methods of Fisher and his followers, the book should prove to be a valuable reference work. Comparison is inevitable in this connection with Fisher's well known text, *Statistical Methods for Research Workers*,[1] for Rider covers somewhat the same theoretical ground and will appeal to much the same group of readers. Although there is something to be said for Fisher's omission of almost all mathematical notation when writing for these readers, it seems to the reviewer that wherever there is overlapping between the two books, Rider definitely excels in organization of material and clarity of presentation. This is not meant to imply that Rider should supplant Fisher in the statistical workers' library. Fisher's book contains a great many valuable suggestions, explanations, and warnings (and also dogmatic assertions) concerning experimental technique which are not to be found in Rider. But as a companion volume, and as a sort of translation of some of the more obscure passages in Fisher's book, Rider should immediately find an important place.

J. H. CURTISS

The Mathematical Theory of Huygens' Principle. By B. B. Baker and E. T. Copson. Oxford, Clarendon Press, 1939. 7+155 pp.

This work is the first of a series of monographs planned by the authors on the mathematics of physics. Each monograph is to be complete in itself and deal with some special topic in the theory of the partial differential equations of mathematical physics not adequately treated in existing books.

The aims of the authors are admirably achieved in the first monograph which deals with the mathematical theory of Huygens' principle in the propagation of sound and light waves. The theme of the work is the general theory of the solution by Green's method of the partial differential equations governing these phenomena.

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[1] Oliver and Boyd, Edinburgh and London.

Huygens' principle is a well known elementary method for treating the propagation of waves. The method assumes that a spherical wavelet starts out with velocity c from each point of a given wave front at time $t=0$. Each wavelet will have a radius ct at time t, and the envelope of these wavelets is taken to be the resulting wave front at this later time t. A difficulty arises in that this geometrical construction would give a wave traveling backward, as well as one traveling forward. To avoid this difficulty it is necessary to generalize Huygens' principle by having recourse to an analysis of the partial differential equation governing the wave motion and of the boundary conditions to be satisfied.

Chapter I deals with an analytical formulation of Huygens' principle for wave fields described by a single wave function. The physical example taken to illustrate this theory is the propagation of sound waves in which the wave function is the scalar velocity potential. Poisson's solution of the wave equation is shown to justify Huygens' construction for isolated spherical or plane sound waves generated by an initial temporary disturbance. Similarly Helmholtz's solution is shown to justify Huygens' principle for periodic trains of waves. Finally sound waves of any structure and origin are treated by means of a general theorem due to Kirchhoff. The chapter closes with a discussion of the solutions of the equation of cylindrical waves by Weber and Volterra. In this connection the recent important work of Professor Marcel Riesz is described in detail.

Chapter II gives an account of the diffraction of light by a black screen on the basis of Kirchhoff's theory according to which the field of a train of light waves may be described by a single scalar potential. Maggi's transformation is introduced by means of which the wave function of the diffracted light is expressed as a line integral along the rim of the diffracting screen. The diffraction of plane and spherical waves by a black half-plane is considered in detail.

The phenomenon of the polarization of light can be accounted for on the basis of the electromagnetic theory of light according to which the field of radiation can be described as a vector field. Chapter III is devoted to a formulation of Huygens' principle for electromagnetic waves in a vacuum. The Larmor-Tedone formulation and Kottler's formulation are compared. The diffraction of polarized light by a black screen is discussed from Kottler's point of view.

Chapter IV is devoted to an account of Sommerfeld's theory of the diffraction of polarized light by a perfectly reflecting half-plane and of

Voigt's theory of the diffraction of polarized light by a black half-plane. The method of many-valued wave-functions and Riemann surfaces is employed.

The standard of knowledge expected of the reader of this work is that of a graduate student who has completed the usual courses in analysis and electromagnetic field theory. Twenty-three exercises are provided in the first three chapters. The text is replete with footnote references to papers that have appeared in the literature up to 1939. By rigor of logical treatment and careful attention to detail the authors have produced a critical treatise which will undoubtedly become a standard reference work.

W. E. Bleick

Modern Elementary Theory of Numbers. By Leonard Eugene Dickson. Chicago, University Press, 1939. 305 pp.

The first few chapters of this book contain, with minor exceptions, the same material as the corresponding chapters of Dickson's *Introduction to the Theory of Numbers.* This may lead those familiar with the earlier book to think that this is a new edition of that book. It is much more than that. Where the topics are the same the explanations are lengthened, more proofs included, examples worked to give clarity to the text and the number of exercises increased. Beginning with the fifth chapter the book is almost completely rewritten. New material and modern topics are introduced. Here Dickson has been able to offer more simply some of the work in theory of numbers that has been in the literature in recent years and to obtain some new results.

Chapters I through IV deal with divisibility, congruences and their solutions, quadratic residues and binary quadratic forms. Dickson states in his preface that these with a few chosen topics from the chapters on indefinite ternary quadratic forms and Diophantine equations would provide a brief elementary course.

Quadratic forms are the subject of several of the later chapters. In Chapter V a study is made of the numbers represented by various ternary quadratic forms with numerical coefficients. A table is given consisting of 102 regular forms and all positive integers not represented by each form. Chapter VIII treats of indefinite ternary quadratic forms, universal and zero forms. Here the problem of representation of integers by indefinite forms whose coefficients involve parameters is studied. The necessary and sufficient conditions for integral solutions of indefinite quadratic forms in four or more variables where the form equals zero are found in Chapter IX. And Chapter

XIII gives a brief discussion of positive quadratic forms in n variables.

In Chapters VI and VII Dickson has considered universal theorems involving cubes and sums of nine values of a cubic function. He says that "simplifications are made in the present exposition which also obtains more than 6000 universal forms, each a sum of 9 products of a cube by a positive integer." Here and in Chapter XII are proved particular cases of Waring's problem.

The conditions for the solution of a quadratic and linear function in 4, 5, 6, 7 and 8 variables are found in Chapter X and theorems on polygonal numbers.

Chapter IX gives a general theory of homogeneous, quadratic diophantine equations. This is a new topic and many exercises are given.

In the appendix, after a brief study of infinite series is a proof of the infinitude of primes in an arithmetical progression. This theorem is assumed in some of the earlier chapters.

CAROLINE A. LESTER

Principles of the Mathematical Theory of Correlation. By A. A. Tschuprow. Translated by M. Kantorowitsch. London, Hodge; New York, Nordemann, 1939. 10+194 pp.

In 1925 Tschuprow published his *Grundbegriffe und Grundprobleme der Korrelationstheorie.* The foundation of the book was a series of lectures given in the insurance seminar of the University of Christiania (now Oslo). It was not intended to be a guide to the calculation of measures of relationship but to provide a logical foundation for the theory of correlation, to clarify fundamental notions and assumptions and to link up the theory of correlation with the theory of probability.

This German edition, which has since become a very well known work, was reviewed in the Bulletin in 1926 (vol. 32, p. 561), so we are concerned here only with the translation into English. It is a straightforward translation with the exception of the "Notes and Bibliography." Here the translator substituted a survey of contemporary English literature on correlation for the author's introductory notes. One may question the advisability of a translation of a 15 year old book on a statistical subject, but the modern student after reading a random sample of the chapters on correlation in our recent text books will find the reading of Tschuprow's book a real tonic. The translation is very well done, accurate but not too literal. Though it is regrettable that the translation was so long in coming, the book will fill a real need, even if it does not contain more recent developments.

A. R. CRATHORNE

Variationsrechnung im Grossen (*Theorie von Marston Morse*). By H. Seifert and W. Threlfall. Leipzig and Berlin, Teubner, 1938. 115 pp.

The purpose of this book is to give an expository account of Morse's theory of the calculus of variations in the large. The approach is, naturally, that of the topologist, and not the analyst. No attempt is made to present the theory in its most comprehensive form; generality is often successfully sacrificed on behalf of clearness of exposition. For example, in the critical point theory, singular cycles are used instead of Vietoris cycles, and a continuous function is used where lower semicontinuity or even less would do, while in the analysis more differentiability conditions than necessary are assumed in order to avoid analytic complications. Topologically the book is self-contained; whereas results from the calculus of variations are assumed. Things such as the "index form," which constitute a bridge between classical calculus of variations and the new developments in the large, are omitted. As a result, the book is excellent both for the calculus of variations student who wishes to learn about the modern developments in his field, and for the topologist who wishes to become acquainted with an important and fascinating application of his field to analysis without becoming involved in too many analytic details.

S. B. MYERS

NOTES

The Society for the Promotion of Engineering Education will hold its 48th Annual Meeting at the University of California in Berkeley, June 24 to June 28.

The Francoeur Prize of the French Academy of Sciences has been awarded to Professor Brelot of the University of Bordeaux for his work on harmonic functions.

The Petit D'Ormoy Prize of the French Academy of Sciences has been awarded to Professor G. Valiron of the University of Paris for his work in mathematics.

Mr. C. E. Shannon, assistant in mathematics at the Massachusetts Institute of Technology, has been awarded the Alfred Noble Prize of $500 by the American Institute of Electrical Engineers.

Professor J. L. Coolidge of Harvard University will retire on September 1, 1940, with the title Professor of Mathematics Emeritus and Master of Lowell House Emeritus. Professor Coolidge has been a member of the department of mathematics for forty years and master of Lowell House for ten years.

Professor Arnold Emch of the University of Illinois, whose retirement was announced on page 22 of the current volume of this Bulletin, has been given the title of professor emeritus.

The School of Mathematics of the Institute for Advanced Study is building up its own library in the fields of mathematics and theoretical physics. Any offers of secondhand books or sets of periodicals from private owners or of duplicates from libraries, addressed to the Librarian, Institute for Advanced Study, Princeton, N. J., will receive careful consideration.

The following ninety-nine doctorates, with mathematics or mathematical physics as a major subject, were conferred during 1939 in universities in the United States and Canada; the major subject is mathematics unless otherwise specified. The university, month in which the degree was conferred, minor subject (other than mathematics), and the title of the dissertation are given in each case if available.

H. W. Alexander, Princeton, June, *The rôle of the mean curvature in the immersion theory of surfaces.*

Warren Ambrose, Illinois, June, minor in astronomy, *Some properties of measurable stochastic processes.*

H. A. Arnold, California Institute of Technology, June, minor in physics, *The theory of operational equations and differential equations in Kantorovitch spaces.*

Margaret E. P. Baxter, Michigan, February, *On the geometry of the Dirac equations.*

Marjorie H. Beaty, Colorado, August, *On the complex roots of algebraic equations.*

Dorothy L. Bernstein, Brown, June, *The double Laplace integral.*

E. E. Betz, Pennsylvania, June, *Accessibility and separation by simple closed curves.*

C. J. Blackall, Cornell, September, *On volume integral invariants of non-holonomic dynamical systems.*

R. C. Blackwell, North Carolina, June, *Cubics having r-point contact with a plane curve.*

Augusto Bobonis, Chicago, December, *Differential systems with boundary conditions involving the characteristic parameter.*

Achille Capecelatro, Rensselaer Polytechnic Institute, June, minors in physics and engineering, *The complementary function of fractional integration.*

W. B. Caton, Yale, June, *A class of inequalities.*

C. L. Critchfield, George Washington, June, major in mathematical physics, *The theory of nuclear forces and their spin-dependence under the pair-emission hypothesis.*

A. W. Davis, Iowa State, July, minors in physics and mathematical physics, *Differentiable and continuity properties of solutions of certain partial differential equations of applied mathematics.*

M. M. Day, Brown, June, I. *Regularity of function-to-function transformations.* II. *Linear methods of summability.*

R. P. Dilworth, California Institute of Technology, June, minor in physics, *The structure and arithmetical theory of non-commutative residuated lattices.*

T. C. Doyle, Princeton, June, *Tensor decomposition: with applications to the contact and complex groups.*

P. L. Dressel, Michigan, June, *A study of statistical seminvariants and their estimates with particular emphasis on their relation to algebraic seminvariants.*

R. J. Dunholter, Cincinnati, June, *A sequence of factors which leaves a class of double Fourier series unchanged.*

J. J. Eachus, Illinois, June, minor in physics, *Regular solutions of non-linear q-difference systems.*

J. E. Eaton, Yale, June, *Associative multiplicative systems.*

R. A. Favila, California (Berkeley), May, *On the projective differential geometry of certain systems of linear homogeneous partial differential equations of the first order, with special application to stratifiable congruences.*

J. W. Foust, Michigan, February, major in education, *The construction of a standardized test in functional thinking.*

R. H. Fox, Princeton, June, *On the Lusternik Schnirelmann category.*

Cleota G. Fry, Purdue, June, minor in physics, *Asymptotic developments of certain integral functions.*

B. E. Gatewood, Wisconsin, October, *Thermal stresses in long cylindrical bodies.*

R. E. Greenwood, Jr., Princeton, June, *Hankel series.*

Harriet Griffin, New York, June, minor in physics, *The abelian quasi-group.*

Leon Gropper, New York, February, major in mathematical physics, *Quantum theory of the equation of state of a gas.*

G. L. Gross, Iowa State, July, minors in physics and mathematical physics, *Use of functionals in obtaining approximate solutions of linear operational equations.*

P. J. Hart, Iowa State, July, major in applied physics, minors in mathematics and mathematical physics, *The determination of elastic constants by Piezo-elastic methods.*

Morris Hendrickson, Ohio State, September, *Some properties of arbitrary functions.*

H. H. Howe, George Washington, February, major in mathematical physics, *The rotational Raman effect for the asymmetric-top molecule.*

S. A. Jennings, Toronto, June, *On the structure of the group-ring of a p-group over a modular field.*

R. N. Johanson, Chicago, June, *Ruled surfaces with intersecting generators.*

Laura L. Johnson, Colorado, June, *On the diophantine equation* $x(x+1) \cdots (x+n-1)=y^k$.

R. P. Johnson, Pittsburgh, June, *Conical roulettes.*

I. N. Kagno, Columbia, March, *Perfect subdivision of surfaces.*

Wilfred Kaplan, Harvard, June, *Regular curve-families filling the plane.*

Chosaburo Kato, Ohio State, December, *Configurations* n_3.

E. S. Kennedy, Lehigh, October, *Exponential analogues of Lambert series.*

J. M. Kingston, Toronto, June, *Special abstract groups.*

Fulton Koehler, Minnesota, June, minor in physics, *Systems of orthogonal polynomials on certain algebraic curves.*

C. F. Kossack, Michigan, June, *The existence of collectives in abstract space.*

K. S. Kunz, Cincinnati, June, major in mathematical physics, *On the classical theory of radiating electrons.*

A. W. Landers, Jr., Chicago, December, *Invariant multiple integrals in the calculus of variations.*

Mary K. Landers, Chicago, December, *The Hamilton-Jacobi theory for the problems of Bolza and Mayer.*

Yuan Lay, Michigan, June, *The imbedding of the skew part of a bilinear function in linear associative algebras.*

R. A. Leibler, Illinois, June, minor in astronomy, *Analytic theory of non-linear singular differential systems whose associated systems are of Fuchsian type.*

A. O. Lindstrum, Jr., Illinois, June, minor in physics, *Functions of a complex variable determined by their values on non dense sets.*

A. T. Lonseth, California (Berkeley), May, *The problem of Plateau in hyperbolic space.*

Samuel Lubkin, New York, June, minor in physics, *Stability of columns under periodically varying loads.*

W. C. McDaniel, Wisconsin, June, *The deflection of an orthotropic plate.*

W. G. McGavock, Duke, June, minor in education, *Annihilators of quadratic forms with applications to Pfaffian systems.*

Brockway McMillan, Massachusetts Institute of Technology, June, minor in physics, *The calculus of the discrete homogeneous chaos.*

Roy MacKay, Michigan, February, *Planarity of peano spaces in terms of homology bases.*

Bella Manel, New York, June, minor in physics, *Conformal mapping of multiply connected domains on the basis of Plateau's problem.*

J. C. Mark, Toronto, June, *On the modular representations of the group* $GLH(3, p)$.

W. H. Myers, Stanford, September, minor in physics, *The Sylow subgroups of odd order of the primitive simple linear groups in five variables.*

H. M. Nahikian, North Carolina, August, *Conditions that a matrix* B *be expressible as a polynomial in a set of partial idempotent and nilpotent elements of a matrix* A, *with applications to the solution of matric equations.*

Sara L. Nelson, Cornell, June, *Cremona transformations belonging to a family of cubic curves.*

E. A. Nordhaus, Chicago, August, *The problem of Bolza for double integrals in the calculus of variations.*

Katharine E. O'Brien, Brown, June, *Some problems in interpolation by characteristic functions of linear differential systems of the fourth order.*

L. F. Ollmann, Michigan, June, *On joining finite subsets of planar Peano spaces by simple closed curves.*

H. V. Park, North Carolina, June, *Conditions on two singular matrices* A *and* B *such that* AB *and* BA *may have the same reduced characteristic function.*

R. S. Phillips, Michigan, June, *Integration in a convex linear topological space.*

Martha H. Plass, Massachusetts Institute of Technology, October, minor in physics, *Ruled surfaces in Euclidean four space.*

E. D. Rainville, Michigan, June, *Linear differential invariance under operators related to the Laplace transformation.*

Adrienne S. Rayl, Chicago, August, *Stability of permanent configurations in the problem of four bodies.*

P. V. Reichelderfer, Ohio State, June, *Some properties of continuous transformations in the plane.*

H. J. Riblet, Yale, June, *Algebraic differential fields.*

V. N. Robinson, Chicago, June, *Generalized intersector varieties.*

D. H. Rock, Iowa State, June, minors in physics and physical chemistry, *Finite strain analysis in elastic theory.*

Clarence Ross, Washington, August, *Periodic orbits for straight-line solutions of the n-body problem.*

Abraham Schwartz, Massachusetts Institute of Technology, June, minor in physics, *The Gauss-Codazzi-Ricci equations in Riemannian manifolds.*

H. M. Schwartz, Pennsylvania, June, *A study of a certain class of continued fractions.*

C. E. Sealander, Iowa, August, *An irregular third-order boundary value problem and the associated series*, II.

C. L. Seebeck, Jr., North Carolina, June, *Vectors associated with a curve in a Riemann n-space.*

C. B. Smith, Wisconsin, June, minor in physics, *Thermo-elastic behavior of composite plates.*

C. V. L. Smith, Harvard, June, *The fractional derivative of a Laplace integral.*

W. S. Snyder, Ohio State, June, *On functions of squares.*

D. E. South, Michigan, February, *The characteristics of correlation surfaces which are the sum of two normal components.*

K. H. Stahl, Pittsburgh, June, *A note on plane collineations.*

Ellen C. Stokes, Chicago, December, *Applications of the covariant derivative of Cartan in the calculus of variations.*

R. E. Street, Harvard, February, *Non-linear equations of electrodynamics based upon Birkhoff's postulates.*

W. C. Strodt, Columbia, May, *Irreducible systems of algebraic differential equations.*

B. J. Tepletsky, Ohio State, June, *Stability and periodicity of solutions of Mathieu's equation.*

G. R. Thurman, Missouri, August, *The geometry of spheroidal spaces and properties of pseudo-spheroidal sets.*

L. W. Tordella, Illinois, June, minor in philosophy, *A classification of groups of order* P^6, *P an odd prime.*

Esther M. Torrance, Brown, June, *Superposition on monotonic functions.*

H. C. Trimble, Wisconsin, March, *On the ring of matrices commutative with a given matrix.*

J. W. Tukey, Princeton, June, *On denumerability in topology.*

E. P. Vance, Michigan, June, *Generalizations of non-alternating and non-separating transformations.*

Mary C. Varnhorn, Catholic University, June, minors in education and philosophy, *Some properties of the quartic function of one variable.*

R. K. Wakerling, California (Berkeley), May, *On the loci of the* $(k+1)$*-secant* k*-spaces of a curve in* r*-space.*

A. D. Wallace, Virginia, June, *On interior and related transformations.*

J. A. Ward, Wisconsin, October, *A theory of analytic function and linear associative algebras.*

F. J. Weyl, Princeton, June, *Analytic curves.*

G. A. Whetstone, Washington, August, minor in mechanical engineering, *Some differential systems which remain passive under transformation.*

The following doctorate was conferred in 1938, but was not included in the list in the preceding volume of this Bulletin (vol. 45, pp. 350–353).

B. J. Topel, Notre Dame, August, *A set of postulates for the ratio of division.*

Dr. Marjorie Heckel Beaty has been appointed to an assistant professorship at the University of South Dakota.

Associate Professor H. R. Brahana of the University of Illinois has been promoted to a professorship.

Professor C. T. Bumer of Kenyon College has been appointed Peabody Professor of Mathematics.

Dr. Josephine H. Chanler of the University of Illinois has been made an associate in mathematics.

Professor Abraham Cohen of Johns Hopkins University has retired.

Dr. Myrtle Collier has been made chairman of the department of mathematics at Immaculate Heart College, Los Angeles.

Mr. J. L. Kelley of the University of Virginia has been appointed to an assistant professorship at the University of Notre Dame.

Rev. J. H. Kenna of the University of Notre Dame has been promoted to an assistant professorship.

Assistant Professor D. H. Lehmer of Lehigh University has been appointed to an assistant professorship at the University of California at Berkeley.

Dr. Norman Levinson of Massachusetts Institute of Technology has been promoted to an assistant professorship.

Dr. Hans Lewy of the University of California at Berkeley has been promoted to an associate professorship.

Dr. A. N. Milgram of the University of Notre Dame has been promoted to an assistant professorship.

Professor John von Neumann of the Institute for Advanced Study has been appointed to a professorship in mathematics at the University of Washington for the first term of the 1940 summer quarter.

Professor J. O. Patterson of St. Charles Seminary, Overbrook, Philadelphia, has retired.

Dr. P. M. Pepper of the University of Notre Dame has been promoted to an assistant professorship.

Professor L. J. Rouse of the University of Michigan is on leave of absence this semester.

Professor J. D. Tamarkin of Brown University and Professor Emil

Artin of the University of Indiana will lecture during the 1940 summer quarter at Stanford University.

Professor J. H. Taylor of George Washington University is on leave of absence this term.

Associate Professor W. J. Trjitzinsky of the University of Illinois has been promoted to a professorship.

Associate Professor C. C. Wylie of the University of Iowa has been promoted to a professorship of astronomy.

The following appointments to instructorships are announced: Allegheny College, Meadville, Pennsylvania: Mr. J. A. Joseph; Brooklyn College: Dr. Moses Richardson; Catholic University of America: Dr. J. F. Daly; Johns Hopkins University: Dr. C. H. Dowker; Massachusetts Institute of Technology: Dr. Eric Reissner, Dr. D. C. Spencer: University of Notre Dame: Dr. C. V. Robinson; Purdue University: Mr. R. C. Davis; University of Virginia: Mr. P. A. White; University of Washington: Dr. H. S. Zuckerman; Woodrow Wilson Junior College, Chicago: Dr. Bernard Friedman.

Professor L. Hopf of the University of Aachen died December 21, 1939.

The death of Dr. Johannes Tropfke of Berlin on November 10, 1939, is reported.

Professor Emeritus U. S. Hanna of Indiana University died on February 18, 1940, at the age of seventy-five years. He had been a member of the Society for thirty-two years.

Professor A. W. Smith of Colgate University died on February 11, 1940, at the age of sixty-three years. He had been a member of the Society since 1905.

ABSTRACTS OF PAPERS

SUBMITTED FOR PRESENTATION TO THE SOCIETY

The following papers have been submitted to the Secretary and the Associate Secretaries of the Society for presentation at meetings of the Society. They are numbered serially throughout this volume. Cross references to them in the reports of the meetings will give the number of this volume, the number of this issue, and the serial number of the abstract.

203. J. W. Calkin: *A quotient ring over the ring of bounded operators in Hilbert space.* I.

Let $\mathfrak{B}$ denote the ring of bounded everywhere defined operators in Hilbert space $\mathfrak{H}$. The subset $\mathfrak{T}$ of totally continuous operators is a two-sided ideal in $\mathfrak{B}$ in the ordinary algebraic sense, and the quotient ring $\mathfrak{B}/\mathfrak{T}$ can be constructed in the usual way; moreover, since $\mathfrak{T}$ is closed with respect to the operation *, this operation can be defined in $\mathfrak{B}/\mathfrak{T}$ too. Defining a norm in $\mathfrak{B}/\mathfrak{T}$ by the equation $|\alpha| = \text{g.l.b.}|A|$, A in α, where $|A|$ is the bound of the operator A in $\mathfrak{H}$, the author shows that $\mathfrak{B}/\mathfrak{T}$ is a complete metric space. Further, he shows that there exists a unitary space $\mathfrak{L}$ (nonseparable, however) and a set $\mathcal{M}$ of bounded everywhere defined operators in $\mathfrak{L}$ which is a $(+, \cdot, {}^*)$-isomorphism of $\mathfrak{B}/\mathfrak{T}$. In addition, if $T(\alpha)$ denotes the element of $\mathcal{M}$ corresponding to α in $\mathfrak{B}/\mathfrak{T}$, the bound of $T(\alpha)$ is $|\alpha|$. Thus the correspondence $\mathfrak{B}/\mathfrak{T} \longleftrightarrow \mathcal{M}$ is an isometry, and $\mathcal{M}$ is an algebraic ring of operators closed in the uniform topology. Other results are: If $\mathfrak{J}$ is an arbitrary two-sided ideal in $\mathfrak{B}$, either $\mathfrak{J} \leqq \mathfrak{T}$ or $\mathfrak{J} = \mathfrak{B}$. Every self-adjoint transformation $T(\alpha)$ in $\mathcal{M}$ has a pure point spectrum. (Received February 24, 1940.)

204. J. W. Calkin: *A generalization of a theorem of Weyl.*

The paper defines the augmented resolvent set of a bounded operator A in Hilbert space $\mathfrak{H}$ as the set of values of λ such that $\mathfrak{R}(A-\lambda I)$, the range of $A-\lambda I$, is closed, while $\mathfrak{H} \ominus \mathfrak{R}(A-\lambda I)$ and the manifold of zeros of $A-\lambda I$ each have a finite dimension number. It then proves, by recourse to very simple properties of the homomorphism $\mathfrak{B} \rightarrow \mathcal{M}$ defined in abstract 46-5-203, that two operators A and B whose difference is totally continuous have the same augmented resolvent set. Since, for a self-adjoint A, the complement of this set is precisely the set of Häufungspunkte of the spectrum of A in the sense of Weyl (Rendiconti del Circolo Matematico di Palermo, vol. 27 (1909), pp. 373–392), Weyl's theorem to the effect that the latter set is the same for any two self-adjoint operators whose difference is totally continuous follows at once. (Received February 24, 1940.)

205. J. W. Calkin: *Functions of several variables and absolute continuity.* I.

The author studies various properties of real- and complex-valued functions of n real variables which are potential functions of their generalized derivatives in the sense of G. C. Evans. It is shown that every such function is equivalent to a function differentiable with respect to each variable almost everywhere (previously proved by

Evans for the case $n=2$) and various kinds of approximation to such functions are considered. In particular, it is shown that a sequence $\{f_n\}$ of such functions defined on an open set, which converges there together with the sequences of partial derivatives in the mean of order $p\geqq 1$, has a limit which is a function of the same type. (Received February 24, 1940.)

206. Roy Dubisch: *Non-cyclic algebras of degree four and exponent two with pure maximal subfields.*

In a paper published in this Bulletin (vol. 44 (1938), pp. 576–579), A. A. Albert proved the falsity of the converse of the well known proposition that a cyclic normal division algebra contains a quantity j whose minimum equation is $x^n=j$ in the base field of the algebra. His proof consisted of an example of a non-cyclic normal division algebra of degree and exponent four over a non-modular field containing a quantity j as described above. It is the purpose of this paper to show that the exponent does not affect the property, by constructing similarly an algebra of degree four and exponent two over the same field. (Received February 17, 1940.)

207. M. H. Heins: *A note on a theorem of Radó.*

Radó has shown that there are no $(1, m)$ conformal maps of a multiply-connected region of finite connectivity, the boundary of which consists of p disjoint continua, onto itself for $m>1$. His proof is based on the possibility of mapping such a region one-to-one and conformally onto a region bounded by p disjoint circles. It is the object of the present note to prove this theorem without recourse to the possibility of mapping the given region one-to-one and conformally onto a region of canonical type. The technique employed consists of a systematic use of the theory of iteration and of a simple extension of Nevanlinna's principle of harmonic measure. This technique may be also used to establish the theorem that the number of $(1, 1)$ conformal maps of a region of finite connectivity p (greater than 2) onto itself is necessarily finite. (Received February 20, 1940.)

208. E. V. Huntington: *Congressional reapportionment by the method of smallest divisors.*

It was shown in a former paper (Transactions of this Society, vol. 30 (1928), pp. 85–110) that there are five possible methods of apportioning representatives in Congress, and that the method of equal proportions has the property of putting each state as nearly as may be on a par with every other state, both with respect to congressional districts and with respect to individual shares. It is shown in the present note that the method of smallest divisors has the property of minimizing the sum of the congressional districts. This method may therefore be called the method of minimum (total) load. (Received February 24, 1940.)

209. E. J. McShane: *On the second variation in certain anormal problems of the calculus of variations.*

If a curve E_{12} is a minimizing curve for a Lagrange problem with variable end points, and the order of anormality of E_{12} does not exceed 1, there are multipliers with which E_{12} satisfies the Euler equations and transversality conditions and the necessary conditions of Weierstrass, Clebsch and Jacobi. Except for the Jacobi condition, this has been previously proved without restriction on the order of anormality. In problems with a single side condition the order of anormality cannot exceed 1, so for such problems there are always multipliers with which the full set of necessary conditions is satisfied. By an example it is shown that the restriction on the order of anormality cannot be weakened. (Received February 21, 1940.)

210. Karl Menger: *On Green's formula and the integral of derivatives.*

We define a multiple integral as the limit of its Weierstrass sums, for example $\iint(\partial p/\partial y-\partial q/\partial x)dxdy$ as the limit of the sums $W(N)=\sum\{[p(x_i, y_{j+1})-p(x_i, y_j)]/(y_{j+1}-y_j)-[q(x_{i+1}, y_j)-q(x_i, y_j)]/(x_{i+1}-x_i)\}(x_{i+1}-x_i)(y_{j+1}-y_j)$ for rectangular nets $N\{x_i, y_j\}$, when the diagonals of the meshes $[(x_{i+1}-x_i)^2+(y_{j+1}-y_j)^2]^{1/2}$ converge toward O. Whenever the Riemann double integral exists, our limit exists and is equal to the integral. For a rectangular domain of integration R, whenever p and q are continuous on R', the boundary of R, except perhaps a set of linear measure O, then our limit exists and is equal to $\int pdx+qdy$ along R', thus satisfying Green's formula regardless of the behavior of p and q in the interior of R. For, by cancelling the "interior" terms of $W(N)$ we see that $W(N)$ is a Weisertrass sum of $\int pdx+qdy$ along R'. The simple integral $\int_a^b f'(x)dx$ considered as the limit of its Weierstrass sums $\sum\{[f(x_{i+1})-f(x_i)]/(x_{i+1}-x_i)\}(x_{i+1}-x_i)$ is $f(b)-f(a)$ for each finite function f, since each of the Weierstrass sums has this value. (Received February 15, 1940.)

211. D. D. Miller: *Hereditary properties of continuous transformations.* Preliminary report.

Necessary and sufficient conditions are found for non-topological continuous transformations of various types (for example, monotone, non-separating, non-alternating, and so on) to have their defining properties on all subcontinua, or on certain special subcontinua, of the space S on which they are defined. Conversely, conditions are found under which continuous transformations on a space S, having these properties on certain collections of subcontinua, have the same properties on S. (Received February 24, 1940.)

212. Rufus Oldenburger and A. E. Porges: *The minimal numbers of binary forms.*

It was previously proved by one of the authors that the range of the minimal number of binary forms of degree n for a field K of characteristic greater than n is either $1, 2, \cdots, n$ or $1, 2, \cdots, n+1$. In the present paper by a study of certain determinants it is shown that the minimal number does not exceed n for finite fields of sufficiently high order. Since the minimal number of $x^{n-1}y$ is n, there exists for each ρ in the set $1, 2, \cdots, n$ a form with minimal number ρ. These results have application to the theory of equivalence and factorability of binary forms. (Received February 24, 1940.)

213. Rufus Oldenburger: *Polynomials in several variables.*

Various aspects of the theory of minimal representation of forms are developed in the present paper. It is proved in particular that for almost every field the range of the minimal number of forms of degree q in n essential variables overlaps the range of this number for forms of lower degree in the same number of essential variables. Adding a term λL^q, where L is linear and λ is in the given field, to a form of degree q changes the minimal number by at most 1. The coefficients in a minimal representation of a form comprise an invariant minimal class. Two forms are equivalent with respect to a given field if and only if their minimal classes with respect to this field are identical. If the minimal number of a form equals the number of essential variables in the form, the form has a unique minimal representation, and the problem of its equivalence to another form can be answered in a simple fashion. (Received February 27, 1940.)

214. Rufus Oldenburger: *On a class of non-negative matrices.*

In the present paper the existence of the infinite powers of a fairly general class of non-negative matrices arising in the Hardy Cross balancing process of engineering is proved. It follows that the associated balancings converge. (Received February 27, 1940.)

215. Gordon Pall: *On the arithmetic of quaternions.*

This article includes the results of abstract 39-5-137, somewhat generalized. Conditions for quaternions to have the same right or left factors of a given norm are obtained. Among other applications the relation between the number of classes of binary quadratic forms and representations by sums of three squares is derived in a simple way. (Received February 13, 1940.)

216. Everett Pitcher: *Homology groups under continuous maps.* I.

An analogue and extension of the Mayer-Vietoris formulas in the formulation of Alexandroff and Hopf (*Topologie*, I, pp. 287–299) is obtained as follows. Let f be a simplicial map on a complex K to a complex L which covers L. Call a chain of K *vanishing* if its image under f is null. Let U^k denote the group of vanishing k-cycles modulo the subgroup which bound vanishing chains. Let N^k be the subgroup consisting of those classes of cycles which bound on K. Let $B^k(K)$, $B^k(L)$ denote the groups of homology classes of k-cycles on K and L. Let S^k denote the group of homology classes of k-cycles of L which are images under f. Let T^k be the group of homology classes of k-cycles of K whose images under f bound. The formulas are the following. I. $B^k(L) \bmod S^k = N^{k-1}$. II. $B^k(K) \bmod T^k = S^k$. III. $U^k \bmod N^k = T^k$. Of course, II is well known. Applications and connections with familiar material seem numerous. (Received February 6, 1940.)

217. L. B. Robinson: *The solution of a functional equation on the contour of the unit circle.*

The functional equation (I) $u'(x) = a(x)u(x^2)$ admits a solution converging within the unit circle and on its contour. Also the equation (II) $u'(x) = a(x)\{u(x^2)\}^2$ admits a solution converging within the unit circle. Does this solution always converge on the contour? The following example shows that the answer is negative. A solution of the equation $u'(x) = (1+x)^2\{u(x^2)\}^2$ is $u(x) = 1/(1-x) = 1+x+x^2+\cdots$, and this series diverges on the contour. It is indeed possible to demonstrate that the solution of (II) is lacunary, but, as the above example shows, only after making the assumption $a(x) \neq 0$ when $|x| = 1$. (Received February 14, 1940.)

218. Wolfgang Sternberg: *The general limit theorem for probability densities.*

Let x_i ($i = 1, \cdots, n$) be independent chance variables, v_i their probability densities, a_i, b_i, c_i their mean values, standard deviations and absolute moments of the third order, respectively. Denote the probability density of $x_1 + \cdots + x_i$ ($i = 2, \cdots, n$) by w_i, mean value and standard deviation by A_i and B_i, respectively. In order to simplify the formulas, assume that $a_i=0$ for all i, so that $A_i=0$, and $B_n=1$. Then, n being a large number, $w_n(x)$ lies near the Gaussian probability density $\phi(x)=(2\pi)^{-1/2}e^{-x^2/2}$, if very general assumptions concerning the v_i are made. Assume that the quotients c_i/b_i are "very small" and that the $v_i(x)$ have continuous derivatives up to the fourth order for all x and state the theorem as follows: For every positive ϵ there exists a positive δ, such that the inequality $|w_n(x)-\phi(x)|<\epsilon$ holds uniformly in x if the conditions $c_i/b_i<\delta$ $(i=1, \cdots, n)$ are satisfied. To prove this, introduce a function $\psi(x, y)$ defined as the solution of $\partial\psi/\partial y=(1/2)\cdot\partial^2\psi/\partial x^2$ assuming the values $w_r(x)$ on the characteristic $y=B_r$. The number r is a certain integer $(1<r<n)$ depending on ϵ. It is shown first that $|\psi(x, 1)-\phi(x)|<\epsilon/2$ because B_r is a small number (this inequality has a simple physical meaning), and second, by means of the recursion formula of the $w_i(x)$ and the properties of $\psi(x, y)$, that $|w_n(x)-\psi(x, 1)|<\epsilon/2$. (Cf. abstract 45-9-380.) (Received February 3, 1940.)

219. H. S. Wall: *A class of power series bounded in the unit circle.*

Let $f(x)=c_0-c_1x+c_2x^2-\cdots$ wherein $c_0, c_1, c_2, \cdots$ is a totally monotone sequence. The radius of convergence of the series is at least 1 so that $f(x)$ is analytic for $|x|<1$. Let $M(f)$ be the least upper bound of $|f(x)|$ for $|x|<1$. It is shown that $M(f)$ is finite if and only if there exists a positive constant h such that $hf(x)$ has a continued fraction representation of the form $g_1/1+(1-g_1)g_2x/1+(1-g_2)g_3x/1+\cdots$, $0\leqq g_n\leqq 1$, $(n\leqq 1)$, it being agreed that the continued fraction terminates with the first vanishing partial quotient; or, if and only if $f(x)$ has a Stieltjes integral representation of the form $\int_0^1(1-u)d\phi(u)/(1+xu)$ in which $\phi(u)$ is monotone non-decreasing; or, if and only if the series $\sum c_i$ converges, whereupon $M(f)=\sum c_i$. If $M(f)\leqq 1$, $w=f(x)$ maps the region $|x|<1$ into the region $|w-1/2|<1/2$, and if w_0 is any point in the latter region, there exists a function $f(x)$ and a point x_0, $|x_0|<1$, such that $w_0=f(x_0)$. It is shown that if $M(f)\leqq 1$, the functions $f_{n+1}(x)=[\gamma_n-f_n(x)]/x[1-\gamma_nf(x)]$, $\gamma_n=f_n(0)$, $f_0=f$, $(n=0, 1, 2, \cdots)$, obtained by the algorithm used by Schur are all functions of the same character as $f(x)$. (Received February 5, 1940.)

220. J. L. Walsh: *Note on overconvergent power series.*

Ostrowski has proved the remarkable fact that in a power series $\sum a_nz^n$ possessing overconvergent partial sums, the sequence of coefficients must exhibit Hadamard gaps, that is to say, gaps whose relative lengths are bounded from zero. The present note emphasizes the fact that these gaps cannot be considered as sharply terminating. The coefficients have moduli decreasing gradually at the beginning of the gaps and increasing gradually at the end. In particular the radius of convergence of the series must be a ***non-isolated*** point of the derivative of the set $\{|a_n|^{-1/n}\}$. (Received February 7, 1940.)

221. G. T. Whyburn: *Mapping theorems.*

A slight modification of a theorem published by the author (American Journal of Mathematics, vol. 53 (1931), p. 753) is shown to yield not only the theorem that any compact locally connected continuum is the continuous image of an interval but also the proposition that any compact metric space is the continuous image of a closed 0-dimensional subset of an interval. Also it is noted that the latter of these results may be obtained readily from the former by an independent method. Further, the author's treatment of the Mazurkiewicz relative distance transformation (American

Journal of Mathematics, vol. 54 (1932), pp. 367–376) is modified so as to yield conclusions in general spaces from which it results that if R is a plane region with a connected boundary B there exists a complete space C of the form $C=S+A$ where $S\cdot A=0$ and a continuous transformation $W(C)=R+B'$ such that: (1) S maps topologically onto R; (2) A maps onto the set B' of all points of B which are accessible from R; (3) the transformation $W(A)=B'$ is non-alternating in a certain natural sense; (4) A can be mapped by a (1-1) continuous transformation onto a dense subset of a circle. (Received February 8, 1940.)

222. G. T. Whyburn: *On non-alternating transformations.*

A new treatment of this subject is given which makes possible the extension to semi-locally-connected compact continua of all major results originally established by the author (American Journal of Mathematics, vol. 56 (1934), pp. 298–300) for locally-connected continua and of some of the principal ones to arbitrary compact continua. For example, if $f(A)=B$ is non-alternating where A is a compact continuum, then for any simple link E_b in B there exists one and only one simple link E_a in A such that $f(E_a)\supset E_b$ and for any other simple link E_a' of A, $f(E_a')$ contains at most one point of E_b. Also it is shown that if A is connected and is either (a) compact and semi-locally-connected or (b) locally-connected, a continuous transformation $f(A)=B$ will be non-alternating if and only if (i) for each $y \in B$, $f^{-1}(y)$ contains every point which separates two points of $f^{-1}(y)$ in A and (ii) f is non-alternating on each cyclic element of A. (Received February 8, 1940.)

223. Y. K. Wong: *Interchange of J-processes in bilinear forms involving non-modular matrices.*

When κ^{12} is of type $\mathfrak{M}^1\overline{\mathfrak{M}}^2$ subject to the basis $(\mathfrak{A}^D, \mathfrak{P}^1, \mathfrak{P}^2, \epsilon^1, \epsilon^2)$, then κ^{12} is said to have the property I^{12} over $\mathfrak{M}_0^1$, $\mathfrak{M}_0^2$ (contained in $\mathfrak{M}^1$, $\mathfrak{M}^2$) in case the iterated integrals $J^1\overline{\mu}_0^1 J^2\kappa^{12}\mu_0^2$, $J^2(J^1\overline{\mu}_0^1\kappa^{12})\mu_0^2$ exist and are equal for every μ_0^1, μ_0^2 in the aforementioned subsets. Equivalent conditions for κ^{12} to have the property I^{12} defined above are obtained. By restricting $\mathfrak{M}_0^1$, $\mathfrak{M}_0^2$ to be the four pairs of subsets in the $(1\ \kappa^*\ 2)$-, $(2\ \kappa\ 1)$-domains, we have sixteen possible modes of I^{12} properties. It is proved that there are but four distinct modes of I^{12} properties, and their relations are discussed. The problem is then formulated in terms of idempotent matrices and positive hermitian modular matrices. Equivalent conditions in terms of certain double integrals are established. We then define the property $I(\epsilon^1\ \epsilon^2)$ for κ^{12} and show that $I(\epsilon_\kappa^1\ \epsilon_{\kappa^*}^2)$ and $I(\epsilon_{\kappa^*}^1\ \epsilon_\kappa^2)$ are equivalent properties. (Received February 1, 1940.)

224. Y. K. Wong: *On non-modular matrices.*

E. H. Moore's generalized Fourier theory shows that $\mathfrak{M}^{1\kappa}$, $\mathfrak{M}^{2\kappa}$ are in one-to-one orthogonal correspondence via $J^1\kappa^{*21}$, $J^2\kappa\kappa^{12}$. This paper studies the invariance of density and closure properties for corresponding subsets in $\mathfrak{M}^{1\kappa}$, $\mathfrak{M}^{2\kappa}$. By using the basis $(\mathfrak{A}^D, \mathfrak{P}^1, \mathfrak{P}^2, \epsilon^1, \epsilon^2, \kappa^{12}$ by columns of $\mathfrak{M}^1)$, the $(1\ \kappa^*\ 2)$-domain is defined to be the class of all μ^1 (modular as to ϵ^1) such that $J^1\kappa^{*21}\mu^1$ is modular as to ϵ^2. The intersection $\mathfrak{M}^{2\cap 2\kappa}$ of $\mathfrak{M}^2$ and $\mathfrak{M}^{2\kappa}$ is in one-to-one correspondence with the $(1_\kappa\ \kappa^*\ 2)$-domain. The $(1\ \kappa^*\ 2)$-domain is the linear extension of the logical sum of the $(1_\kappa\ \kappa^*\ 2)$-domain and the orthogonal complement of $\mathfrak{M}^{1\kappa}$. Density and closure properties of the $(1\ \kappa^*\ 2)$-domain and the set $\mathfrak{M}^{2\cap 2\kappa}$ are studied. When κ^{12} is of type $\mathfrak{M}^1\overline{\mathfrak{M}}^2$, the intersection $\mathfrak{M}^{1\kappa\cap 1\kappa^*}$ of $\mathfrak{M}^{1\kappa}$ and $\mathfrak{M}^{1\mu^*}$ and various domains associated with κ^{12} are discussed. (Received February 1, 1940.)

225. J. C. Abbott: *Projective theory of congruence in non-euclidean geometry.* Preliminary report.

Most assumptions made in non-euclidean geometry about congruence of segments can be derived from the projective postulates for parallelism and order of Jenks and the law of Pappus. If, in addition, Pascal's law is assumed for asymptotic hexagons, then also non-euclidean properties of perpendicularity and congruence of angles can be derived which, without this assumption, break down. The theory of congruence of segments, however, is independent of this assumption. (Received March 12, 1940.)

226. R. P. Agnew: *Some remarks on a paper entitled "General Tauberian theorems."*

The Tauberian classes T, T', T_α, T'_α, T_β, and T'_β defined by H. R. Pitt (*General Tauberian theorems*, Proceedings of the London Mathematical Society, (2), vol. 44 (1938), pp. 243–288) each fail to have the following property: if $s(x)$ is a function in the class, then, for each constant A, the function $s(x)-A$ is in the class. This fact has a bearing on the connection between the Tauberian oscillation theorems and the Tauberian convergence theorems of Pitt's paper. (Received March 11, 1940.)

227. S. P. Avann: *Lattice automorphisms.*

It is shown in this paper that every finite abelian group is the automorphism group of some lattice, in fact of a finite distributive lattice. A necessary and sufficient condition is obtained that a point lattice have the symmetric group as its automorphism group. The group of a distributive lattice is considered from the standpoint of symmetrically placed generators. Also included is a test for modularity of a lattice in terms of the number of coverings of the elements. Further study of lattice automorphisms is in progress by the author. (Received March 7, 1940.)

228. Reinhold Baer: *Nets and groups.* II.

The nets, derived from division systems with unit, satisfying $x(yz)=1$ if, and only if $(xy)z=1$, are characterized by certain local symmetry properties. (Received March 11, 1940.)

229. Reinhold Baer: *Sylow theorems for infinite groups.*

This paper investigates the results which one obtains in applying the methods used for proving the theorems of the Sylow type upon infinite groups. Particular attention is given to locally finite groups, that is, groups all of whose finite subsets are contained in finite and normal subgroups, since for these groups certain conditions turn out to be necessary which in general are only sufficient, but not necessary. (Received March 11, 1940.)

230. R. C. F. Bartels and R. V. Churchill: *An extension of Duhamel's theorem.*

The method of the Laplace transformation is applied in establishing an extension of Duhamel's theorem to a general boundary-value problem with a partial differential equation of the parabolic type having discontinuous coefficients. It is shown that the solution of a problem having boundary conditions dependent on the variable time can be expressed in terms of the solution of a simpler problem with fixed boundary conditions. Conditions on the solution of the latter problem which are sufficient to insure a solution of the general problem are discussed. The derivation makes use of a generalization of the composition integral (Faltung) in the theory of the Laplace transformation. Some properties of this generalized form are considered. (Received March 15, 1940.)

231. Harry Bateman: *Hulthen's integral equation*. Preliminary report.

Let $V(x, y)$ be a potential function regular for $y \geqq 0$. The integral equation in question is for the determination of $V(x, 0)$ and may be written in the form $f(x) = V(x, 0) + kV(x, a)$ where $a > 0$ and $f(x)$ is continuous for real values of x. If $U(x, 0)$ is the potential regular for $y \geqq 0$ which is equal to $f(x)$ for $y = 0$, the solution obtained by the method of successive approximations is $V(x, 0) = U(x, 0) - kU(x, a) + k^2U(x, 2a) - \cdots$ and if $|f(x)| \leqq f$ this series is convergent for $|k| < 1$. The case $k = 1$ may be treated by expanding $V(x, y)$ in a trigonometric series of multiples of $2\pi y/a$, but the sine terms are left undetermined by the equation. The homogeneous integral equation has in this case solutions and the nonhomogeneous equation has solutions only if $f(x)$ is a function of a certain form. Potential problems in which a correspondence between the points of two boundaries occurs in the boundary condition give rise to an interesting type of integral equation. (Received March 16, 1940.)

232. R. A. Beaumont: *Projections of non-abelian groups upon abelian groups*. Preliminary report.

A function f of the subgroups of a group G upon the subgroups of a group H is called a projectivity of G upon H if f is a single valued monotone increasing function with a single valued inverse. Groups which have projections upon abelian groups naturally fall into two classes: groups without elements of infinite order, and groups with elements of infinite order. In this paper the latter are studied. Baer has shown that if an abelian group G contains at least two independent elements of infinite order, then every projection of G is induced by an isomorphism. Hence in a survey of groups containing elements of infinite order which are projective with abelian groups, only abelian groups G where $G/F(G)$ is of rank one need be considered. The author gives a survey of such groups. The discussion of the case where $G/F(G)$ is an infinite cyclic group is important for the general problem. (Received March 12, 1940.)

233. E. T. Bell: *Note on a certain type of diophantine system*.

The system is $a_iP_i(x) = c_iy_i^{n_i}$, $(i = 1, \cdots, r)$, in which a_i, c_i, n_i are any given constant integers greater than 0; $P_i(x)$ is a polynomial of degree m_i (greater than 0) in x with integer coefficients; $P_i(x)$ has no constant term; the coefficient of the lowest power of x in $P_i(x)$ is b_i. Necessary and sufficient conditions that the system have a solution in integers $x, y_1, \cdots, y_r$ are obtained. These conditions concern $b_1, \cdots, b_r$ alone; when they are satisfied, solutions are immediately obtainable. Very special cases of such systems have been discussed by several writers, without attention to necessity or sufficiency. (Received March 11, 1940.)

234. E. T. Bell: *Postulational basis for the umbral calculus*.

As the umbral calculus for vectors has been misunderstood, a full postulational treatment is given. (Received March 11, 1940.)

235. Stefan Bergman: *On the surface integrals of functions of two complex variables.*

The author considers functions $f(z_1, z_2)$ of two complex variables meromorphic in domains $\mathfrak{M}^4$, $\mathfrak{M}^4$ being bounded by a finite number of segments of analytic hypersurfaces (see, for details, Mathematische Zeitschrift, vol. 39 (1937), p. 76, and Recueil Mathématique, vol. 1 (43) (1936), p. 851). Each domain $\mathfrak{M}^4$ possesses a distinguished boundary surface $\mathfrak{F}^2$. From an extension of the methods used in the papers mentioned, it can be shown that a surface integral over $f(z_1, z_2)\chi(z_1, z_2)$ along $\mathfrak{F}^2$ is equal to the sum $\sum_\nu R(P_\nu; f)$ of expressions (residues) $R(P_\nu; f)$ which are associated with certain points P_ν lying in $\mathfrak{M}^4$ and on the singularity surface of f. The function $\chi(z_1, z_2)$ is a weight function which depends upon $\mathfrak{M}^4$ and upon the intersection of $f(z_1, z_2)=0$ with the boundary of $\mathfrak{M}^4$. (Received March 25, 1940.)

236. B. A. Bernstein: *Groups in terms of addition and negation.*

The author defines groups and abelian groups by sets of postulates expressed in terms of addition and negation. The postulates are simple and readily yield the basic propositions of groups. The sets for abelian groups embrace, in each case, a set for groups in general. (Received March 8, 1940.)

237. R. P. Boas: *Expansions of analytic functions.*

This paper is a contribution to the problem of expanding an analytic function $f(z)$ in a series of the form $\sum c_n z^n [1+h_n(z)]$, where the $h_n(z)$ are "small." (For references to the literature, see G. S. Ketchum, Transactions of this Society, vol. 40 (1936), pp. 208–224.) The central problem is to obtain as large a circle of convergence as possible for this expansion. Most of the known criteria are obtained as consequences of a new one: if the functions $h_n(z)$ are analytic in $|z| < r$, vanish at $z=0$, and have (for large n) a common majorant $h(z)$, and if $f(z)$ is analytic in $|z| < s$, the above series represents $f(z)$ in $|z| < \min(s, t)$ if $h(t) < 1$. The method used in this paper avoids rearrangements of power series, using instead arguments depending on Cauchy's theorem. Applications are made to the study of the distribution of the values taken by an analytic function and its derivatives. A more detailed abstract of part of the paper has appeared in the Proceedings of the National Academy of Sciences, vol. 26 (1940), pp. 139–143. (Received March 18, 1940.)

238. R. P. Boas: *Univalent derivatives of entire functions.*

The following theorem is proved: If $f(z)$ is a transcendental entire function of order one and exponential type k (that is, if $|f(z)| < C \exp(k|z|)$), and $k < \log 2$, then an infinite number of the derivatives of $f(z)$ are univalent in $|z| < 1$. The proof depends on an expansion theorem of the type described in the preceding abstract. (Received March 18, 1940.)

239. Richard Brauer: *On the Cartan invariants of a group of finite order.*

E. Cartan introduced an important set of invariants $c_{\kappa\lambda}$ ($\kappa, \lambda=1, 2, \cdots, k$) of an algebra with a principal unit. If G is a group of finite order, these $c_{\kappa\lambda}$ may be formed for the group ring of G taking the field of reference as a modular field K. It is shown that the determinant $|c_{\kappa\lambda}|$ is a power of the characteristic p of K, and the exponent of p is determined. (Received March 27, 1940.)

240. J. L. Brenner: *Well-orderable abelian groups whose elements have prime-power order.*

The author gives a new, short proof of the Kronecker decomposition theorem for finite abelian groups and a partial decomposition theorem for arbitrary torsion groups. The elements of the basis are picked out one by one. (Received January 19, 1940.)

241. H. H. Campaigne: *A lower limit on the number of hypergroups of a given order.*

Let ν_ω be the number of hypergroups of order ω. Then it can be shown that $\nu_{\omega+1} > 7\nu_\omega$. The method of proof is to give constructions, in terms of an arbitrary hypergroup G of order ω, for seven different hypergroups of orders $\omega+1$. Six of these have G as a subhypergroup. As a corollary we have $\nu_\omega > 7^{\omega-1}$. (Received March 15, 1940.)

242. H. H. Campaigne: *Multiplication systems.* Preliminary report.

A set M of elements $m, n, \cdots$ is a multiplication system if for every pair of elements m, n there is uniquely determined a subset mn of M. This is a generalization of the notion of group. The multiplication system C is complementary to M if the elements of C are in one-to-one correspondence, $c_\eta \longleftrightarrow m_\eta$, with those of M such that $c_\lambda \varepsilon c_\mu c_\nu$ if and only if m_λ is not in $m_\mu m_\nu$. It is evident that M is isomorphic to M' if and only if C is isomorphic to C'. The set of all automorphisms of M forms a group. The group of automorphisms of C is isomorphic to that of M. Conditions on M are found that are necessary and sufficient that C be a hypergroup. It is shown that these conditions are satisfied by groups, quasi-groups, Mischgruppen, Brandt groupoids, and defective groups. Thus we have a method of uniquely representing these systems by means of hypergroups. (Received March 15, 1940.)

243. Leonard Carlitz: *A set of polynomials.*

Put $f_m(t) = \prod_c (t + c_0 u_0 + \cdots + c_{m-1} u_{m-1})$, the product extending over all sets $(c_0, \cdots, c_{m-1})$ in the finite field $GF(p^n)$; $t, u_0, \cdots, u_{m-1}$ are indeterminates. For $u_i = x^i$, $f_m(t)$ becomes $\psi_m(t)$ (Duke Mathematical Journal, vol. 1 (1935), p. 137). The purpose of the present note is to extend known formulas involving ψ_m to the more general f_m. In particular, an explicit formula representing an arbitrary "linear" polynomial $\sum \alpha_i t^{p^{ni}}$ is derived in terms of $f_m(t)$; also the inverse function to f_m is constructed. Application is made to the evaluation of $\sum (c_0 u_0 + \cdots + c_m u_m)^{p^{nk}-1}$ and $\sum (c_0 u_0 + \cdots + c_m u_m)^{-(p^{nk}-1)}$. (Received March 28, 1940.)

244. Leonard Carlitz: *On certain exponential sums.*

Certain sums, which may be briefly described as generalized Gauss sums, are defined relative to a special modular system $M = (m, A_1, \cdots, A_k)$, where A_k is a polynomial in k indeterminates. The generalized sums are reduced to ordinary sums by means of a method used previously in a special case; a generalized Jacobi symbol (quadratic character) appears incidentally. (Received March 28, 1940.)

245. Randolph Church: *Numerical analysis of certain free distributive structures.*

The elements of the free distributive structure based on n elements fall into disjoint sets of conjugates under the permutations of the symmetric group of degree n.

The 208 sets of conjugates and the structure inclusion relations among the elements constituting them have been obtained for $n=5$. The number of conjugate elements in these sets, and their rank in the structure, are here presented for $n\leqq 5$. (Received March 28, 1940.)

246. R. V. Churchill: *A problem in the conduction of heat.*

The following physical problem is treated. Find the temperatures in a semi-infinite solid $x\geqq 0$ composed of a layer $0\leqq x\leqq a$ of one material initially at uniform temperature A in contact with a semi-infinite solid $x\geqq a$ of another material initially at temperature zero, when the face $x=0$ is kept insulated. A complete formulation of this problem as a boundary value problem is determined so that just one solution exists. The temperature formula is obtained in the form of a simple and practical series whose terms involve probability integrals. It is completely established as the solution. The Laplace transformation is used both to obtain the solution and to establish its uniqueness. (Received March 15, 1940.)

247. R. V. Churchill: *Termwise integration of Sturm-Liouville expansions*. Preliminary report.

The following theorem is established. Let $y=\phi_n(x)$, $(n=1, 2, \cdots)$, be the normalized characteristic functions of the Sturm-Liouville system $y''-[q(x)+\lambda]y=0$, $y(0, \lambda)=0$, $y(1, \lambda)=0$, where $q(x)$ is continuous in x in the interval (0, 1). Let $c_n=\int_0^1 F(x)\phi_n(x)dx$, where $F(x)$ is any bounded and Riemann integrable function. Then for any function $G(x)$ of bounded variation the series $\sum_1^\infty c_n\int_0^x G(x)\phi_n(x)dx$ converges uniformly for all x in the interval (0, 1) to the function $\int_0^x F(x)G(x)dx$. The Parséval equality for this system follows at once: $\sum_1^\infty c_n d_n=\int_0^1 F(x)G(x)dx$, where $d_n=\int_0^1 G(x)\phi_n(x)dx$. The method of proof is that of integration in the complex plane of the parameter λ. The result is extended to the termwise integration of the expansion based on the equation $(ry')'-[q+\lambda p]y=0$. (Received March 15, 1940.)

248. Paul Civin: *Inequalities for trigonometric integrals*. Preliminary report.

This paper concerns transforms of functions expressible as $f(x)=\int_{-R}^{R}e^{ixt}d\alpha(t)$, where $\alpha(t)$ is a function of bounded variation, and $|f(x)|\leqq M$ for all x. These transforms are of the form $T[f(x)]=g(x)=\int_{-R}\mu(t)e^{ixt}d\alpha(t)$. The problem is to find a bound for $|g(x)|$ depending only on $\mu(t)$ and M. The results obtained include generalizations of many of those in the literature for trigonometric polynomials: for example, Bernstein's theorem (where $g(x)=f'(x)$), results of G. Sokolov (Bulletin de l'Académie des Sciences de l'URSS, (7), Classe des Sciences Mathématiques et Naturelles, 1935, pp. 857–882), and results of G. Szegö (Schriften der Königsberger gelehrten Gesellschaft, vol. 5 (1928), pp. 59–70). In particular, a bound is obtained for the fractional derivative $f^{(\alpha)}(x)$, $0<\alpha<1$; $f^{(\alpha)}(x)=T[f(x)]$ where the multiplier $\mu(t)=(it)^\alpha$. This bound is $(4\alpha^{-1}+2)R^\alpha M$. A more precise but more complicated bound is also obtained. (Received March 25, 1940.)

249. A. G. Clark: *Two sample problem concerning Poisson and binomial distributions*. Preliminary report.

In an article published in Biometrika, 1939, J. Przyborowski and H. Wilenski have developed a test based upon best critical regions of the hypothesis that two variables with distributions given by the Poisson exponential limit have equal means.

The author has independently obtained some of the results given in the article quoted and has developed more extensive tables for the power of the test in both its symmetric and asymmetric forms. In addition, the effect of the relative size of the two samples upon the power of the test is determined. A corresponding test is constructed where the distribution of the variables is binomial. In this case the power of the test is expressed as a finite hypergeometric series. It is shown how the critical region may be identified when the samples are of equal size. It is hoped to find a feasible method of calculating the power of the test in the binomial case. (Received March 12, 1940.)

250. J. M. Clarkson: *An involutorial line transformation associated with a quadric.*

Given a fixed quadric H and a point O not on H, an arbitrary line t meets H in two points P_1, P_2 which are projected from O into two points P_1', P_2' on H. The line $t' \equiv P_1' P_2'$ is the transform of t by the line transformation T. The transformation T is involutorial. The invariant lines and the singular lines of T are discussed, and also the changes in the configurations when H is a cone and when H is composite. (Received March 22, 1940.)

251. Nancy Cole: *The index theorem for a calculus of variations problem in which the integrand is discontinuous.*

In this paper Morse's index theorem (Duke Mathematical Journal, vol. 4 (1938), pp. 231–246) is established for a problem in euclidean m-space in which the integrand is discontinuous. The basic curve g is a broken extremal with a finite number of corners, at each of which g is cut across by a regular $(m-1)$ manifold of class C^2, not tangent to either arc of g at the corner. At each corner g satisfies a set of "primary incidence relations." The conjugate points of a fixed point a of g are defined in terms of the zeros of the conjugate determinant $D_g(t, a)$, whose m columns represent certain linearly independent solutions of the Jacobi equations determined by g. It is also proved that if a and b are any two fixed points of g, the numbers of zeros of $D_g(t, a)$ and $D_g(t, b)$ on any interval (open or closed) of g differ by at most $m-1$. (Received March 28, 1940.)

252. H. S. M. Coxeter: *The binary polyhedral groups, and other generalizations of the quaternion group.*

The chief result of this paper is that the (largest) group defined by $R^l = S^m = T^n = RST$ is of order $4|l^{-1}+m^{-1}+n^{-1}-1|(|l|^{-1}+|m|^{-1}+|n|^{-1}-1)^{-2}$, whenever $|l|$, $|m|$, $|n|$ and $|l|^{-1}+|m|^{-1}+|n|^{-1}$ are all greater than 1. (This covers all non-trivial cases, since the group is obviously infinite when $|l|^{-1}+|m|^{-1}+|n|^{-1} \leq 1$.) Well known cases of this group $\langle l, m, n\rangle$ are the dicyclic groups $\langle 2, 2, n\rangle$ (when $n=2$, the quaternion group), and the binary polyhedral groups $\langle 2, 3, n\rangle$ ($n=3, 4, 5$). The investigation of $\langle 2, 3, n\rangle$ for negative n was proposed by W. Threlfall as a problem (Jahresbericht der Deutschen Mathematiker-Vereinigung, vol. 46 (1936), p. 80). In most cases $\langle l, m, n\rangle$ is the direct product of $\langle |l|, |m|, |n|\rangle$ and a cyclic group; an interesting exception is the group $\langle 2, 3, -3\rangle$, of order 72. (Received March 29, 1940.)

253. H. B. Curry: *A formulation of recursive arithmetic.*

The term "recursive arithmetic," as used here, means the theory of the natural integers developed by the use of primitive recursive definitions; the emphasis is not so much on the construction of such definitions (as in work by Gödel, Péter and

Kleene) as on the technique of proving equations involving such functions and free variables. Previous formalizations of this sort of arithmetic (for example, that by Skolem and Hilbert-Bernays) have been based on a logical calculus, which is regarded as already formulated. In the present paper the theory is set up in a rigorously formal manner as an entirely independent logical system; it is then shown that the rules of the logical calculus are valid in the system, and that the system is equivalent to that of Hilbert-Bernays. (Received March 9, 1940.)

254. G. B. Dantzig: *Existence of the Neyman-Pearson unbiased test of type* A.

Consider the case where it is known that the elementary probability law $p(E|\theta)$ of a system E of n observable variables depends on only one parameter θ, the value of which is unknown, and where it is desired to test a hypothesis H which assumes that $\theta=\theta_0$. Let w denote a region which is used as a critical region to test H, such that, if $E\,\varepsilon\, w$, then H is rejected. Denote by $\beta(\theta)$ the power function of w, that is, $\beta(\theta)$ is the probability of $E\,\varepsilon\, w$ calculated under the assumption that θ is the true value of the unknown parameter. The region w is said to be unbiased of type A (Newman-Pearson, Statistical Research Memoirs, vol. 1 (1936), pp. 1–37, and ibid., vol. 2 (1938)), if it satisfies the following conditions: (1) $\beta(\theta_0)=\alpha$, (2) $d\beta/d\theta|_{\theta_0}=0$ and (3) $d^2\beta/d\theta^2|_{\theta_0}$ has for the region w a value at least equal to that corresponding to any other region w having properties (1) and (2). The existence of unbiased critical regions of type A was proved by Neyman and Pearson for a particular type of functions $p(E|\theta)$. This result is now extended by the present author to all elementary probability laws $p(E|\theta)$ which admit two differentiations under the sign of the integral over any fixed region. (Received March 9, 1940.)

255. G. B. Dantzig: *On the non-existence of tests of "Student's" hypothesis, the power function of which would be independent of* σ.

One is given a sample of n independent, normally distributed variables with a common mean ξ and a standard deviation σ. It is wished to test the hypothesis that the mean ξ is 0 when the standard deviation σ is unknown. Student's test consists in rejecting this hypothesis whenever the ratio of the mean of the sample, $\bar{x}$, to the standard deviation of the sample, s, is larger than a fixed value. The important property of this test is that the probability of rejecting the hypothesis $\xi=0$, when it is true, is independent of the unknown standard deviation σ. On the other hand, the probability of rejecting the hypothesis when some alternative hypothesis $\xi=\xi_1$ is true, depends on the value of σ. This causes difficulties in applications. It is natural, then, to consider whether there exists some other test such that the latter probability is independent of the unknown σ. This problem reduces to finding a region w in the sample space $(x_1, x_2, \cdots, x_n)$ such that the integral over w of the function $p=(\sigma(2\pi)^{1/2})^{-n}\exp\left[-\sum_{i=1}^{n}(x_i-\xi)^2/2\sigma^2\right]$ is independent of σ whatever be the value of ξ. It appears, however, that the integral of p over any such region is also independent of ξ. From the point of view of statistics such a region is unsuitable for testing the hypothesis that $\xi=0$, because it will reject this hypothesis when it is wrong and when it is correct equally frequently. (Received March 9, 1940.)

256. P. H. Daus: *Bisecting circles.*

This paper discusses from a geometric point of view the circles which bisect and the circles which are bisected by a given circle. Attention is given to the one-parameter families which bisect two circles, which are bisected by two circles, and other families which involve other conditions such as orthogonality and tangency. The paper considers the circles determined by three such conditions including, as one or more conditions, these bisection properties. It enumerates the linear, quadratic and quartic cases, and in the latter case distinguishes between those which are constructable by ruler and compass and those which are not. (Received March 6, 1940.)

257. M. M. Day: *A norm ergodic theorem.*

With fewer restrictions than in other theorems of this type, a theorem using the method of F. Riesz is proved which includes results of F. Riesz (Journal of the London Mathematical Society, vol. 13 (1938)), Cohen (this Bulletin, abstract 45-5-169), Yosida (Proceedings of the Imperial Academy of Japan, Tokyo, vol. 14 (1938)), and Dunford (Duke Mathematical Journal, vol. 5 (1939)). In fact a simple lemma on convex bodies in euclidean n-space used with the main theorem shows that the arithmetic means used by Dunford can as well be taken over all convex sets of finite nonzero measure ordered by the radius of the largest inscribed sphere rather than only over cubes ordered by edge length. The generality of the theorem lies in the greater latitude allowed in the choice of the parameter set and of the transformations used, since, as Cohen showed for sequences, the arithmetic mean turns out to be only one of a class of transformations for which the norm or "mean" ergodic theorem holds. (Received March 28, 1940.)

258. M. M. Day: *Linear methods of summability.* II.

In a previous paper the author considered the problem of extending the Silverman-Toeplitz theorem to more general limits than those of sequences. These results are now extended to functions on a directed set Y with values in a fixed Banach space B. The extension of the following theorem of Vulich (Kharkov Communications, 1938) is also considered: If B is a Banach space, if $\{a_{mn}\}$ is a matrix of real numbers, and if $U_m(f)=\sum_{n=1}^{\infty}a_{mn}f_n$, where $f=\{f_n\}$ is a convergent sequence of points of B, then $\lim_{m\to\infty} U_m(f)=\lim_{n\to\infty} f_n$ for every convergent sequence of points of B if and only if $\{a_{mn}\}$ satisfies the Toeplitz conditions; that is, regularity on real convergent sequences is equivalent to regularity on every sequence of points from every Banach space: The methods used in the investigation include some study of topologies in spaces of linear operators, and the theory of order among directed sets due to Tukey (Princeton University thesis, 1939). Also, the definition and some properties of a simple sort of a completely additive integral, analogous to the finitely additive integral of Gowurin (Fundamenta Mathematicae, vol. 27 (1936)), are given, and a regularity criterion involving it is given for use in a subsequent paper. (Received March 28, 1940.)

259. J. J. DeCicco: *The affinelong near-Laguerre transformations.*

This paper is concerned with the group of line transformations of the complex plane with respect to the maximum number of circles preserved. A line correspondence is (I) magnilong if it magnifies by a nonzero constant the distance between the two points of contact on the common tangent line of any two curves, (II) affinelong if it is a non-magnilong correspondence which preserves the ∞^1 parallel pencils of lines, (III) general if it is not of types (I) or (II). A general (affinelong, magnilong) transformation preserves at most $2\infty^2$ (∞^2, ∞^1) circles. (*The analogue of the Moebius group in the Kasner plane*, this Bulletin, vol. 45 (1939), pp. 936–943.) In abstract

46-1-59 are found all the magnilong correspondences which preserve exactly ∞^1 circles. The author has now succeeded in finding *all* the affinelong near-Laguerre transformations. The family preserved is a certain quadratic family F which under the group of Laguerre transformations may be classified into the three distinct types: (A) the ∞^2 circles tangent to a fixed line. (B) The ∞^2 circles of the ∞^1 linear pencils (in line geometry) which contain a fixed circle and whose vertices are on a fixed line. (C) The ∞^2 circles tangent to a fixed circle. (Received March 20, 1940.)

260. E. L. Dodd: *The substitutive mean and certain subclasses of this general mean.*

Chisini gave a very general definition of a mean, $m = f_n(x_1, x_2, \cdots, x_n)$, which required that $f_n(c, c, \cdots, c) = c$; but did not require that $\min x_i \leqq m \leqq \max x_i$. This may be called a substitution mean. Nagumo, Kolmogoroff, and B. de Finetti treated the associative mean f_n which remains invariant when each element of a set of r of its elements, with $r < n$, is replaced by the f_r for that set. As a large subclass of substitutive means, the author defines a summational mean as a solution for y of an equation of the form $F\{y, \sum f_1(c_i x_i, y), \cdots, f_k(c_i x_i, y)\} = 0$; and defines, as a subclass of summational means, the quasi-arithmetic mean m determined from $\sum c_i \psi(m) = \sum c_i \psi(x_i)$, with $\sum c_i \neq 0$. The associative mean is then a special case of the quasi-arithmetic mean. By transformations, some functions become means of certain functions of the given elements. When this is applied to curve-fitting with Pearson types, various kinds of means emerge. (Received March 18, 1940.)

261. J. L. Doob and R. A. Leibler: *On the spectral analysis of a certain transformation.* Preliminary report.

The analysis of a one-to-one measure preserving transformation T on an abstract space Ω by means of the spectrum of a corresponding operator U goes back to Koopman. In the present paper a particular such transformation is studied in detail: it is supposed that there is a sequence $\cdots, f_{-1}(\omega), f_0(\omega), f_1(\omega), \cdots$ of functions, such that any measurable function can be expressed as a function of the $f_j(\omega)$, and such that $f_j(T\omega) = f_{j+1}(\omega)$. This transformation is of importance in probability. The operator U transforms ω-functions in L_2 into themselves taking $f_j(\omega)$ into $f_{j+1}(\omega)$. It is found that if $E(\lambda)$ is the resolution of the identity of U $(0 \leqq \lambda \leqq 2\pi)$, then $(E(\lambda)f(\omega), g(\omega))$ is absolutely continuous except for a possible jump at $\lambda = 0$. Necessary and sufficient conditions on $E(\lambda)$ are found that $E(\lambda)$ be the resolution of the identity of an operator corresponding to such a transformation T. (Received March 15, 1940.)

262. Jesse Douglas: *A new special form of the linear element of a surface.*

This paper will be published in full in the Transactions of this Society. (Received March 21, 1940.)

263. F. W. Dresch: *A mathematical model of a dynamic economic system.*

A static economic model due to G. C. Evans is made dynamic by introducing as additional variables the amounts of fixed capital tied up in the production processes of the economy. Assumed production functions and the hypothesis of strict competition permit all quantity variables (those giving rates of production, rates of employment of factors of production and the allocation of such factors to alternative production processes) to be expressed as functions of prices (including the interest rate) and their time-derivatives. Introducing a demand function for consumption goods, a supply function for labor and assuming the value of money constant, one may express all variables as functions of time by equating supply and demand. The expressions thus obtained may be regarded as representing a trend situation for the system. In the general situation for which supply is not equal to demand and for which a variation in stocks is thus possible, the traditional assumption that the rate of change of prices is related to the accumulaton of corresponding stocks (or, in the case of labor, to unemployment) is sufficient to again permit all variables to be expressed as functions of time. If one may regard the general situation as a perturbation of the trend situation, the parameters of the system may be estimated from available economic data provided that the empirical functions entering the equations of the model may be assumed of suitable forms. (Received March 9, 1940.)

264. D. M. Dribin: *The norm residue symbol for solvable algebraic number fields.*

Let $K|k$ be a solvable algebraic number field and $K \supset K_{t-1} \supset \cdots \supset K_1 \supset k$ be a chain of subfields of $K|k$ such that $K_{i+1}|K_i$ is abelian. In terms of the generalized Frobenius-Artin symbol $\{K/\mathfrak{p}\}$ (which is a certain unordered vector composed of sets of automorphisms of $K|k$) and a prescribed definition for $\{K/\mathfrak{a}\}$, where $\mathfrak{a}$ is a composite ideal in k (given by means of a definite ordering $\mathfrak{o}$ of the symbols $\{K/\mathfrak{p}\}$), the generalized Hilbert norm residue symbol $\{\beta, K/\mathfrak{p}\}$ can be defined. It enjoys the usual properties, save that the symbol $\{\beta_1\beta_2, K/\mathfrak{p}\}$ is not in general the same as $\{\beta_1, K/\mathfrak{p}\}\{\beta_2, K/\mathfrak{p}\}$, since multiplication of the symbols is non-abelian. If $\mathfrak{F}_i$ is the conductor of $K_{i+1}|K_i$ and $\mathfrak{f}=N_{K|k}(\prod_{i=0}^{t-1}\mathfrak{F}_i)$, then $\{\beta, K/\mathfrak{p})=1$ if and only if β is a norm residue modulo $\mathfrak{f}_{\mathfrak{p}}$, where $\mathfrak{f}_{\mathfrak{p}}$ is the $\mathfrak{p}$-contribution of $\mathfrak{f}$. (Received March 8, 1940.)

265. R. J. Duffin and J. J. Eachus: *The converse of a closure theorem of Paley and Wiener.*

Paley and Wiener (*Fourier Transforms in the Complex Domain*, American Mathematical Society Colloquium Publications, vol. 19, p. 100) have formulated a criterion for a set of functions $g_n(x)$ to lie close, on the average, to a given orthonormal set $f_n(x)$. From this criterion they show that the functions $g_n(x)$ approximately satisfy Parseval's equation. The authors show that, conversely, the validity of this latter condition on a set of functions $g_n(x)$ is sufficient to insure the existence of at least one orthonormal set $f_n(x)$ which satisfies the original criterion. They prove that the closure of the set $f_n(x)$ implies the closure of the set $g_n(x)$; it is found that the closure of the set $g_n(x)$ implies the closure of $f_n(x)$. (Received March 12, 1940.)

266. Ben Dushnik and E. W. Miller: *On partially ordered sets.*

Circumstances under which a partially ordered set contains an infinite (or non-denumerable) linearly ordered set are investigated. The work is carried out using associated families of sets, and leads to certain questions about real functions of a real variable. The idea of a reversible partial order is introduced. A partial order P is called reversible if "precedes" can be redefined so that, in the resulting partial order P', any two elements a and b are comparable if and only if they are not comparable in P. Not all partial orders are reversible. Necessary and sufficient conditions that a partial order be reversible are given. (Received March 15, 1940.)

267. Samuel Eilenberg: *On spherical cycles.*

A singular n-dimensional cycle of a space X with integer coefficients is called spherical if it is homologous in X to a singular n-sphere. The following two theorems are proved for any r-dimensional simplicial manifold M^r. (1) Given any $(r-n-1)$-dimensional subpolyhedron P^{r-n-1} of M^r, every n-cycle of M^r-P^{r-n-1} which bounds in M^r is spherical. (2) If $r>2n$ then every spherical n-cycle of M^r is homologous to an n-sphere topologically imbedded into M^r, simplicially with respect to some subdivision of M^r. (Received March 13, 1940.)

268. Samuel Eilenberg: *Ordered topological spaces.*

A topological space X is called ordered if an order-relation $<$ for the points of X is given such that: given $x<y$ there are neighborhoods $U(x)$ of x and $U(y)$ of y such that $x<y'$ and $x'<y$ whenever $x' \varepsilon U(x)$ and $y' \varepsilon U(y)$. It is proved that a connected nondegenerate space X can be ordered if and only if the subset $P(X)$ of the cartesian product $X\times X$ consisting of all points $(x, y) \varepsilon X\times X$ such that $x\neq y$ is not connected. The uniqueness of the order in a connected space also is established. Some other questions linking order and topology are discussed. The following application is given: a separable connected locally connected nondegenerate space X is homeomorphic with a subset of the linear continuum if and only if $P(X)$ is not connected. (Received March 13, 1940.)

269. Samuel Eilenberg and R. L. Wilder: *Uniform local contractibility.*

Domains of the n-sphere S_n, which have uniform local connectedness properties in the homology sense have been investigated by Wilder in earlier papers. The present paper is concerned with the stronger property of uniform local contractibility (ULC). Although the results obtained hold for more general spaces, the following summary will suffice to disclose their nature: Let D be a domain of S_n, $M=\bar{D}-D$, and E the interval $0\leqq t\leqq 1$. Then in order that D be ULC it is necessary and sufficient that there exist a continuous mapping $f(D\times E)\subset\bar{D}$ such that $f(x, 0)=x$ and $f(x, t) \varepsilon D$ for $x \varepsilon \bar{D}$ and $t>0$. Consequently if D is ULC then $\bar{D}$ is LC and M is deformation-free in the sense of Wilder (Fundamenta Mathematicae, vol. 21 (1933), p. 163). If D is ULC, M is LC, and the Poincaré group $\pi_1(M)=0$, then $\pi_1(\bar{D})=\pi_1(D)=0$. In particular if M is a topological image of S_{n-1} and D is ULC then $\pi_1(\bar{D})=\pi_1(D)=0$ and $\bar{D}$ can be represented as a *membrane* in the following way: Let Q_n be an open n-element with S_{n-1} as boundary; there is a continuous mapping $f(\bar{Q}_n)=\bar{D}$ such that $f(S_{n-1})=M$ is a homeomorphism and $f(Q_n)=D$. Thus in the case of the Alexander examples (Proceedings of the National Academy of Sciences, vol. 10 (1924), pp. 8–12) the D for which $\pi_1(D)\neq 0$ is not ULC. (Received March 28, 1940.)

270. J. M. Feld: *Whirl-similitudes, euclidean kinematics and non-euclidean geometry.* Preliminary report.

Kasner's whirl group G_3 (American Journal of Mathematics, vol. 33, p. 193) is enlarged to a mixed group $\bar{G}_3$ composed of 4 distinct continuous families of line element transformations by the introduction of the concepts: non-direct whirl, direct whirl correlation and non-direct whirl correlation. Likewise Kasner's whirl-motion group G_6 is enlarged, first to a mixed $\bar{G}_6$ and then to a $\bar{G}_7$ (complete whirl-similitude group). The groups $\bar{G}_6$ and $\bar{G}_7$ are each composed of 8 distinct continuous families of line element transformations; Kasner's G_6 being one of the families in $\overline{G}_6$. The geometry of line elements, flat fields and turbines is investigated and two transfer principles are presented: (1) turbines, line elements and flat fields are mapped respectively on ordered pairs of points, euclidean planar displacements and planar symmetries. (2) Turbines, line elements and flat fields are mapped respectively on lines, points, and planes of pseudo-elliptic 3-space (Blaschke, Zeitschrift für Mathematik und Physik, vol. 60, p. 61); in consequence of which $\overline{G}_6$ is shown to be isomorphic with the group of pseudo-elliptic motions and $\overline{G}_7$ with the group of automorphisms of the pseudo-elliptic absolute. (Received March 15, 1940.)

271. R. E. Gaskell: *A problem in heat conduction and an expansion theorem.*

The problem is that of determining the distribution of heat in a right cylindrical solid, one end of which is in contact with a liquid, with lateral insulation provided so as to make the problem one-dimensional. The solution is obtained by reducing the problem, with the Laplace transformation, to an ordinary second order differential equation with end conditions, both differential equation and end conditions involving a parameter. Inversion is carried out by means of the inversion integral and the result expressed in series form. The convergence of this series at $t=0$ leads to an expansion theorem for the initial temperature distribution function $f(x)$ which is piecewise-continuous and has a bounded, integrable first derivative. The same problem for a solid in contact with a liquid at each end is also considered. (Received March 15, 1940.)

272. Abe Gelbart: *On the growth of a function of two complex variables satisfying certain partial differential equations.*

In this paper the author obtains relations between the growth of max $|U(z_1, z_2)|$ and the coefficients a_{mn} of the Taylor series $\sum_{m,n=0}^{\infty} a_{mn} z_1^m z_2^n$ of an analytic function $f(z_1, z_2)$ of two complex variables, where f belongs to the class of functions F given by the totality of functions satisfying the partial differential equation $\partial^2 U/\partial z_1 \partial z_2 + b_1 \partial U/\partial z_1 + b_2 \partial U/\partial z_2 + b_3 U = 0$ where $b_k = b_k(z_1, z_2)$ are entire functions of two complex variables. A simple form for the upper bound of the growth of max $|U(z_1, z_2)|$ is obtained, which depends on the coefficients b_k, $(k=1, 2, 3)$, of the partial differential equation and on the sequences a_{n0}, a_{0m}, $(n, m=1, 2, \cdots)$, by using the linear operators given by Bergman (Matematischeskii Sbornik, vol. 2 (44) (1937), pp 1169–1197) which transforms the class of analytic functions of one complex variable into the class F. Certain applications are made of these results. (Received March 23, 1940.)

273. H. H. Goldstine: *Linear functionals and integrals in abstract spaces.*

In this paper we consider a linear and non-negative functional I defined on a linear set $\mathfrak{X}$ of real-valued functions x, whose range p is arbitrary. Upper and lower functionals are introduced and it is shown that I may be so extended that a function is in the extension of $\mathfrak{X}$ if and only if its upper and lower functionals are equal. Outer and inner measures are then introduced and every set for which they are equal is measurable in the usual sense. If a continuity assumption is made (this hypothesis is slightly less restrictive than those of Daniell or Banach), then the outer measure is a regular one in the sense of Carathéodory and the integral defined by this measure coincides with the original functional. A generalization of the Lebesgue theorem on termwise integration is then obtained by means of semi-uniform convergence. (Received March 8, 1940.)

274. Michael Golomb: *On the boundary-value problems of the equation $\Delta u = g(x, y)$ in the infinite half-strip.*

In this paper some boundary-value problems of the equation $\Delta u = g(x, y)$ in the infinite half-strip $0<x<1$, $y>0$ are explicitly solved. The studied boundary conditions are: either u or $\partial u/\partial n$ on each of the three sides of the half-strip are given. The method of solution applied makes use of Laplace's transformation, as outlined by G. Doetsch (*Theorie und Anwendung der Laplace-Transformation*, Berlin, 1937, pp. 378–383) for the homogeneous equation, with an incorrect result. The condition that the solution has a Laplace transform, convergent in a certain half-plane, restricts the number of possible solutions, and yields theorems of uniqueness. (Received March 27, 1940.)

275. O. G. Harrold (National Research Fellow): *Characterizations of a class of continua by means of continuous functions.*

Many of the known types of continuous transformations when acting on the unit interval I, $0 \leqq t \leqq 1$, produce only image sets which are topological images of the interval, although the mapping function itself need not be topological. For instance, monotone, non-alternating, and interior mappings (which do not reduce to a single point) have this property. In this paper the nondegenerate continua without continua of condensation are characterized as the class of separable, metric spaces (M) for which there exists a continuous map f defined on the unit interval, $f(I) = M$, satisfying either (a) for X closed in I, $\dim X = \dim f(X)$; or, (b) for a nondegenerate continuum X in I, $f(X)$ contains an open set, and, for a nondegenerate continuum Y in M, $f^{-1}(Y)$ contains an open set. (Received March 28, 1940.)

276. O. G. Harrold (National Research Fellow): *Minimal coverings of Peano spaces by maps of a circle.*

Let M denote a regular continuum in the sense of Menger. There exists a continuous mapping f of the circle (interval) A onto M such that there is a $G_{\delta\sigma}$-set dense in M each point of which has at most two inverses in A. This is related to a theorem due to Nöbeling on mappings of a circle onto a regular curve (Fundamenta Mathematicae, vol. 20, pp. 30–46). An analogous theorem to the above holds for an arbitrary Peano space provided the phrase *at most two* is replaced by *at most a finite number.* For a Peano space M in which there are no free arcs, there exists a mapping f of the unit interval I onto M such that there is a G_δ-set dense in M each point of which has a single inverse on I. Thus there exists a continuous map of the interval onto the fundamental cube, $I^{\aleph_0}$, with the property which has just been mentioned. (Received March 28, 1940.)

277. A. E. Heins: *On the solution of partial difference equations: symmetric boundary conditions.*

It is shown that Nörlund's solution of a linear difference equation with constant coefficients may be reduced to a finite sum if boundary conditions are considered. The solution has two groups of terms. The first group depends on the assigned boundary conditions; the second group depends on the inhomogeneous solution (which now appears as a finite sum). This result is now applied to the solution of a boundary value problem in difference equations. The difference equation is a partial difference equation of parabolc type. (See abstract 45-5-184.) This paper considers (a) vanishing finite boundary conditions, nonvanishing initial condition; (b) infinite boundary conditions, nonvanishing initial condition. Case (a) reduces to a double finite sum of sines and cosines; case (b) reduces to a single finite sum of factorials. Limiting cases are also considered. The Laplace transform is used to reduce the partial difference equation to an ordinary inhomogeneous equation and this equation is solved by classical methods (Nörlund, *Differenzrechnung*). (Received March 11, 1940.)

278. Olaf Helmer: *A new type of transcendence proof.*

This paper is based on the following two theorems from the arithmetic of integral functions. (i) Let S be the set of integral functions of finite order with rational coefficients; the greatest common divisor of two elements f and g of S is expressible in the form $Af+Bg$ where A and B are also in S (this follows from a result obtained by R. Miche in his dissertation, Zurich, 1920). (ii) An irreducible polynomial with rational coefficients is also irreducible in S (cf. the author's abstract in this Bulletin, vol. 44, p. 338). With the help of these theorems it can be shown very easily that certain numbers N (such as π, and $\log_e r$ for rational $r \neq 0$) are either rational or transcendental, and that others (such as arc sin r and arc cos r for rational r) are algebraic at most of degree 2 or transcendental. The method used here is very natural from the standpoint of the algebraist inasmuch as it makes essential use of the fact that the numbers in question are roots of certain integral functions with rational coefficients, thus establishing the algebraic character of the roots in relation to the algebraic character of the coefficient field. (Received March 25, 1940.)

279. L. K. Hua: *On the number of partitions of a number into unequal parts.*

Let $q(n)$ be the number of partitions of an integer n into unequal parts, or into odd parts. Let $\epsilon_{h,k} = \exp\{-\pi i[(h'^2-1)((1-hh')/k-1)/8+h'(1-hh')/8h+h(k+(1-hh')/k)(hh'-2)/24]\}$ for $2|k$, and let $\epsilon_{h,k} = \exp\{-\pi i[(h^2-1)/8-hk/8+(h+h')(hh'k-(hh'-1)/k)/24]\}$ for $2\nmid k$; also let $\omega_{h,k} = \epsilon_{h,k}\exp[(-\pi i/12k)(h+h')]$ for $2|k$ and let $\omega_{h,k} = \epsilon_{h,k}\exp[(-\pi i/24k)(2h-h')]$ for $2\nmid k$ where $hh' \equiv 1 \pmod{k}$ and $2\nmid h'$ for $2\nmid k$. Then $q(n) = 2^{-1/2}\sum_{k=1,2\nmid k}^{\infty}\sum_{(h,k)=1,\ 0<h<k}\{\omega_{h,k}\exp(-2\pi i h n/k)\cdot(d/dn)J_0(i\pi[2(n+1/24)/3]^{1/2}/k)\}$ where $J_0(x)$ is the Bessel function of order 0. (Received March 30, 1940.)

280. L. K. Hua: *On Waring's problem for cubic polynomial summands.*

Let $f(x) = a(x^3-x)/6+b(x^2-x)/2+cx+d$, $a>0$, where a, b, c, d are integers and $(a, b, c) = 1$. Then every sufficiently large integer is the sum of eight values of $f(x)$, $x \geqq 0$. Further, almost all positive integers are the sum of four values of $f(x)$, $x \geqq 0$, except the following two cases: (1) $f(x) \equiv 2(2a'+1)x^3+(2b'+1)x^2+2(2c'+1)x+d' \pmod{16}$; (2) $f(x) \equiv a''x^3+3b''x^2+3c''x+d'' \pmod{9}$, where $b''^2 \equiv a''(c''+1) \pmod{3}$. For the first exceptional case almost all positive integers are the sum of seven values of $f(x)$, $x>0$, and in the second case almost all positive integers are the sum of five values of $f(x)$, $x>0$. (Received March 30, 1940.)

281. Henry Hurwitz: *On certain wave equations analogous to the Dirac equations.*

G. D. Birkhoff (Congrès International des Mathématiciens, Oslo, 1936, p. 213) has shown that wave equations for multiple-component wave functions whose associated multiplier equations have the form $P\equiv(W^2/c^2-p^2-1)(W^2/c^2-p^2-m)=0$ are of special interest because of the wave packet phenomenon they exhibit. In this paper particular examples of such four-component wave equations are obtained for arbitrary m. They may be written $(W/c-\alpha_x p_x-\alpha_y p_y-\alpha_z p_z-\beta_L)\psi_L=0$, where W and p_i have their usual quantum mechanics definitions and α_i are ordinary Dirac matrices (Dirac, *Quantum Mechanics*, 2d edition, Oxford, 1935, pp. 254–255). With each member of the Lorentz group a β_L may be associated which reduces to Dirac's α_m when $m=1$. Since the β_L found are not Hermitian, it is convenient to consider also the equation $(W/c-\alpha_x p_x-\alpha_y p_y-\alpha_z p_z-\beta_L\dagger)\chi_L=0$. Then the charge and current four-vector has the form $\chi_L^*\alpha_i\psi_L+\psi_L^*\alpha_i\chi_L$, $i=0, 1, 2, 3$, where $\alpha_0=1$ and the ψ_L and χ_L are suitably related. A given β_L remains unchanged only in certain types of Lorentz transformations (for example, rotations about, or translations along, a particular axis). Therefore, to obtain a relativistic theory, it is necessary to solve the equations corresponding to all the elements of the Lorentz group subject to identical boundary conditions and then average the results over the group by means of the Hurwitz invariant integral. In a Lorentz transformation, S, $\psi_L'=\gamma_S\psi_{S^{-1}L}$ and $\chi_L'=\gamma_S\chi_{S^{-1}L}$. (Dirac, loc. cit., pp. 257–258.) (Received March 2, 1940.)

282. W. H. Ingram: *A generalization of Erhard Schmidt's solution of the nonhomogeneous integral equation.* Preliminary report.

Using results obtained by Frazer, Duncan and Collar and Fredholm's tentative limiting process, a particular solution of the linear integral equation of the second kind with non-symmetrical kernel is found to be $u(x)=\sum\lambda_r(\lambda_r-\lambda)^{-1}\phi_r(x)\int_a^b\psi_r(\alpha)f(\alpha)d\alpha \div\int_a^b\psi_r\phi_r d\alpha$. (Received March 11, 1940.)

283. F. P. Jenks: *Order and parallelism in the non-euclidean geometry of joining and intersecting.*

If, in the author's foundation for Bolyai-Lobachevsky geometry in terms of joining and intersecting (Reports of a Mathematical Colloquium, (2), issue 1, pp. 45–48), postulate VII is sharpened and another assumption added, then the whole theory of order (including Pasch's axiom) can be derived. Two nonintersecting lines a and b and called parallel if there exists a point P such that through P there is at most one line which intersects neither a nor b. These postulates then yield that parallelism is transitive, that there are not more than two parallels to a line through a point, and other theorems on non-euclidean parallelism. (Received March 12, 1940.)

284. Fritz John: *The Dirichlet problem for a hyperbolic equation.*

Recently D. G. Bourgin and R. Duffin (this Bulletin, vol. 45, pp. 851–858) considered the Dirichlet problem for the equation $u_{xx}-u_{yy}=0$ in the case of a rectangle with sides parallel to the coordinate axes. This paper discusses the Dirichlet problem for the same equation for general contours. The main result is the following: Let C be an arbitrary convex curve, for which the Dirichlet problem can be solved for every sufficiently regular set of boundary values. Then the region bounded by C can be mapped on a rectangle with sides parallel to the coordinate axes in such a way that the differential equation (and hence the Dirichlet problem) is preserved. (Received March 7, 1940.)

285. Wilfred Kaplan: *Singularities of a curve-family on a surface, with applications to differential equations.*

Let F be a family of curves filling an open region on a (2-dimensional) surface S. Let F be regular, that is, locally homeomorphic with parallel lines. Let P be an isolated singularity of F. Then all the possible configurations of F in the neighborhood of P are listed. The method of analysis consists in regarding a deleted neighborhood of P as a doubly-connected region, hence homeomorphic with the surface of a sphere from which two points have been removed. Methods previously developed by the author (see abstract 45-9-328) are then applied to the curve-family on the sphere. As an application, F can be taken as the set of trajectories of a differential equation on S (Received March 14, 1940.)

286. Edward Kasner and J. J. DeCicco: *Families of curves conformally equivalent to circles.*

Any three-parameter family of curves which is obtained by applying a conformal transformation to the ∞^3 circles of the plane is called an Ω family. Several geometric characterizations (in the general case) of such a family are obtained. Thus a family of curves is an Ω family if and only if: (I) the foci of the osculating parabolas of the ∞^1 curves which contain a lineal element E generate a lemniscate, (II) the centers of the orthogonal pairs of equal circles which define the lemniscates of (I) as E is rotated about its point P generate an equilateral hyperbola; and (III) the foci of the equilateral hyperbolas of (II) are connected to the point P by a direct conformal transformation. Also the only curves of an Ω family which are hyperosculated by their osculating circles consists of two orthogonal isothermal families of curves. As a corollary of abstract 46-1-92, those families of curves are determined whose hyperosculated isothermal nets are circles. Geometric characterizations are obtained of two-parameter families of curves which are conformally equivalent to the ∞^2 circles orthogonal to a given circle. Finally, by a study of Schwarzian reflection with respect to these families of curves, many interesting generalizations of ordinary inversion are obtained. (Received March 20, 1940.)

287. J. L. Kelley: *On the hyperspaces of a continuum.*

Let A be a compact continuum and let 2^A and $C(A)$ be respectively the spaces of all closed subsets and of all subcontinua of A, metricized by the Hausdorff metric. It is proved: (1) $C(A)$ is arc-wise connected, (2) 2^A and $C(A)$ are locally p-connected in the sense of Lefschetz for $p>0$, (3) any mapping of a polyhedron into $C(A)$ or 2^A is homeotopic to a constant, (4) the homology groups of 2^A of dimension greater than 0 vanish, (5) (theorem of Wodjslawski) A is locally connected if and only if 2^A (or $C(A)$) is an absolute retract. Two new proofs of this latter result are given, depending on the results previously stated and on characterizations of absolute retracts by Lefschetz and by Borsuk respectively. (Received March 25, 1940.)

288. B. O. Koopman: *Intuitive probability and sequences.*

This paper forms a second part of the study of the foundations of probability regarded as a branch of intuitive logic, the first part having appeared in the author's paper, *The axioms and algebra of intuitive probability* (abstract 45-1-92; Annals of

Mathematics, (2), vol. 41 (1940), pp. 269–292). The object is to set forth its connection with the objective notion of statistical weight or frequency in a sequence of trials. The chief theorems assert that when the values of such frequencies are given (for example, by physical laws) and when the conditions of the trials in each sequence are "essentially similar" (in the precise language of the intuitive theory), then the intuitive probabilities of success are to one another as the frequencies. No general principles beyond those assumed in the earlier paper are required, nor does the theory seek an objective definition of random as do the theories of collectives. The present theory thus reveals itself as perfectly capable of dealing with the objective conception of probability involved in statistics and quantum mechanics. It is shown moreover that every experimental application of collectives presupposes in last analysis the intuitive concept of probability. (Received March 21, 1940.)

289. H. L. Krall: *Orthogonal polynomial solutions of a certain fourth order differential equation.*

It is found that there are four classes of nonclassical orthogonal polynomials which satisfy a differential equation of the type $\sum_{i=0}^{4}\sum_{j=0}^{i} l_{ij}x^j y_n^{(i)}(x)=\lambda_n y_n(x)$. One of these classes is a new set of orthogonal polynomials whose derivatives are also orthogonal polynomials. (Received March 19, 1940.)

290. J. P. LaSalle: *Applications of the pseudo-norm to the study of linear topological spaces.*

The pseudo-norm of Hyers, which may be defined for any linear topological space (l.t.s.), is here used as a "pseudo-metric" to define closure for a l.t.s., and it is shown that this definition of closure is equivalent to that of von Neumann. This of course enables one to state necessary and sufficient conditions for the continuity of functions, convergence, and so on, in terms of the pseudo-norm. In particular a necessary and sufficient condition is given that the weak convergence of a sequence of linear functions on T to T', T and T' l.t.s.'s, implies that the limit of the sequence be linear. Also a characterization of a l.t.s. on which there is defined a non-null linear functional is given in terms of a pseudo-normed linear space in which there is defined an operation of "multiplication." (Received March 9, 1940.)

291. P. E. Lewis: *Characters of abelian groups.*

Any homomorphism of an abelian group A in an abelian group V is termed a character of A in V. The problem under consideration is to characterize those pairs of groups A and V satisfying one of the following conditions: (a) the character group of A in V is isomorphic to A; (b) the character group of B in V is isomorphic to B for every subgroup B of A; (c) the character group of A/B in V is isomorphic to A/B for every subgroup B of A. Among other results it is found that (c) implies the finiteness of A and that (a) likewise implies the finiteness of those groups A which contain only elements of finite order. If condition (b) holds, then A is either finite or a direct sum of a finite number of infinite cyclic groups. In each case an almost obvious condition has to be imposed on V. The theory is developed so as to include not only ordinary abelian groups but also certain classes of groups admitting a ring R (not necessarily commutative) as operator system. (Received March 9, 1940.)

292. D. T. McClay: *On certain manifolds of somas.*

This paper contains the complete classification of chains of somas with respect to a 27-parameter continuous subgroup of the pseudo-conformal group; the results are applied in the consideration of one-parameter families of somas. Analogues are found for the configuration of six linear complexes in involution and other configurations in projective line-geometry. (Received March 28, 1940.)

293. E. J. McShane: *On the theory of relative extrema.*

The problem of minimizing a function while giving assigned values to p other functions is discussed in a sufficiently general setting to be useful in the calculus of variations. The necessary conditions on the first derivatives are obtained. For $p=1$, the necessary conditions on the second derivatives are obtained without assumptions of normality. (Received March 15, 1940.)

294. Saunders MacLane and O. F. G. Schilling: *Normal algebraic number fields.*

Let K be a normal extension with Galois group $\{\sigma, \cdots\}$ over the algebraic number field k. Let p denote the prime divisors of k. A set of local algebra classes $H=\{H_p, \text{all } p\}$ is termed an "ideal algebra" if $H_p \sim k_p$ for all but a finite number of prime divisors. The group of all H contains the subgroup $H(K)$ of all H with $H_p \times K^p \sim K^p$. Thus, $H(K)$ contains the group of all actual algebras S_K which are split by K. Restricting H_K, S_K to subgroups H_K', S_K' which are "relatively prime" to a suitable module M (determined by K) one finds that the index $J=[H_K' : S_K']$ equals the l.c.m. of the orders of the elements σ. The index J can be interpreted in terms of factor sets of ideals. Finally, another expression for J is obtained by direct computation, using the arithmetic theory of index reduction. (Received March 12, 1940.)

295. Margaret P. Martin: *A sequence of tests for the convergence and divergence of infinite series.*

The well known de Morgan and Bertrand sequence of tests for the convergence and divergence of infinite series involves an expression for the ratio r_n of one term of the series being tested to the preceding term. This paper contains a similar sequence of tests involving an expression for the ratio $R_n=r_{n+1}/r_n$ of one ratio to the preceding ratio. The proof is based on one of a series of integral tests developed by R. W. Brink (Annals of Mathematics, (2), vol. 21 (1919), pp. 39–60) and on a method of generalization of Brink's tests given by C. T. Rajagopal (this Bulletin, vol. 43 (1937), pp. 405–412). The applicability of the tests to series which would be difficult to test by other known tests is illustrated. (Received March 26, 1940.)

296. Karl Menger: *On shortest polygonal approximations to a curve.*

In an euclidean space, let A be an arc of length $l(A)$, and let $F_1, F_2, \cdots$ be finite subsets of A getting indefinitely dense in A. If for each n, P_n is a shortest polygon through the set F_n, and λ_n the length of P_n, then the polygons $P_1, P_2, \cdots$ converge toward A, and the numbers $\lambda_1, \lambda_2, \cdots$ toward $l(A)$. The theorem holds for any continuous curve A whose length does not surpass that of any continuous curve passing through all points of A. As was previously proved (Mathematische Annalen, vol. 103, p. 467), each arc is in this sense a shortest path through all its points. (Received March 12, 1940.)

297. A. B. Mewborn: *Abstract local geometry of paths.* II.

In the local geometry of paths defined in terms of distinguished (normal) coordinates (this Bulletin, vol. 46 (1940), p. 31) a new linear connection form, defined by the use of a modified and somewhat more general Fréchet differential in the coordinate Banach space, has been found. The curvature form based on this connection does not vanish identically, and hence this space of paths is not necessarily even locally flat. Also some theorems on the local domains of paths as open sets have been found. (Received March 11, 1940.)

298. A. D. Michal: *Higher order differentials of functions with arguments and values in topological abelian groups.*

The author (see abstract 46-1-25) has already studied first order differentials of functions $f(x)$ of a topological abelian group variable x with values in a topological abelian group. The present paper gives an inductive definition of nth order differentials of $f(x)$ at $x=x_0$ by assuming the existence of the $n-1$ previous higher order differentials at $x=x_0$. Several fundamental theorems are proved—including a unicity theorem for all values of the increments and theorems on higher order differentials of functions of functions. As in the case of first order differentials, the real number system does not enter into the theory and so a new flavor is given to an ancient subject and its generalizations. (Received March 27, 1940.)

299. A. D. Michal and Max Wyman: *Characterization of complex couple spaces.*

In considering generalizations of classical complex analysis for complex Banach spaces it seems desirable to consider complex couple spaces. This paper attempts to characterize such spaces. In this connection the following two theorems can be proved. I. A necessary and sufficient condition that an arbitrary complex Banach space B be a complex couple space is that there exist a function $\bar{Z}$ on B to B with the properties: (a) $\overline{Z_1+Z_2}=\bar{Z}_1+\bar{Z}_2$, (b) $\overline{iZ}=-i\bar{Z}$, (c) $\bar{\bar{Z}}=Z$, (d) $\|\bar{Z}\|=\|Z\|$. II. A necessary and sufficient condition that an arbitrary complex Banach space B be a Hermitian couple space is that: (α) B must possess a Hermitian inner product $[Z, U]$, (β) there exist a function $\bar{Z}$ on B to B such that: (β_1) $[\overline{Z_1, Z_2}]=[\bar{Z}_1, \bar{Z}_2]$, ($\beta_2$) $\bar{\bar{Z}}=Z$. By a Hermitian couple space is meant one that is generated from a real Banach space with a real inner product. (The paper will appear in the Annals of Mathematics.) (Received March 9, 1940.)

300. A. N. Milgram: *On the length of continuous curves.*

A Jordan continuum C in a given metric space may be in many ways considered as a continuous curve, that is, continuous image of a finite interval $[a, b]$. Call $\mu(C)$ the greatest lower bound of the lengths of these curves. If $C \subset C'$ are two Jordan continua, it frequently happens that $\mu(C)>\mu(C')$. It is shown however, that $\mu(C)<2\mu(C')$ and that 2 is the best estimate; that is, for each $\epsilon>0$, there are pairs of continuous curves $C\subset C'$ such that $\mu(C')=1$ and $\mu(C)>2-\epsilon$. If M is a subset of a compact convex metric space S, the Jordan continuum C is called shortest join of M if $C\supset M$ and for each C' containing M gives $\mu(C)\leqq\mu(C')$. Then for each subset M of S there exists a shortest join C of M, and C is obtained by adding an at most denumerable number of straight line segments to the closure of M. (Received March 12, 1940.)

301. A. N. Milgram: *Partially ordered sets and the covering theorems of topology.*

Let P be a partially ordered set with a unit 1, that is, an element such that for each x of P, $x\neq 1$ implies $x<1$. If P has a denumerable separating system and P'

is any upper inductive subset of P such that, for each $x \varepsilon P'$, if $x \neq 1$ there exists $y \varepsilon P'$ such that $x<y$, then P' contains the unit. Let P be the class of open subsets of the space S, let $\{O\}$ be a covering of S and P' the set of sums of denumerably many elements of $\{O\}$. Then, the hypotheses of the theorem hold. The conclusion, that is, that $1 \varepsilon P'$, means that S is the sum of denumerably many sets in $\{O\}$. That is the Borel covering theorem for completely separable spaces. Another theorem about partially ordered sets yields the Heine-Borel covering theorem for compact spaces. The definitions of separating system and of inductiveness are given in an earlier paper of the author (Reports of a Mathematical Colloquium, (2), issue 1 (1939), p. 18). (Received March 9, 1940.)

302. W. L. Mitchell: *Topological rings and infinite matrices.*

It is noted that the set of all continuous linear transformations of a topological group into itself forms a ring. In particular, if the topological group admits a topological ring as a ring of operators, and relative to this ring possesses a basis, finite or infinite, the ring of continuous linear transformations forms a ring of matrices, finite or infinite. Some simple properties of these matrices are developed. Some properties of special topological rings are also discussed. (Received March 14, 1940.)

303. R. L. Moore: *Concerning separability.*

It is shown that if a non-separable space satisfies Axioms 0 and 1 of the author's *Foundations of Point Set Theory* (American Mathematical Society Colloquium Publications, vol. 13, 1932), then it contains uncountably many mutually exclusive domains. (Received March 21, 1940.)

304. D. C. Murdoch: *A characterization of abelian quasi-groups.*

An abelian quasi-group is one which satisfies the generalized associative law $(ab)(cd)=(ac)(bd)$. This paper contains a complete characterization of all quasi-groups of this type. It is shown that every abelian quasi-group is the direct product of a self-unit quasi-group (one in which every element is a right unit) with one which contains an idempotent element. All quasi-groups of the latter type can be constructed by defining certain new operations in abelian groups while all self-unit ones can be formed by defining similar new operations in those quasi-groups, already constructed, in which all elements are idempotent. (Received March 11, 1940.)

305. Lewis Nelson: *An Abel integral equation with constant limits of integration.*

Applying the method of T. Carleman (Mathematische Zeitschrift, vol. 15 (1922), pp. 111–120) this paper shows the uniqueness of and gives the solution $\phi(\xi)$ of the integral equation $\int_0^x (x-\xi)^{-\alpha}\phi(\xi)d\xi-\int_x^1(\xi-x)^{-\alpha}\phi(\xi)d\xi=f(x)$ where $0<\alpha<1$, $0<x<1$, $|x-\xi|^{-\alpha}\phi(\xi)$ is integrable over $0<\xi<1$, and $f(z)$ is analytic in a domain of the complex plane containing the real axis from 0 to 1 of $z=x+iy$. (Received March 9, 1940.)

306. Philip Newman: *The geometry of the* (2, 2) *planar connex.* Preliminary report.

A convex $f(x_1x_2x_3,\ u_1u_2u_3)$ (Clebsch, *Vorlesungen über Geometrie*) of second order and second class defines two quadratic systems of ∞^2 conics each, point and line respectively. The equations $\rho v_i=f_{x_i}$, $\sigma y_i=f_{u_i}$, $i=1, 2, 3$, define an element (y, v) conjugate to the element (x, u). These same equations define a multiple valued contac transformation of the element (u, y) into four elements (x, v). A pencil of lines (u) on the point (y) will be transformed into a series of conics quadritangent to a general quartic. A geometry thus appears in the (x) plane with conic and quartic as dual elements. Properties of the systems of conics and quartics with respect to the discriminant curves are studied. A related $(3, 3)$ connex, the Jacobian of the first polar nets, is introduced. The representation of the system of $(2, 2)$ connexes (35-parameter) on Veronese surfaces in 5-space obtained by projective transformations of two fundamental dual surfaces is also discussed. (Received March 25, 1940.)

307. K. L. Nielsen: *Concerning general boundary value problems for linear differential equations.* Preliminary report.

Trjitzinsky (Acta Mathematica, vol. 67 (1936), pp. 1–50) has obtained asymptotic representation of the solutions of the system, which in matrix notation is written $Y^{(1)}(x, \lambda) = Y(x, \lambda)D(x, \lambda)$, for x in the interval (c, d) and certain extending to infinity regions R for the parameter λ. This system is associated with a linear differential equation of order n, $L(x, \lambda; y) \equiv \sum_{k=0}^{n} \lambda^{H(n-k)} a_{n-k}(x, \lambda) y^{(k)} = 0$. The two-point boundary value problem is formulated by considering the above system with the boundary condition $Y(c, \lambda)W_c + Y(d, \lambda)W_d = 0$, where W_c and W_d are matrices of constant terms. The author obtains theorems determining the values of λ for which the nonhomogeneous boundary problem is possible when (c, d) is the interval for which Trjitzinsky's existence theorem holds. He first considers the two-point problem; then the problem formulated by taking the conditions at a finite number of points in the x interval (c, d); that is, $\sum_{i=1}^{n} Y(a_i, \lambda) W_{a_i} = 0$; and finally obtains the restrictions necessary when considering the conditions at a denumerably infinite number of points in the interval (c, d). (Received March 4, 1940.)

308. Rufus Oldenburger: *Binary forms.*

In the present paper the theory of minimal numbers and representations introduced elsewhere is developed for binary forms, and is used to give a solution of the problem of equivalence of these forms. The range of the minimal number for binary forms of degree n and field K, subject to minor restrictions, is $1, 2, \cdots, n$. The maximum value n is attained for the complex field if and only if the form has a repeated linear factor of degree $n-1$. The minimal number of a binary form F for a field K exceeds a class of 2-way ranks of the form which have maximum values. For the whole set of these ranks, formally studied elsewhere, no simple applications were found until the present paper. The linear forms in a minimal representation of F with respect to K are factors of a form transforming covariantly with F. If the minimal number of a form is not too large, the associated representation is unique, in which event there is a simple answer to the problem of the equivalence of this form to another form. For the complex field the minimal number can be expressed in terms of resultants and 2-way ranks only. (Received March 28, 1940.)

309. C. D. Olds: *On the number of representations of the square of an integer as the sum of an odd number of squares.*

Let $N_r(n^2)$ denote the number of representations of the square of a positive integer n as the sum of r squares. By analytical means it is possible to derive formulas for $N_r(n^2)$ when $r=3, 5, 7$ which involve only the divisors of n (see G. Pall, Journal of the London Mathematical Society, vol. 5 (1930), pp. 102–105). In this paper it is shown that these results can be obtained in an elementary manner by using only arithmetical reasoning. Use is made of certain fundamental identities for arbitrary parity functions due originally to Liouville and which have since been proved arithmetically. When $r=3$ and 7, the method is an extension of that used by A. Hurwitz (*Mathematische Werke*, vol. 2, pp. 5–7) in his solution of the case when $r=5$. When $r=3$, a second derivation is given, the method being a generalization of T. J. Stieltjes' solution of the case when $r=3$ and $n=p^k$ where p is a prime number identical to 1 (mod 8). (See *Correspondance d'Hermite et de Stieltjes*, vol. 1, pp. 89–94.) (Received March 8, 1940.)

310. Oystein Ore: *Remarks on structures and group relations.*

This paper gives the conditions for three elements in an arbitrary structure to form a Dedekind structure. The results are applied to show that two normal subgroups and an arbitrary third always form a Dedekind structure. (Received March 25, 1940.)

311. Oystein Ore: *The extension problem for groups.*

This paper contains a discussion of the properties of extensions and factor sets and it is shown that by a special reduction method any extension may be reduced to abelian extensions. Among the applications are criteria for splitting extensions. (Received March 25, 1940.)

312. E. W. Paxson: *Strictly convex metric spaces.*

Extending an idea used by J. A. Clarkson in linear spaces (Transactions of this Society, vol. 40 (1936), pp. 396–414), a metric space is called strictly convex if triangles degenerate uniquely, that is, if the equations $\rho(x, u)+\rho(u, y)=\rho(x, y)$, $\rho(x, u)=\alpha\leqq\rho(x, y)$, for example, have a unique solution $u(x, y; \alpha)$. It is then shown that multiplication by real numbers and addition may be defined, the latter via a parallelogram construction. These defined entities verify the properties of those in a linear space. The only exceptional point is that associativity of addition is equivalent to the trisection theorem on the medians of a triangle, which may be postulated in a purely metric way. (Received March 15, 1940.)

313. P. M. Pepper: *Concerning pseudo-planar-quintuples.*

A pseudo-planar-quintuple is a metric space of five points not congruently imbeddable into the euclidean plane, although each four point subset is so imbeddable. A convex space into which each member of the five-parameter family of pseudo-planar-quintuples in congruently imbeddable consists of three euclidean semi-planes joined along a common line. The triangular dihedra are a three-parameter family of convex spaces such that each pseudo-planar-quintuple is congruently imbeddable into some member and a two-parameter family of the quintuples into each member. (Received March 29, 1940.)

314. I. E. Perlin: *Sufficient conditions that polynomials in several variables be positive.*

In this paper the author considers polynomials in n variables with real coefficients. By inserting parameters he obtains polynomials in the parameters which satisfy certain recursion relations. Sufficient conditions that the original polynomial in the n variables be positive for all real values of the arguments are obtained. These conditions are expressed in terms of the coefficients of the given polynomial and the roots of the polynomials in the parameters. (Received March 14, 1940.)

315. Sam Perlis: *Scalar extensions of algebras with exponent equal to index.*

If A is a normal simple algebra with equal index and exponent, one may inquire whether all scalar extensions A_K also have this property. It is shown, in essence, that this property is preserved for all separable extension fields K of finite degree if and only if this is true for all cyclic fields K of prime degree. For purely inseparable extensions K an example is constructed in which A is a cyclic division algebra of index and exponent four over an appropriate field of characteristic two, but A_K has index four and exponent two. The example also shows that for every integer $r>1$ there exists a modular field of degree of imperfection r such that not all p-algebras over this field have equal index and exponent. This is in contrast to the case $r=1$. (Received March 5, 1940.)

316. W. T. Puckett: *On arc-preserving transformations.*

Let $T(M)=M'$ be an arc-preserving transformation (G. T. Whyburn, American Journal of Mathematics, vol. 58 (1936)). In case M is cyclicly connected and contains a simple closed curve J such that $T(J)$ is not an arc, the inverse transformation $T^{-1}(M')$ is single-valued. If, in addition, M is strongly arcwise connected, then T is continuous and consequently topological. The set M is said to be strongly arcwise connected, provided every infinite collection of its points contains an infinite subcollection which lies on an arc in M. (Received March 11, 1940.)

317. H. A. Rademacher and A. L. Whiteman: *On Dedekind sums.*

The sums under consideration appear in the transformation formula of log $\eta(\tau)$, and were first investigated by Dedekind in his *Erläuterungen zu den Riemannschen Fragmenten über die Grenzfälle der elliptischen Modulfunktionen* (Riemann's *Werke*, 1876, pp 438–447). The present paper consists of three parts. The first part contains proofs of arithmetical formulas which Dedekind partly derived by analytic methods and partly stated without proofs. The second part verifies the equivalence of Riemann's and Dedekind's results. The last part contains new arithmetical formulas involving Dedekind sums. These formulas are used to obtain simple proofs of Lehmer's results about the sums $A_k(n)$ which appear in the theory of partitions (Transactions of this Society, vol. 43 (1938), pp. 271–295). (Received March 25, 1940.)

318. H. J. Riblet: *Factorization of differential ideals in an algebraic differential field.*

This paper gives conditions sufficient to insure the unique factorization of differential ideals. The problem of extending these properties to algebraic differential fields is discussed and sufficient conditions are given for a factorization theorem. (Received March 25, 1940.)

319. J. H. Roberts: *A theorem on dimension.*

Hurewicz has shown that if X is a compact metric space, and $Y=f(X)$, where f is an at most $(k+1)$-to-1 continuous mapping, then dim $Y\leq$ dim $X+k$. He has raised the following question: Given a compact metric space Y of dimension n $(n>0)$, does there exist for every k $(0 \leq k \leq n)$ a compact metric space X of dimension $n-k$ and an at most $(k+1)$-to-1 continuous mapping f such that $f(X)=Y$? The present paper gives an affirmative answer to this question. This result, combined with the above theorem of Hurewicz, yields the following characterization of dimension: The compact metric space Y is of dimension n $(n>0)$ if and only if Y is the image of an $(n-1)$-dimensional compact metric space X under a continuous, at most 2-to-1, mapping. The hypothesis that X and Y are *compact* can be replaced throughout by the hypothesis that they are *separable*, provided that f is restricted to be reciprocally continuous (*beiderseits stetig*). (Received March 29, 1940.)

320. Barkley Rosser: *An additional criterion for the first case of Fermat's last theorem.*

It has been proved in a succession of papers due to numerous authors that if p is an odd prime and $a^p+b^p+c^p=0$ has a solution in integers prime to p, then $m^{p-1}\equiv 1 \pmod{p^2}$ for each prime $m \leq 41$. The methods which work for $m \leq 41$ lead to excessive computation when applied to $m=43$. In this paper a new method especially adapted to m's of the form $6n+1$ is developed, and applied in the case of $m=43$ to prove the result quoted above for this case also. (Received March 28, 1940.)

321. A. C. Schaeffer and Gabor Szegö: *Inequalities for harmonic polynomials in two and three dimensions.*

The first part deals with two dimensional, the second part with three dimensional harmonic polynomials. I. The basic result is a formula of interpolatory character representing a given linear combination of the coefficients of a harmonic polynomial $U(n, \phi)$ of degree n in terms of $U(1, \phi)$ at $2n$ equidistant points. Here only the following application should be mentioned: let $U(1, \phi) \geq 0$ and let $\sigma_n(\phi)$ be the nth Cesàro means of $U(1, \phi)$. Then $n\sigma_n(\phi) \geq |\text{grad } U(1, \phi)|$. This theorem includes Szegö's "gradient theorem" which states that $|\text{grad } U(1, \phi)| \leq n$ provided $|U(1, \phi)| \leq 1$. II. The second part deals with the following problem. Consider all three dimensional harmonic polynomials $U(P)=U(x, y, z)$ of degree n satisfying the condition $|U(P)| \leq 1$ for $x^2+y^2+z^2 \leq 1$. If Q is a fixed point in space at a distance $R>1$ from the origin, an explicit expression for the maximum of $|U(Q)|$ is obtained in terms of R and n. The asymptotic value of this maximum as $n \rightarrow \infty$ is $(\pi^{1/2}/2)(1-R^{-2})^{1/2}n^{1/2}R^n$, whereas the corresponding maximum in the two dimensional case (as already known) is exactly R^n. (Received March 9, 1940.)

322. G. E. Schweigert: *Equivalence of pointwise periodic and interior transformations on dendrites.*

Given that $T(A)=B$ is continuous, A and B are dendrites, and $T^{-1}(b)$ is finite for each point b in B, what are the conditions under which it is possible to define a pointwise periodic homeomorphism $h(A)=A$ such that for each b in B and each x in $T^{-1}(b)$ the point-orbit of x is exactly the set $T^{-1}(b)$? Since an interior transformation S which carries A into the orbit space is induced whenever h exists, one looks to the preservation of open sets and other known necessary conditions to answer this question. A sufficient set of such conditions is found and h is effectively defined to within the choice of permutation for certain inverse sets. The conditions are simple and represent a kind of symmetry necessary in A; they may be varied slightly to suit a given case. Given T the h defined induces T; given h the induced S allows us to rediscover h to within the occasional choice of permutation in an orbit. Special care was taken throughout to avoid methods that would not be useful in a similar problem for Peano spaces. (Received March 20, 1940.)

323. R. W. Shephard: *The length of production and related dynamic aspects of a simplified economic system.* Preliminary report.

By consideration of mean values and totals of integrals over the length of production, the equations of equilibrium of an economic system can be replaced approximately by those of a simplified system involving a time lag. The plans of the producers, taken with reference to certain technical functionals, are given in terms of current and expected future prices. Under somewhat general assumptions concerning the relation of stocks to the demand for capital and consumption goods, it is found that the quantities of the system may be completely determined. The character of this determination is investigated by a study of small changes. (Received March 11, 1940.)

324. Max Shiffman: *The Morse relations in the Plateau problem for several boundaries.*

Let $\Gamma_1, \Gamma_2, \cdots, \Gamma_k$ be k prescribed non-intersecting Jordan curves in space. This paper establishes the Morse relations for minimal surfaces bounded by these k curves if degenerate as well as nondegenerate surfaces are included. The space $\mathfrak{R}_k$ of circular domains of representation with k boundaries are introduced and those points on each circle which are mapped into three specified points of Γ_ν are marked and considered parts of the domain. Degenerate domains are also introduced and limit is defined. It is shown that $\mathfrak{R}_k$ is a metric space and its connectivity numbers are determined. The paper then deals with the behavior of the Dirichlet functional over the space $\mathfrak{P}$ of potential surfaces defined over the domains of $\mathfrak{R}_k$. This behavior permits carrying over the results obtained by the author in previous work on the case of one boundary (Annals of Mathematics, (2), vol. 40 (1939), pp. 834–854). A similar theory was developed independently by Morse and Tompkins (Annals of Mathematics, (2), vol. 40 (1939), pp. 443–472). (Received March 29, 1940.)

325. W. S. Snyder: *Functions of simple figures.* Preliminary report.

Let F be a function defined on a class S of subsets of euclidean n-space. Assuming certain simple properties of the elements of S it is possible to develop a comprehensive theory of the derivatives and the Burkill integrals of F. These results include and extend the principal results of Banach, Burkill, Saks and Kempisty. The purpose of the paper is to establish a theory which is sufficiently general to cover the most important applications known at the present time. (Received March 15, 1940.)

326. D. C. Spencer: *On finitely mean valent functions.*

Suppose that $f(z)$ is regular in the unit circle $|z| < 1$, and that $W(R)$ is the area (multiply covered regions being counted multiply) of the portion of the transform of $|z| < 1$ by f which lies in the circle $|w| \leqq R$. Then if $W(R) \leqq p\pi R^2$ for all $R > 0$, where p is a positive number (not necessarily integral), $f(z)$ is said to be mean p-valent. Let $A(r)$ be the area of the transform of $|z| < r$ by $f(z)$, $M(r, f)$ the maximum modulus of $f(z)$ on the circle $|z| = r$. Functions f satisfying the condition (less restrictive than the one given above) that $A(r) \leqq p\pi M^2(r, f)$ are described as having weak mean valency p. These definitions were first suggested to the author by Professor J. E. Littlewood. In this paper it is shown that many of the properties of p-valent functions are possessed by mean p-valent functions, but *not* by the wider class of weak mean p-valent functions. Inequalities are obtained for the mean values of a mean p-valent function f and its derivative analogous to those already known for p-valent functions, and bounds are deduced from these inequalities for the coefficients in the power series of f. (Received March 18, 1940.)

327. D. C. Spencer: *On finitely mean valent functions.* II.

This paper is a sequel to one of the same title of which the abstract appears above. Here the rate of growth of mean p-valent functions is discussed. It is shown, for example, that if f is mean p-valent and $0 \leqq r < 1$, then $|f|$ on the circle $|z| = r$ is of order $(1-r)^{-2p}$ at most. Another (and related) result is that *schlicht* functions which fill only an infinitesimal part of the plane are of infinitesimal order. Theorems concerning the behavior of mean p-valent functions on arbitrary paths tending to the circumference $|z| = 1$ are also included. The results of the paper remain true under hypotheses less restrictive than that of mean p-valency. If the theorems of this paper are combined with those of the paper above, it is found, for example, that if f is mean p-valent and of the form $f=\sum_{\nu=0}^{\infty} a_{\nu k+1} z^{\nu k+1}$, then $|a_n| = O(n^{2p/k-1})$ when $p > k/4$ (the restriction that $p > k/4$ is actually necessary). (Received March 18, 1940.)

328. Alvin Sugar: *On a result of Hua for cubic polynomials.*

L. K. Hua showed that every integer can be additively represented by seven values of certain cubic polynomials (Tôhoku Mathematical Journal, vol. 41 (1935–1936), pp. 361–366). The writer by a shorter method, with more general cubic polynomials, proves that five, and in some cases four, values suffice. (Received March 30, 1940.)

329. J. L. Synge: *On the electromagnetic two-body problem.*

The system under discussion consists of two charged particles (hydrogenic atom). The argument is relativistically invariant (in the sense of the special theory), but classical in the sense that quantum mechanics is not involved. The field due to a particle is that of the usual retarded potential, and the force on a particle the usual ponderomotive force, without any "radiation" term. The problem has been treated previously in two limiting cases: (1) small velocities (Darwin), (2) small mass-ratio (Sommerfeld). Neither of these approximations gives any degeneracy of the motion (radiation of energy). In the present paper a general method of successive approximations is sketched, and applied to the Sommerfeld approximation, but with inclusion of higher-order terms arising from the motion of the nucleus. A full calculation is carried out for the case where the orbit of the lighter particle is approximately circular. It is found that the orbit decreases steadily in radius, remaining circular. Thus there is degeneracy, but the radiation of energy is much less than that usually quoted, which is derived from a computation of flux of energy at infinity. (Received March 16, 1940.)

330. Olga Taussky and John Todd: *On determinants of quaternions.*

The usual definitions of determinants whose elements are (real) quaternions do not ensure the reality of a "hermitian" determinant. A definition which does this can be given. With this definition the so-called Gram's determinants are investigated and the results are applied to extend recent work of H. and F. J. Weyl on unitary metrics in projective space (Annals of Mathematics, (2), vol. 39 (1938), pp. 516–538 and vol. 40 (1939), pp. 141–148 and 634–635). The metric thus obtained has the interesting property that a point need not have a zero distance from itself. (Received March 25, 1940.)

331. H. P. Thielman: *On the convex solution of a certain functional equation.*

In this paper the following theorem is proved: The only convex solution of the functional equation $1/f(x+a)=x^pf(x)$ $(x>0, a>0, p>0)$ is $F(x)=[\mathrm{B}(x/2a, 1/2)/(2a\pi)^{1/2}]^p$. This is an analogue of a theorem by E. Artin (*Einführung in die Theorie der Gammafunktion,* Hamburger mathematische Einzelschriften, no. 11, 1931, p. 12), which states that if $f(x)$ is a continuous solution of $f(x+1)=xf(x)$, and if $\log f(x)$ is convex, then $f(x)=a\Gamma(x)$, where a is an arbitrary constant. A. E. Mayer (Acta Mathematica, vol. 70 (1938), pp. 61, 62) has shown that Artin's theorem cannot be improved by replacing the convexity of $\log f(x)$ by the mere convexity of $f(x)$. He has given examples of functions which are convex, satisfy the equation $f(x+1)=xf(x)$, but are essentially different from $a\Gamma(x)$, where a is any constant. For the result of this paper it was not necessary to assume the convexity of the logarithm of the solution. The special case $a=1$, $p=1$ of the present theorem was given by Mayer in the paper referred to above. (Received March 14, 1940.)

332. R. M. Thrall: *A note on a theorem by Witt.*

Let G^c denote the cth member of the lower central series of any group G. If F is the free group with n generators, E. Witt (Journal für die reine und angewandte Mathematik, vol. 177 (1937), pp. 152–160) has shown that F^c/F^{c+1} is a free abelian group with N generators and $N=1/c\sum\mu(c/d)n$ where μ is the Möbius μ-function and the summation is over all divisors of c. Let $Q=F/H_q$ where H_q is the smallest group containing all qth powers in F. In this note it is proved that if p is a prime less than c and $q=p$, then Q^c/Q^{c+1} is abelian and of order q^N. This result is applied to the commutator calculus for groups of small class. (Received March 14, 1940.)

333. Gerhard Tintner: *Fourier integrals as Bessel differential operators.* Preliminary report.

Let $g(v)=\int[\exp(-ivx)]f(x)dx$, where the limits are $-\infty$ and $+\infty$ and $f(x)$ can be developed into a Taylor series in terms of $1/x$, $f(x)=h(1/x)=\exp(z/x)\rightarrow h(t)$ for $t=0$, where $z=d/dx$. Then by Cauchy's theorem: $g(v)=z\sum_{k=0}^{\infty}(-izv)^k/k!(k+1)!\rightarrow g(t)\big|_{t=0}$ $=(z/(-izv)^{1/2})I_1(2(-izv)^{1/2})\rightarrow g(t)\big|_{t=0}$ where I_1 is a Bessel function for purely imaginary argument. Let also $G(v_1, v_2, \cdots, v_n)=\int\cdots\int(\exp-\sum_{j=1}^{u}iv_jx_j)F(x_1, x_2, \cdots, x_n)dx_1dx_2\cdots dx_n$, where each integral runs from $-\infty$ to $+\infty$. Define $z_j=\partial/\partial x_j$. Then by a similar argument $G(v_1, v_2, \cdots, v_n)=\prod_{j=1}^{n}z_j\sum_{k=0}^{\infty}(-iz_jv_j)^k/k!(k+1)!$ $\rightarrow H(t_1, t_2, \cdots, t_n)\big|_{t_j=0}$, which can also be expressed as the product of n Bessel differential operators. It is believed that this method may yield useful approximations in the evaluation of statistical distributions by the method of characteristic functions. (Received March 14, 1940.)

334. W. J. Trjitzinsky: *Developments in the analytic theory of algebraic differential equations.*

This work to appear in the Acta Mathematica presents an extensive analytic theory of algebraic differential equations $F=0$, where F is a polynomial in $y, y^{(1)}, \cdots, y^{(n)}$. In F the coefficients are polynomials in x or, more generally, they are functions, analytic in certain regions extending to infinity and therein asymptotic to series of the form $x^m[a_0+a_1x^{-1}+\cdots]$ (integer m). The case is also considered when the coefficients are continuous in x on an interval and contain a parameter. The main results amount to the following. When the equation $F=0$ has a formal solution s of the same type as occurs in the corresponding linear case, one can always construct regions R and "actual solutions" y of $F=0$ for which $y^{(i)}\sim s^{(i)}$ ($i=0,\cdots,n$; in R; to $k(t)$ terms; $k(t)\rightarrow\infty$ with t). Essentially, the regions are determined by the character of a certain linear problem associated with $F=0$. (Received March 19, 1940.)

335. V. J. Varino: *A note on matrices over a principal ideal ring.*

Let A and B be square matrices with elements in a principal ideal ring $\mathfrak{P}$. An algorithm for finding the greatest common right divisor and least common left multiple of A and B is given. It is also shown that any two greatest common left divisors are left associates, and similarly for any two least common left multiples. (Received March 14, 1940.)

336. C. W. Vickery: *On cyclically invariant graduation.*

Let $f(t)$ be a function of a real variable t. Let Z be a random variable symmetrically distributed about the mean $\mathcal{E}(Z)=0$ and having a (cumulative) distribution function $F(z)$. The graduation $f^*(t)$ of $f(t)$ is defined as follows: $f^*(t)=A\cdot\mathcal{E}\{f(t+z)\}=A\int_{-\infty}^{\infty}f(t+z)dF(z)$. If $A=\{\mathcal{E}(\cos Z)\}^{-1}=\{\int_{-\infty}^{\infty}(\cos z)dF(z)\}^{-1}$, then $\sin t$ and $\cos t$ are invariant with respect to this operation. In applications, Z may be assumed to have a normal distribution, a symmetrical Bernoulli distribution, and so on. The coefficient A corrects Spencer's and similar graduation formulae so that $\sin t$ and $\cos t$ are invariant with respect to their application. (Received March 9, 1940.)

337. C. W. Vickery: *On spaces ($\mathcal{E}$) and Moore spaces.*

In order that a space S be a Moore space (F. B. Jones, this Bulletin, vol. 43 (1937), p. 675) it is necessary and sufficient that it satisfy the following conditions: (1) S is a space ($\mathcal{E}$) of Fréchet (*Les Espaces Abstraits*, p. 214); (2) the derived set of every point set is closed; (3) S is regular. These conditions are independent. (Received March 9, 1940.)

338. T. L. Wade: *Subgeometries of projective geometries as theories of tensors.*

This paper is concerned with the consideration of a subgeometry of classical (flat) projective geometry as the theory of a tensor in lieu of the customary consideration of such a geometry as the theory of a subgroup of the general linear projective transformation group. Affine, euclidean (parabolic), elliptic, and hyperbolic geometries are each treated as the theory of a tensor. Fundamental theorems are given whereby all algebraic concomitants in these geometries in 2-space can be constructed as tensors of order zero; also a large number of examples connecting the tensor-invariant method with the existent literature are given. Further, it is pointed out how a presentation of the metric subgeometries of projective geometry as the theories of tensors gives additional insight into their differences. "Distance tensors" for the three metric geometries are introduced. (Received March 28, 1940.)

339. A. D. Wallace: *An analysis of non-alternating transformations.*

If $T(A)=B$ is a non-alternating transformation on the locally connected continuum A, then there exists a unique A-set A_0 such that A_0 is irreducible with respect to the property of mapping onto B under T. If $r(A)=A_0$ is the hereditarily monotone transformation which retracts A onto A_0 and t is the transformation T restricted to A_0, then $T=tr$ and $t(A_0)=B$ is a non-alternating transformation which maps cut-points into cut-points. Further if $T=T_2T_1$ and $t=t_2t_1$ are factored in accordance with the Whyburn-Eilenberg theorem, then T_2 and t_2 are equivalent as are T_1 and t_1r. Moreover the transformation T_2 takes cut-points into cut-points. (Received March 25, 1940.)

340. A. D. Wallace: *On certain classes of subcontinua.*

In what follows S will denote a metric continuum and Roman numerals will be used to designate classes of closed sets in S defined as I: all subcontinua of S; II: all closed sets X in S such that if $K \varepsilon$ I then $X \cdot K \varepsilon$ I; III: all retracts $X=r(S)$ such that if $K \varepsilon$ I and $X \cdot K \neq 0$, then $X \cdot K = r(K)$; IV: all hereditary monotone retracts of S; V: all sets X such that if Y is a closed set in X which separates X between two points, then Y separates S between these same points. In this paper alternative characterizations are given for these classes of sets and the transformations involved in their definition. The invariance of these classes is proved for certain continuous transformations. These classes may be ordered in accordance with the theorem: *For any continuum S we have* V $\subset$ IV $\subset$ III $\subset$ II $\subset$ I. *If S is locally connected, the inclusions (except the first) become equalities, but this does not hold in general.* (Received March 11, 1940.)

341. J. L. Walsh and W. E. Sewell: *On the degree of convergence of harmonic polynomials to harmonic functions.*

Let R be the interior of an analytic Jordan curve C, and let the function $u(x, y)$ be harmonic in R, continuous in $R+C$, and satisfy a Lipschitz condition of order $\alpha<1$ on C. Then there exist harmonic polynomials $p_n(x, y)$ of respective degrees n such that (1) $|f(x, y)-p_n(x, y)| \leqq M/n^\alpha$ for (x, y) on $R+C$, where M is independent of (x, y) and n. Conversely, inequality (1) implies the continuity of $u(x, y)$ on C and a Lipschitz condition of order α there. (Received March 27, 1940.)

342. M. S. Webster: *Maximum of certain fundamental Lagrange interpolation polynomials.*

This paper extends some of the results given in a previous paper (this Bulletin, vol. 45 (1939), pp. 870–873). Additional properties are also given for the maximum of certain fundamental Lagrange interpolation polynomials based on the zeros of Jacobi polynomials. Important use is made of the asymptotic expressions given by Szegö. (Received March 9, 1940.)

343. Louis Weisner: *Moduli of the roots of polynomials and power series.*

Let $f_m(z)=a_0z^{n_0}+ \cdots +a_mz^{n_m}$ $(0 \leqq n_0 < \cdots < n_m)$ be a polynomial with zero coefficients suppressed, and let $\nu_s=n_s-n_{s-1}$. For a fixed k, let $p_1, \cdots, p_m$ be positive numbers, $p_0=1$, satisfying the inequality $p_0+p_0p_1+ \cdots +p_0p_1 \cdots p_m \leqq 2p_0p_1 \cdots p_k$, and let $u_s=|p_sa_{s-1}/a_s|^{1/\nu_s}$. The author proves that if $R \leqq u_s$ $(s=1, \cdots, k)$, $R \geqq u_s$ $(s=k+1, \cdots, m)$, then the circle $|z|<R$ includes just n_k roots of $f_m(z)$. Various applications are made to the moduli of the roots of polynomials and power series. (Received March 27, 1940.)

344. Louis Weisner: *Power series, the roots of whose partial sums lie in a sector.*

With the aid of a generalization of the familiar inequality concerning the harmonic, geometric and arithmetic means of a number of positive numbers, the author proves the following theorem: If, for each $n \geqq n_0$, the roots of the partial sum of degree n of the formal power series $f(z) = \sum a_n z^n$ lie in some sector with vertex at the origin and aperture $\alpha < \pi$, then $f(z)$ is an entire function of order zero. This theorem was proved for the case $\alpha = 0$ by G. Pólya (Rendiconti di Palermo, vol. 36 (1913)). (Received March 26, 1940.)

345. D. W. Western: *A method for analytic continuation.*

The results obtained in this paper provide an expansion for the analytic continuation of a function of a complex variable with finite singularities beyond the limits of the ring of convergence of the Laurent series. The region of convergence is determined by definition of major and minor circles, a notion introduced by Flora Streetman and L. R. Ford (American Mathematical Monthly, vol. 38 (1931), pp. 198–201). In the limiting position about the origin, the region of convergence has an inner boundary composed of the smallest circles through the origin and the inner singular points. The outer boundary has as its limiting form the straight lines through the outer singular points perpendicular to the radius vectors of these points. The expansion involves the parameter used in construction of the major and minor circles. It is in the form of a double series of integral terms, one series being a polynomial in z and the other a rational fractional function of z. (Received March 28, 1940.)

346. F. J. Weyl: *On the defect relation for meromorphic curves.*

The theory of meromorphic curves (see Herman and F. J. Weyl, Annals of Mathematics, (2), vol. 39 (1938), pp. 516–538) is an extension of R. Nevanlinna's theory of meromorphic functions. Quantities $m^*(r; a)$, closely resembling in their behaviour the Nevanlinna defects of meromorphic functions, can be attached to the points a of the complex k-dimensional space $\mathfrak{R}$ in which a meromorphic curve is defined. The theory of meromorphic curves culminates in a stringent estimate (analogous to R. Nevanlinna's second main theorem) of the sum of such defects over any finite number of points a in $\mathfrak{R}$. In its previous formulation the validity of this so-called defect relation depends upon the assumption that the sum be extended over points which satisfy no accidental linear relations, that is, any $(k+1)$ of which are linearly independent. In the present paper the defect relation has been reformulated so as to make this restriction unnecessary without sacrificing essential parts of the original stringency. (Received March 29, 1940.)

347. P. A. White: *On certain relatively non-alternating transformations.*

This paper considers certain properties of "G-non-alternating transformations," as defined by A. D. Wallace (this Bulletin, vol. 46 (1940), p. 56). It is shown that ordinary non-alternating transformations and G-non-alternating transformations are the same when defined on a locally connected continuum A: (1) when G is the collection of all finite sets, if and only if A is unicoherent; (2) when G is the collection of all continua, if and only if A is a boundary curve. The question of the equivalence of G-n.a. and monotone transformations is also answered for the collections defined above. Finally, it is shown that the property of being a linear graph is invariant under a G-n.a. transformation, where G is the collection of all sets containing at most two points. (Received March 27, 1940.)

348. Hassler Whitney: *On regular families of curves.*

A new condition that a family of curves be "regular" is given. This is the same as that in Annals of Mathematics, (2), vol. 34 (1933), p. 244, Theorem 7A, except that the relation (2) is omitted. (Received March 19, 1940.)

349. R. L. Wilder: *Characterization of the lower dimensional generalized manifolds by positional properties in S_n.*

In the earlier papers the generalized closed manifold of dimension $n-1$ (g. c. $(n-1)$-m.) has been characterized by its positional topological properties in n-space. The g. c. 1-m. (simple closed curve) in n-space has been similarly characterized by P. Alexandroff (Annals of Mathematics, (2), vol. 36 (1937), p. 19). In the present paper the following characterization is obtained for all dimensions k: In order that a closed point set M in the n-sphere S_n should be a g. c. k-m., it is necessary and sufficient that (1) $p^{n-k-1}(S_n-M)=1$ and $p^{n-k-1}(F)=0$ for all closed proper subsets F of M; (2) S_n-M be uniformly locally i-connected for $i>n-k-1$; and (3) S_n-M be uniformly locally connected in terms of its bounding $(n-k-1)$-cycles. (Received March 15, 1940.)

350. A. R. Williams: *On a certain Cremona transformation between two $(n-1)$-spaces in S_n.*

The author takes a linear system of ∞^{n-1} quadrics in n-space. Then if a point P describes a $(n-1)$-space, its conjugate P' with respect to all the quadrics of the system describes a variety V of order n and dimension $n-1$. But this variety is also the locus of the poles of the $(n-1)$-space with respect to the individual quadrics of the system. Hence P' is the pole of the $(n-1)$-space with respect to some quadric of the system, and since the latter may be represented by a point P'' in a $(n-1)$-space one has a Cremona transformation between two $(n-1)$-spaces, which if one wishes may be taken coincident. Thus V appears in two aspects and the loci on it occur in pairs. To a locus of order r on V will correspond two loci of orders r_1 and r_2 on the two $(n-1)$-spaces. If r_1 is not equal to r_2, there will be another locus also of order r on V corresponding to loci of orders r_2 and r_1, respectively, in the two $(n-1)$-spaces. The study of the loci on V is facilitated by the Cremona transformation between the two $(n-1)$-spaces. (Received March 9, 1940.)

351. J. R. Woolson: *The mean of the iteration of linear operators in reflexive Banach spaces.* Preliminary report.

There exists a complex valued bilinear interspace inner product $[F, f]$ on (B), B to the complex numbers. B is a complex Banach space and (B) is the space of linear functionals on B of bound $\|F\|$. (B) is a space of the same type as B. If $[F,f]=0$ for all F, then $f=0$, and if $[F,f]=0$ for all f, then $F=0$. Let B and (B) be reflexive and define the adjoint of a linear operator A on B to B as the operation A^* on (B) to (B) such that $[A^*F, f]=[F, Af]$. Define a projection as a linear operation such that $P^2=P$. Using these concepts it is possible to demonstrate that $n^{-1}\sum_{i=1}^{n} A^i f$ approaches weakly a linear operation which is a projection. Weak convergence is functional convergence. (Received March 9, 1940.)

352. J. W. T. Youngs: *On parametric representations of surfaces.* Preliminary report.

The paper is concerned with the study of certain concepts in topology and area. A continuous vector function $\mathfrak{x} = \mathfrak{x}(a)$, defined on the surface of a sphere A (say) with values in 3-space, is a representation. If B is the image of A, then the totality of components of the inverse sets $\mathfrak{x}^{-1}(b)$, $b \varepsilon B$, constitutes an upper semi-continuous collection Σ. The set Σ can be topologized, and, by the Whyburn factor theorem, $\mathfrak{x}(A) = L(M(A))$, where $M(A) = \Sigma$ is monotone, and $L(\Sigma) = B$ is light. Two representations $\mathfrak{x}_1$ and $\mathfrak{x}_2$ are K-equivalent (Kerékjártó, Acta Szeged, vol. 3 (1927), pp. 49–67) if there exists a homeomorphism $H(\Sigma_1) = \Sigma_2$ such that $L_1(\Sigma_1) = L_2(H(\Sigma_1))$. They are F-equivalent if the Fréchet distance between them is zero. This paper discusses the interdependence of these ideas and affiliated concepts in the theory of area. Kerékjártó has shown that F-equivalence implies K-equivalence. Here it is shown that the theorem is still true if A and B are Peano spaces. Kerékjártó and Morrey (American Journal of Mathematics, vol. 57 (1935), pp. 17–50) both prove the converse in special cases. Unfortunately an example shows this to be false even in the simple cases they consider. (Received March 11, 1940.)

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EXISTENCE AND DIFFERENTIABILITY THEOREMS FOR THE SOLUTIONS OF VARIATIONAL PROBLEMS FOR MULTIPLE INTEGRALS[1]

CHARLES B. MORREY, JR.

I. INTRODUCTION

1. The object of the research [1] with which this address is concerned is twofold: first, to demonstrate, by direct methods, the existence of solutions, perhaps in some generalized sense, for a wide class of variational problems for multiple integrals, and second, to investigate further differentiability properties of the generalized solutions thus obtained. We consider only problems in nonparametric form.

2. Both of these goals have been achieved in the recent well known solutions of the problem of Plateau and in the solutions of the Dirichlet problem by variational methods. In more general problems this program has not been carried through completely except in very restricted cases. However, there are very important known results in connection with each separate aim. In connection with the existence theory, very important work has been done in the nonparametric case by Tonelli [2] and in the parametric case by McShane [3]. In the parametric case, practically nothing is known concerning the differentiability of the solutions obtained. In the nonparametric case, it has been proved by E. Hopf [4] that if $f(x, y, z, p, q)$ is of class C_α'' (that is, if f is of class C'' and its second derivatives satisfy a uniform Hölder condition[2] with exponent α, $0<\alpha<1$, on any bounded portion of space) and is the integrand in a regular variational problem (that is, if $f_{pp}f_{qq}-f_{pq}^2>0$, $f_{pp}>0$) and if z_0 is continuous on $\bar{G}$ and is of class C_β' on each region D with $\bar{D}\subset G$ and minimizes $\iint_G f(x, y, z, p, q)dxdy$ among all functions, continuous on $\bar{G}$, of class C' in G, and coinciding with z_0 on G^*, then z_0 is of class C_γ'' on any region D as above for any $\gamma<\alpha$. The author has proved in a previous paper [5] that the same result holds if z_0 merely satisfies a uniform Lipschitz condition[3] on regions $\bar{D}$ interior to G. The result of E. Hopf

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[1] An address delivered before the meeting of the Society in Pasadena on December 2, 1939, by invitation of the Program Committee.

[2] A function $f(P)$ is said to satisfy a *uniform Hölder condition* on a set S if $|f(P_1)-f(P_2)|\leq C\cdot|P_1P_2|^\alpha$, $0<\alpha<1$, for each pair of points (P_1, P_2) in S; C is called the *coefficient* and α is called the *exponent* of the Hölder condition and $|P_1P_2|$ denotes the distance from P_1 to P_2; C and α are supposed to be independent of P_1 and P_2.

[3] A function $f(P)$ is said to satisfy a uniform Lipschitz condition on a set S if $|f(P_1)-f(P_2)|\leq C\cdot|P_1P_2|$ for every pair (P_1, P_2) on S, C independent of P_1, P_2.

can be extended readily to the case where f is a function of $(x, y, z, \cdots, z_N)$ while the methods used in the author's previous paper [5] do not permit this extension.

3. In connection with the existence theory, Tonelli proves in the two papers mentioned that if $f(x, y, z, p, q)$ satisfies certain continuity restrictions, gives rise to a quasi-regular variational problem, and if it satisfies

$$(3.1) \qquad f(x, y, z, p, q) \geqq m(p^2 + q^2)^{\alpha/2} + N, \quad m > 0, \alpha > 2,$$

or (essentially)

$$(3.2) \qquad f(x, y, z, p, q) \geqq m(p^2 + q^2), \quad f(x, y, z, 0, 0) = 0, \quad m > 0,$$

then the integral

$$I(z, G) = \iint_G f(x, y, z, p, q)dxdy$$

takes on its minimum among all functions z which are absolutely continuous in his sense and assume given continuous boundary values, provided $I(z, G)$ is finite for some such function and G is properly restricted; more general boundary value problems are also considered. The proof consists in showing first that $I(z, G)$ is lower semicontinuous with respect to uniform convergence in his class, and then in replacing any minimizing sequence by one in which the functions are equicontinuous so that a uniformly convergent minimizing subsequence is obtained which can be shown to converge to a function in his class. If the α in (3.1) is less than 2, such a procedure is not possible in general. In fact, when we try to generalize these results to functions of N independent variables, we find that Tonelli's procedure is impossible unless $\alpha \geqq N$ (see §12). As the gap $1 < \alpha < N$ grows wider as N increases, it seems desirable to obtain existence theorems of some sort assuming merely that $\alpha > 1$; for $\alpha = 1$, there are examples, even for $N = 1$, of problems which have no solutions.

Evidently we must abandon the use of continuity and equicontinuity in obtaining our existence theorems but must certainly retain absolute continuity in some sense. G. C. Evans [6] has introduced a class of functions, which he calls "potential functions of their generalized derivatives," which would seem to form a proper class of admissible functions; we shall call these functions "functions of class $\mathfrak{P}$" (see below). Surprisingly enough, these functions were introduced before those of Tonelli, and Evans [7] has investigated the connection between the two classes of functions (we state the results below).

4. Before proceeding to a discussion of these functions and their applications, we must introduce certain notations and terminology. The N-tuples $(x_1, \cdots, x_N)$, $(\xi_1, \cdots, \xi_N)$, $\cdots$, will be denoted by the single letters x, ξ, $\cdots$, and the closed cell $a_i \leqq x_i \leqq b_i$, $i=1, \cdots, N$, will be denoted by $[a; b]$. The functional notation $f(x_1, \cdots, x_N)$ will be abbreviated to $f(x)$ and the Lebesgue integrals of $f(x)$ over $[a, b]$ or a set E will be denoted by

$$\int_a^b f(x)dx, \qquad \int_E f(x)dx.$$

It is frequently desirable to consider the behavior of a function $f(x)$ with respect to a particular coordinate variable x_k or with respect to the $(N-1)$ variables $(x_1, \cdots, x_{k-1}, x_{k+1}, \cdots, x_N)$. We then denote the latter variables by x_k', we write (x_k', x_k) for x and $f(x_k', x_k)$ for $f(x)$: thus $f(a_k', x_k)$ denotes the function $f(a_1, \cdots, a_{k-1}, x_k, a_{k+1}, \cdots, a_N)$ of the single variable x_k (a_k' being fixed) and $f(x_k', a_k)$ denotes the function $f(x_1, \cdots, x_{k-1}, a_k, x_{k+1}, \cdots, x_N)$ (a_k being fixed). The projection of the cell $[a, b]$ on $x_k=0$ is denoted by $[a_k', b_k']$ and

$$\int_{a_k'}^{b_k'} f(x_k', a_k)dx_k'$$

denotes the $(N-1)$-dimensional integral of $f(x_k', a_k)$ over $[a_k', b_k']$. If G is a region (open connected set), G^* denotes its boundary and $\overline{G}$ its closure. All integrals are Lebesgue integrals.

II. The admissible functions

5. We now define the various classes of functions of which we shall speak.

Definition 5.1. *A function $f(x)$ is of class $\mathfrak{P}$ on a region G if*

(i) *it is summable on each closed cell interior to G, and*

(ii) *there exist functions $v_1(x), \cdots, v_N(x)$, satisfying* (i) *and such that*

$$\int_{a_k'}^{b_k'} [f(x_k', b_k) - f(x_k', a_k)]dx_k' = \int_a^b v_k(x)dx$$

for almost all cells $[a, b]$ interior to G (that is, all such cells such that the point $(a_1, \cdots, a_N; b_1, \cdots, b_N)$ does not belong to a set of $2N$-dimensional measure zero).

Definition 5.2. *A function $f(x)$ is of class $\mathfrak{P}'$ on G if*

(i) *it satisfies* (i) *of Definition* 5.1, *and*

(ii) *if* $[a, b]$ *is any cell interior to* G, $f(x_k', x_k)$ *is A.C.* (*absolutely continuous*) *in* x_k *for* $a_k \leqq x_k \leqq b_k$ *for almost every* x_k' *in* $[a_k', b_k']$, *and*

(iii) $\partial f/\partial x_k$ (*which exists almost everywhere and is measurable by* (i) *and* (ii)) *is summable on each cell interior to* G.

DEFINITION 5.3 [8]. *A function* $f(x)$ *is absolutely continuous in the sense of Tonelli* (*A.C.T.*) *on* G *if*

(i) *it is continuous on* G,

(ii) *it satisfies* (ii) *of Definition* 5.2, *and*

(iii) *if* $[a, b]$ *is any cell in* G, *then the variation of* $f(x_k', x_k)$ *as a function of* x_k *alone on* $[a_k, b_k]$ *is a summable* (*known to be lower semicontinuous from* (i)) *function of* x_k'.

DEFINITION 5.4. *A function* $f(x)$ *is of class* $\mathfrak{P}''$ *on* G *if it is of class* $\mathfrak{P}$ *there and is continuous.*

It is easily shown that a function of $f(x)$ of class $\mathfrak{P}$ determines the functions $v_k(x)$ of Definition 5.1 to within a null function (a function which vanishes almost everywhere). Hence it defines completely the set functions $\int_e v_k(x)dx$ $(k=1, \cdots, N)$ which are A.C. on any bounded set E whose closure is in G. Accordingly we propose, with Evans [6], the following definition:

DEFINITION 5.5. *The generalized derivative* $D_{x_k}f$ *of* f *with respect to* x_k *is defined as the Lebesgue derivative* [9] *of the set function* $\int_e v_k(x)dx$ *defined above, wherever this derivative exists.*

It is well known that this derivative exists almost everywhere and coincides with $v_k(x)$ almost everywhere. For simplicity, we shall use the notation f_{x_k} to denote either $D_{x_k}f$ or $\partial f/\partial x_k$ as desired (see §6 for a justification of this usage).

6. The following theorems concerning the above classes of functions are easily demonstrated:

THEOREM 6.1. *If* $f(x)$ *is of class* $\mathfrak{P}$ *on a region* G, *any function* $f^*(x)$ *which is equivalent to* $f(x)$ *on* G (*that is, differs from* $f(x)$ *by a null function*) *is also of class* $\mathfrak{P}$ *on* G *and the generalized derivatives of* f *and* f^* *exist at exactly the same points and coincide at these points.*

THEOREM 6.2. *Any two functions of class* $\mathfrak{P}$ *whose respective generalized derivatives coincide almost everywhere differ at most by a constant and a null function.*

THEOREM 6.3. *Any function of class* $\mathfrak{P}'$ *is of class* $\mathfrak{P}$ *and corresponding generalized and partial derivatives coincide almost everywhere.*

THEOREM 6.4. *Any function of class* $\mathfrak{P}$ *is equivalent to a function of class* $\mathfrak{P}'$.

THEOREM 6.5. *The classes in Definitions* 5.3 *and* 5.4 *coincide.*

THEOREM 6.6. *A necessary and sufficient condition that* $f(x)$ *be of class* $\mathfrak{P}$ *on* G *is that it satisfy* (i) *of Definition* 5.1 *and that for each cell* $[a, b]$ *interior to* G, *there exist summable functions* $v_k(x)$ *and a sequence* $\{f_n(x)\}$, *each* f_n *satisfying a uniform Lipschitz condition on* $[a, b]$, *such that*

$$\lim_{n\to\infty}\int_a^b\left[\,|f_n - f| + \sum_{k=1}^{N}|f_{n,x_k} - v_k|\,\right]dx = 0.$$

7. We now define the most important of all of our classes of admissible functions. The theorem below and those in the next five sections would make it appear that these functions should play an important rôle in many other branches of analysis as well as in the present research.

DEFINITION 7.1. *A function* $f(x)$ *is said to be of class* $\mathfrak{P}_\alpha$ ($\mathfrak{P}_\alpha'$ *or* $\mathfrak{P}_\alpha''$), $\alpha \geqq 1$, *on* G *if it is of class* $\mathfrak{P}$ ($\mathfrak{P}'$ *or* $\mathfrak{P}''$) *there and if* $|f|^\alpha$ *and* $|D_{x_k}f|^\alpha$ *are summable over* G, $k=1, \cdots, N$.

The following theorem concerning these functions is of fundamental importance.

THEOREM 7.1. *The space* $\mathfrak{P}_\alpha$ *of classes of equivalent functions of class* $\mathfrak{P}_\alpha$ *on a region* G *is a Banach space* (*space of type* B) [10] *if we define, for instance,*

$$(7.1)\qquad \begin{aligned} \|f\|^\alpha &= \overline{D}_\alpha(f,G) \equiv D_\alpha(f,G) + \int_G |f|^\alpha dx,\\ D_\alpha(f,G) &\equiv \int_G\left[\sum_{i=1}^{N}|f_{x_i}|^2\right]^{\alpha/2}dx, \end{aligned}$$

$\|f\|$ *denoting the norm of* f *in* $\mathfrak{P}_\alpha$ *on* G.

8. Before proceeding further with the discussion of these functions, we propose two more definitions as follows:

DEFINITION 8.1. *A transformation* $x=x(y)$ *of a set* T *into a set* S *is Lipschitzian if it is* 1-1 *and continuous and if the functions* $x_i(y)$ *and* $y_i(x)$ (*of the inverse*) *all satisfy uniform Lipschitz conditions on each closed subset of* T *and* S *respectively. If the Lipschitz conditions are uniform over the whole of* T *and* S, *we say that the Lipschitzian transformation is regular.*

Definition 8.2. *A region G is said to be Lipschitzian if $\overline{G}$ can be covered by a finite number of sets $\gamma_1, \cdots, \gamma_L$, each of which is the product of $\overline{G}$ with an open set and is the image under a regular Lipschitzian transformation of either the open unit cube $|y_i| < 1$ or the part of this cube for which $y_N \leqq 0$; in the latter case we assume that the transformation sets up a 1-1 correspondence between the points of $\gamma_i \cdot G^*$ and the points of the half-cube for which $y_N = 0$.*

9. Rademacher [11] has shown that a function satisfying a Lipschitz condition on a region G possesses a total differential almost everywhere and has also proved the usual theorems on transformations of multiple integrals to hold for Lipschitzian transformations. With these definitions and results in mind, we may generalize a result of Evans [12] as follows:

Theorem 9.1. *Let $f(x)$ be of class $\mathfrak{P}$ (or $\mathfrak{P}''$) on a region G and let $x = x(y)$ be a Lipschitzian transformation of a region H into G. Then the transformed function $f[x(y)] = g(y)$ is of class $\mathfrak{P}$ (or $\mathfrak{P}''$) on H. Moreover if $x_0 = x(y_0)$ where y_0 is a point of H at which all the $x_i(y)$ possess total differentials and if all the generalized derivatives of f exist at x_0, then all the generalized derivatives of g exist at y_0 and are connected with those of f at x_0 by the usual formulas. If f is of class $\mathfrak{P}_\alpha$ on G and the transformation is regular, then g is of class $\mathfrak{P}_\alpha$ on H.*

Concerning the functions of class $\mathfrak{P}'$, we have proved the following theorem:

Theorem 9.2. *Any function of class $\mathfrak{P}$ on G is equivalent to a function of class $\mathfrak{P}'$ on G which is transformed into a function of class $\mathfrak{P}'$ by any regular Lipschitzian transformation.*

Not every function of class $\mathfrak{P}'$ has this property, and even if it does, an example of Saks [13] shows that the statement of Evans concerning the generalized derivatives is not in general true for the partial derivatives.

10. We now come to a discussion of the boundary values of functions of class $\mathfrak{P}_\alpha$ on bounded regions. If $f(x)$ is of class $\mathfrak{P}_\alpha$ on a Lipschitzian region G ($\alpha \geqq 1$), we may state the following fundamental results:

Theorem 10.1. *There exists a sequence $\{f_n(x)\}$ of functions, each satisfying a uniform Lipschitz condition on $\overline{G}$, which converges strongly, accordingly to the norm (7.1) to f on G.*

Theorem 10.2. *There exists a boundary value function f^* of class L_α*

on G^* *(with respect to hyper-area on* G^**) such that* f_n *converges strongly in* L_α *on* G^* *to* f^*, $\{f_n\}$ *being any sequence as in Theorem* 10.1.

THEOREM 10.3. *If* $f^*=0$ *almost everywhere, the sequence* $\{f_n(x)\}$ *of Theorem* 10.1 *may be chosen so that each* f_n *vanishes on and near* G^*.

THEOREM 10.4. *If* $x'=x'(x)$ *is a regular Lipschitzian transformation of* G *into* G' *and if* f' *and* f'^* *are the transforms of* f *and* f^*, *then* f'^* *is the boundary value function for* f' *on* G'^*.

THEOREM 10.5. *Let* $\bar{f}(x)$ *be of class* $\mathfrak{P}_\alpha'$ *and equivalent to* $f(x)$ *on* G. *Then there exist sets* Z_k, $k=1, \cdots, N$, *each of* $(N-1)$*-dimensional measure zero, such that if* $\bar{x}_k'$ *is not in* Z_k, $\bar{f}(\bar{x}_k', x_k)$ *is A.C. on each segment* $x_k^{(1)} \leqq x_k \leqq x_k^{(2)}$, *such that* $(\bar{x}_k', x_k)$ *is in* G *and* $(\bar{x}_k', x_k^{(i)})$ *is on* G^*, $i=1, 2$, *and* $\bar{f}(\bar{x}_k', x_k)$ *tends to* $f^*(\bar{x}_k', x_k^{(i)})$ *as* x_k *tends to* $x_k^{(i)}$ *along this segment,* $i=1, 2$, $k=1, \cdots, N$.

If G is merely an arbitrary bounded region, we do not have such explicit results but we may follow a procedure suggested by Courant [14]. This procedure suggests the following definition:

DEFINITION 10.1. *A function* $f(x)$ *of class* $\mathfrak{P}_\alpha$ *on* G *is said to vanish on* G^* *if there exists a sequence* $\{f_n(x)\}$ *of functions, each satisfying a uniform Lipschitz condition on* $\bar{G}$ *and vanishing on and near* G^*, *such that* $\|f_n-f\| \to 0$. *Two functions* f_1 *and* f_2 *of class* $\mathfrak{P}_\alpha$ *are said to coincide on* G^* *if their difference vanishes on* G^* *in the above sense.*

From Theorem 10.3, it is clear that the above definition of vanishing coincides with the condition that $f^*=0$ in the case that G is Lipschitzian. Using this very general terminology, we may state the following very important "substitution theorem":

THEOREM 10.6. *Let* $f(x)$ *be of class* $\mathfrak{P}_\alpha$ *on a region* G, *let* $g(x)$ *be of class* $\mathfrak{P}_\alpha$ *on a subregion* D *of* G *and coincide with* $f(x)$ *on* D^*, *and let* $h(x)$ *be defined equal to* f *on* $G-D$ *and equal to* g *on* D. *Then* $h(x)$ *is of class* $\mathfrak{P}_\alpha$ *on* G, *coincides with* $f(x)$ *on* G^*, *and* $D_{x_k}h=D_{x_k}f$ *almost everywhere on* $G-D$ *(*D *is open) and* $D_{x_k}h=D_{x_k}g$ *on* D *wherever either exists,* $k=1, \cdots, N$.

11. As the spaces $\mathfrak{P}_\alpha$ have been seen to be Banach spaces, we may inquire into the nature and consequences of weak convergence [15] in $\mathfrak{P}_\alpha$. In this connection, the following interesting results have been proved:

THEOREM 11.1. *The most general linear functional* [16] $F(f)$ *defined on* $\mathfrak{P}_\alpha$ *is of the form*

$$F(f) = \int_G \left[A_0(x)f(x) + \sum_{i=1}^{N} A_i(x) D_{x_i} f \right] dx,$$

where the A_i *are essentially bounded if* $\alpha = 1$ *and are in* L_β *with* $\alpha^{-1} + \beta^{-1} = 1$ *if* $\alpha > 1$ *(the* $A_i(x)$ *are not uniquely determined by* $F(f)$ *however).*

THEOREM 11.2. *A necessary and sufficient condition that* f_n *tend weakly to* f *in* $\mathfrak{P}_\alpha$ *is that* f_n *and* $D_{x_i} f_n$ *tend weakly to* f *and* $D_{x_i} f$, *respectively.*

THEOREM 11.3. *Weak convergence is preserved by regular Lipschitzian transformations.*

THEOREM 11.4. *If* f_n *tends weakly to* f *in* G, *the convergence holds on any subregion and* f *is uniquely determined.*

THEOREM 11.5. *If the* f_n *all vanish on* G^*, f *vanishes on* G^*.

THEOREM 11.6. *If* G *is Lipschitzian and* f_n *tends weakly in* $\mathfrak{P}_\alpha$ *on* G *to* f, *then* f_n *tends strongly in* L_α *to* f *on* G *and the boundary value functions* f_n^* *tend strongly in* L_α *on* G^* *to* f^*.

THEOREM 11.7. *If* $\alpha > 1$, *a necessary and sufficient condition that a family of functions* $\{f\}$ *in* $\mathfrak{P}_\alpha$ *be compact (perhaps not closed) with respect to weak convergence in* $\mathfrak{P}_\alpha$ *is that their norms be uniformly bounded. If* $\alpha = 1$ *we must also have a convex function* $\phi(p_1, \cdots, p_N)$ *with*

$$\lim_{|p| \to \infty} \phi(p_1, \cdots, p_N)/|p| = +\infty, \qquad |p|^2 = p_1^2 + \cdots + p_N^2,$$

such that

$$\int_G \phi[D_{x_1} f, \cdots, D_{x_N} f] dx$$

is uniformly bounded.

THEOREM 11.8. *A necessary and sufficient condition that the norms in* $\mathfrak{P}_\alpha$ *of each function of a family be uniformly bounded is that* $D_\alpha(f, G)$ *be uniformly bounded, each function* f *of the family coinciding on* G^* *with a function* g *of another family whose norms are uniformly bounded.*

THEOREM 11.9. *If* G *is Lipschitzian, the above condition may be replaced by either the condition that* $\bar{D}_\alpha(f, \tau)$ *be uniformly bounded for some cell* τ *in* G *or the condition that* $\int_E |f^*|^\alpha ds(e)$ *be uniformly bounded,* E *being some set open on* G^* *and* $s(e)$ *being the hyper-area set function on* G^*.

12. Of less interest here are Green's theorem for functions of class $\mathfrak{P}_1$ on a Lipschitzian region G and theorems hinted at above (in §3) which show that functions of class $\mathfrak{P}_\alpha$ with $\alpha > N$ are equivalent to continuous functions. If $\alpha \leqq N$, then this is not usually true as the examples

$$f(x) = \log \log (1 + r^{-1}), \qquad \alpha = N, r^2 = x_1^2 + \cdots + x_N^2,$$

$$f(x) = r^{-\beta}, \qquad \alpha < N, 0 < \beta, \alpha(\beta + 1) < N,$$

show.

III. THE EXISTENCE THEORY

13. Having studied our admissible functions at some length, we can now easily derive some very general existence theorems. We consider integrals

$$I(z, G) = \int_G f(x_1, \cdots, x_N, z_1, \cdots, z_P, p_1^1, \cdots, p_N^1, \cdots, p_1^P, \cdots, p_N^P) dx$$

$$= \int_G f(x, z, p) dx, \qquad p_j^i = D_{x_j} z_i,$$

in which $f(x, z, p)$ is continuous all over the space of its arguments, is convex in p for each (x, z), and satisfies a condition

$$f(x, z, p) \geqq \phi(p), \quad \lim_{|p| \to \infty} |p|^{-1} \phi(p) = +\infty, \quad |p|^2 = \sum_{i=1}^{N} \sum_{j=1}^{P} (p_i^j)^2,$$

$\phi(p)$ being convex. It is straightforward to prove that such an integral $I(z, G)$ is lower semicontinuous with respect to weak convergence in $\mathfrak{P}_1$. From our study of the admissible functions, we see that any family of functions z for which $I(z, G)$ is uniformly bounded will be compact with respect to weak convergence in $\mathfrak{P}_1$, provided merely that the norms are uniformly bounded. This may be ensured by satisfying one of the conditions in Theorems 11.8 or 11.9. If this can all be done for a minimizing sequence, we may then pick out a subsequence which converges weakly in $\mathfrak{P}_1$ to some function *which is also in* $\mathfrak{P}_1$. From the lower semicontinuity of $I(z, G)$ we may conclude that our limit function is a minimizing function. It is clear that a great variety of existence theorems with variable as well as fixed boundary values can be proved with very little difficulty. For example, we may conclude the existence of a minimal surface part of whose boundary is to be a fixed Jordan arc or is to lie on a bounded closed manifold. Of course our existence theorems do not allow us to conclude continuity of the surface on the boundary and, indeed, this is quite difficult to prove and may not be true in many problems.

IV. CONTINUITY PROPERTIES OF THE MINIMIZING FUNCTIONS

14. Without placing further restrictions on the integrand $f(x, z, p)$, one cannot expect to be able to deduce many further properties of the solutions obtained from the above existence theory. This type of study seems quite difficult and is almost unexplored. Accordingly we have restricted ourselves to the case $N=2$ and have required that our function (N is the previous P)

$$f(x, y, z^1, \cdots, z^N, p^1, \cdots, p^N, q^1, \cdots, q^N) = f(x, y, z, p, q)$$

be of class C_α'' over the space of its arguments and that it satisfy the further conditions

$$\begin{aligned} &m_1(p^2+q^2) \leqq f(x, y, z, p, q) \leqq M_1(p^2+q^2), \\ &\qquad p^2+q^2 = \sum_{i=1}^N [(p^i)^2+(q^i)^2], \quad 0 < m_1 \leqq M_1, \end{aligned} \tag{14.1}$$

$$\begin{aligned} &m_2(R)(\xi^2+\eta^2) \leqq f_{p^\alpha p^\beta}\xi^\alpha\xi^\beta + 2f_{p^\alpha q^\beta}\xi^\alpha\eta^\beta + f_{q^\alpha q^\beta}\eta^\alpha\eta^\beta \\ &\qquad \leqq M_2(R)(\xi^2+\eta^2), \quad \xi^2+\eta^2 = \sum_{i=1}^N [(\xi^i)^2+(\eta^i)^2], \\ &\sum_{i=1}^N \Big[f_{p^i x}^2 + f_{p^i y}^2 + f_{q^i x}^2 + f_{q^i y}^2 + |f_{z^i x}| + |f_{z^i y}| \\ &\qquad + \sum_{j=1}^N \{ f_{p^i z^j}^2 + f_{q^i z^j}^2 + |f_{z^i z^j}| \} \Big] \leqq M_3(R)(p^2+q^2), \\ &0 < m_2(R) \leqq M_2(R), \quad x^2+y^2+(z^1)^2+\cdots+(z^N)^2 \leqq R^2, \end{aligned} \tag{14.2}$$

where in (14.1) we assume m_1 and M_1 to be independent of (x, y, z, p, q) and where $m_2(R)$, $M_2(R)$, and $M_3(R)$ depend only on R.

This type of integrand, although quite restricted, is not trivial. For instance, the integrand obtained in the problem of Plateau on a Riemannian manifold (if Courant's method [17] for the ordinary problem of Plateau is used) is

$$g_{\alpha\beta}(z^1, \cdots, z^N)(p^\alpha p^\beta + q^\alpha q^\beta), \tag{14.3}$$

which satisfies all the conditions (14.1) and (14.2) on the cell of the z space in which the $g_{ij}(z)$ are defined, provided merely that the part of the manifold thus represented is regular and the g_{ij} are of class C_α''. Moreover, the existence theory for such an integrand does not follow from the results of Tonelli as the corresponding restrictions (3.2) for his methods to be extended to N dependent variables $z^1, \cdots, z^N$ would be (essentially)

$$f(x, y, z, p, q) \geqq m(p^2 + q^2), \qquad m > 0,$$

$$f(x, y, z^1, \cdots, z^N, p^1, \cdots, p^{k-1}, 0, p^{k+1}, \cdots, p^N, q^1, \cdots, q^{k-1}, 0, q^{k+1}, \cdots, q^N) = 0, \quad k = 1, \cdots, N,$$

such a set of equalities not being assumed in our case. In fact it seems very difficult to show the existence of solutions using methods which involve equicontinuity.

We have used the tensor summation convention in this section, summing only over repeated Greek indices, and we shall continue its use throughout the rest of the paper.

15. Of fundamental importance in the continuity and differentiability theory are the following four theorems:

THEOREM 15.1. *If the vector function* z (*that is, each component* z^i, $i=1, \cdots, N$) *is of class* $\mathfrak{P}_2$ *on* G *and if*

$$D_2[z, C(P, r)] \leqq M(r/a)^\lambda \tag{15.1}$$

for each circle $C(P, r)$ *in* G, *then* z (*is equivalent to a function which*) *satisfies a uniform Hölder condition with exponent* $\frac{1}{2}\lambda$ *on each region* D *with* $\bar{D} \subset G$, *the coefficient in which depends only on* M, λ, *and the distance of* $\bar{D}$ *from* G^*; a *denotes the distance of* P *from* G^*.

THEOREM 15.2. *If, in Theorem* 15.1, (15.1) *holds only for circles with center at a fixed point* P, *then the average of* z *over circles* $C(P, r)$ *tends to a limit* $\bar{z}(P)$ *as* $r \to 0$. *If* z *vanishes on* G^*, *then*

$$|\bar{z}(P)| \leqq J(M, \lambda, a, \delta), \tag{15.2}$$

where J *is a number depending only on the indicated quantities,* δ *being the diameter of* G *and* a *being the distance of* P *from* G^*.

THEOREM 15.3. *Let* z *be of class* $\mathfrak{P}_2$ *on* G *and satisfy*

$$D_2(z, \Delta) \leqq K \cdot D_2[H(z, \Delta), \Delta], \qquad K \geqq 1, \tag{15.3}$$

for each region Δ *in* G, $H(z, \Delta)$ *being the harmonic function in* Δ *coinciding with* z *on* Δ^* (*known to exist by our existence theorems and easily proved harmonic*). *Suppose* G *is bounded by a finite number of nonintersecting simple closed curves and suppose that* z *coincides with a function of class* $\mathfrak{P}_2''$ *on* G *and continuous on* $\bar{G}$.

Then z *is continuous on* $\bar{G}$ (*or equivalent to such a function*).

THEOREM 15.4. *If z is of class $\mathfrak{P}_2$ on G and satisfies*

$$D_2[z, C(P, r)] \leqq K \cdot D_2[H\{z, C(P, r)\}, C(P, r)] + L \cdot (r/a)^{\lambda}, \quad 0 < \lambda < K^{-1}, \tag{15.4}$$

for all circles in G with center at a fixed point P $(0 \leqq r \leqq a)$, then z satisfies (15.1), *the M of* (15.1) *depending only on K, L, and λ.*

Some of these theorems have generalizations to vector functions of n $(n \geqq 2)$ variables. To generalize Theorems 15.1 and 15.2, we need only to replace the λ of (15.1) by $n-2+\lambda$, $\lambda > 0$. Theorem 15.3 has no immediate generalization for all numbers K and Theorem 15.4 yields an exponent $(n-1)/K$ which is not greater than $n-2$ unless K is restricted to be quite near to unity. As we use Theorem 15.4 to obtain a condition (15.1), it will thus appear to the reader that the methods used in this research will not generalize immediately to integrands of the type described in §14 where x and y are replaced by n independent variables, essential use being made of the bound (15.2) and the continuity results of Theorem 15.1, both of which require the exponent $n-2+\lambda$, $\lambda > 0$.

16. Now, if z minimizes our integral $I(z, G)$ among all functions of class $\mathfrak{P}_2$ which coincide with it on G^*, then it follows from (14.1) that z satisfies the condition in Theorem 15.3 with $K = M_1/m_1$; for if some D exists where this does not hold, we can define $Z(x) = z(x)$ on $G - D$ and define $Z(x) = H(z, D)$ on D; then $Z(x)$ is of class $\mathfrak{P}_2$ on G and we have (using Theorem 10.6)

$$\begin{aligned} I(Z, G) &= I(z, G - D) + I[H(z, D), D] \\ &\leqq I(z, G - D) + M_1 D_2[H(z, D), D] \\ &< I(z, G - D) + m_1 D_2(z, D) \leqq I(z, G), \end{aligned} \tag{16.1}$$

so that z would not minimize $I(z, G)$. Thus, from the theorems in §15, we see that $D_2[z, C(P, r)] \leqq (r/a)^{\lambda} \cdot D_2(z, G)$, $\lambda = m_1/M_1$, for each circle in G with $r \leqq a$, that z satisfies a uniform Hölder condition, with exponent $\frac{1}{2}\lambda$ on closed regions interior to G, and if the boundary values are continuous and G is bounded by a finite number of simple closed curves, then z is continuous on $\bar{G}$.

17. To proceed with the examination of our minimizing function z, we next can extend Haar's well known lemma [18] to show that z satisfies

$$\int_{R^*} (f_{p^i} dy - f_{q^i} dx) = \int\int_R f_{z^i} dx dy \tag{17.1}$$

for almost all rectangles R. If D is any simply connected subregion of G, we may find functions w^i and W^i of class $\mathfrak{P}_2''$ which satisfy

$$\begin{gathered} w^i_x - W^i_y = -f_{q^i}, \qquad w^i_y + W^i_x = f_{p^i}, \\ \int_{R^*} (W^i_x dy - W^i_y dx) = \iint_R f_{z^i} dx dy, \end{gathered} \tag{17.2}$$

almost everywhere and for almost all R on D. We also note that if D is any simply connected region with $\overline{D} \subset G$ and z is any minimizing function for $I(z, G)$ (whether continuous on the boundary or not), then (referring to (14.2) and various lemmas in potential theory) it follows that

$$\begin{gathered} D_2[z, D\cdot C(P, r)] \leqq M_4 r^\lambda, \qquad D_2[w, D\cdot C(P, r)] \leqq M_4 r^\lambda, \\ D_2[W, D\cdot C(P, r)] \leqq M_4 r^\lambda, \qquad\qquad \lambda > 0, \end{gathered} \tag{17.3}$$

and (14.2) holds with $R = \max_{x,y \in D} [x^2 + y^2 + (z^1)^2 + \cdots + (z^N)^2]$ which is bounded on such a region $\overline{D}$.

18. We next resort to a device due to Lichtenstein [19] and used by E. Hopf [4]. This consists in subtracting equation (17.1) formed for a rectangle $(a, c; b, d)$ $(a \leqq x \leqq b,\ c \leqq y \leqq d)$ from the same equation for the rectangle $(a+h, c; b+h, d)$ or the rectangle $(a, c+h; b, d+h)$ and then dividing by h. We confine ourselves to a region Δ such that $\overline{\Delta} \subset D \subset \overline{D} \subset G$. If we use the first rectangle above, we find that if h is fixed and sufficiently small,

$$\begin{aligned} \int_{R^*} & (a_{i\beta} u^\beta_x + b_{i\beta} u^\beta_y + d_{i\beta} u^\beta + g_i) dy \\ & - (b_{\beta i} u^\beta_x + c_{i\beta} u^\beta_y + e_{i\beta} u^\beta + k_i) dx \\ & = \iint_R (d_{\beta i} u^\beta_x + e_{\beta i} u^\beta_y + f_{i\beta} u^\beta + l_i) dx dy, \quad i = 1, \cdots, N, \end{aligned} \tag{18.1}$$

for almost all rectangles R in Δ where the coefficients are measurable functions and a_{ij}, for instance, is given almost everywhere by the formula

$$\begin{aligned} a_{ij}(x, y) = \int_0^1 f_{p^i p^j} [& x + th, y, z^i(x, y) + t\{z^i(x + h, y) - z^i(x, y)\}, \\ & p^i(x, y) + t\{p^i(x + h, y) - p^i(x, y)\}, \\ & q^i(x, y) + t\{q^i(x + h, y) - q^i(x, y)\}] dt \end{aligned} \tag{18.2}$$

and the other coefficients are given by similar formulas, and the $u^i(x, y)$ are the functions

$$u^i(x, y) = \frac{z^i(x + h, y) - z^i(x, y)}{h}. \tag{18.3}$$

It is easily seen, using (14.2), (17.3), and (18.2) that there exists an $h_0 > 0$ and numbers m, M, P, and μ with $0 < m \leqq M$ such that the coefficients satisfy (almost everywhere) the inequalities

$$\begin{gathered} m(\xi^2 + \eta^2) \leqq a_{\alpha\beta}\xi^\alpha\xi^\beta + 2b_{\alpha\beta}\xi^\alpha\eta^\beta + c_{\alpha\beta}\eta^\alpha\eta^\beta \leqq M(\xi^2 + \eta^2), \\ \iint_{\Delta \cdot C(P,r)} \sum_{i=1}^{N} \left[|l_i| + \sum_{j=1}^{N} \{ d_{ij}^2 + e_{ij}^2 + |f_{ij}| \} \right] dxdy \leqq Pr^\mu, \\ \iint_{\Delta \cdot C(P,r)} (g^2 + k^2) dxdy \leqq Pr^\mu, \qquad \mu > 0, \end{gathered} \tag{18.4}$$

for all circles $C(P, r)$ if $|h| < h_0$. By taking the rectangle $(a, c+h; b, d+h)$, we obtain similar equations and the same bounds as in (18.4) if h_0 is small enough, but in this case we have

$$u^i(x, y) = \frac{z^i(x, y + h) - z^i(x, y)}{h}. \tag{18.5}$$

We emphasize that the bounds m, M, P, and μ are independent of h if $|h| < h_0$. If D is simply connected, we can carry out simultaneously a similar process on the equations (17.2) and we get

$$\begin{gathered} v^i_x - V^i_y = -(b_{\beta i}u^\beta_x + c_{i\beta}u^\beta_y + e_{i\beta}u^\beta + k_i), \\ v^i_y + V^i_x = a_{i\beta}u^\beta_x + b_{i\beta}u^\beta_y + d_{i\beta}u^\beta + g_i, \\ \int_{R^*} V^i_x dy - V^i_y dx = \iint_R (d_{\beta i}u^\beta_x + e_{\beta i}u^\beta_y + f_{i\beta}u^\beta + l_i) dxdy, \\ i = 1, \cdots, N, \\ v^i(x, y) = \frac{w^i(x + h, y) - w^i(x, y)}{h}, \cdots, \\ V^i(x, y) = \frac{W^i(x + h, y) - W^i(x, y)}{h}, \cdots, \end{gathered} \tag{18.6}$$

almost everywhere and for almost all R. From (17.3) we have numbers Q and ν, independent of h if $|h| < h_0$ such that

$$\iint_{\Delta \cdot C(P,r)} (u^2 + v^2 + V^2)\,dxdy \leqq Qr^{\nu}, \qquad \nu > 0, \tag{18.7}$$

for all $C(P, r)$.

19. In the course of this research, we have demonstrated the following theorem concerning the system (18.6):

THEOREM 19.1. *Any solution* (u, v, V) *of* (18.6) *which satisfies* (18.7) *on a region* Δ *is of class* $\mathfrak{P}_2''$ *and satisfies a uniform Hölder condition on each region* H *with* $\overline{H} \subset \Delta$; *the coefficient and exponent of this Hölder condition depend only on the diameter of* H, *the distance of* $\overline{H}$ *from* Δ^*, *and on the numbers* m, M, P, μ, Q, *and* ν *of* §18.

Thus if H is any region with $\overline{H} \subset G$, we may let $h \to 0$ in §18 and we then conclude (since we may then choose regions Δ and D with $\overline{H} \subset \Delta \subset \overline{\Delta} \subset D \subset \overline{D} \subset G$) that *the first derivatives of* z *satisfy uniform Hölder conditions on* $\overline{H}$. This result plus that of E. Hopf mentioned in §2 and those in §16 lead us to the final theorem:

THEOREM 19.2. *If the integrand* $f(x, y, z, p, q)$ *of* §§13 *and* 14 *satisfies the conditions of those two sections and if* z *minimizes* $I(z, G)$ *among all functions of class* $\mathfrak{P}_2$ *on a region* G *which coincide with* z *on* G^*, *then* z *is of class* C_β'' *on any region* H *with* $\overline{H} \subset G$ *for each* $\beta < \alpha$, α *being the* α *of* §14. *Moreover if* G *is bounded by a finite number of simple closed curves and if the boundary values of* z *are continuous on* G^*, *then* z *is continuous on* $\overline{G}$.

This result together with those of §13 completes the program announced in §1 for the important class of variational problems described in §14.

It should be remarked that S. Bochner [20] has been able to solve the *Euler differential equations* for the integrand (14.3) (probably by means of the topological methods of Leray and Schauder [21]) and to prove certain properties of the solutions. He has been unable, however, to demonstrate the minimizing property of his solutions. The advantage of the procedure in this paper for this purpose is evident. By using direct methods, the author [22] has recently solved the Plateau problem (one contour case) on a Riemannian manifold of considerable generality, the surface thus obtained then being shown, using the results in this paper, to be of class C''.

V. A STUDY OF THE SYSTEMS (18.1) AND (18.6)

20. The most interesting part of this research, however, is concerned with the study of the systems (18.1) and (18.6) (which is

equivalent to (18.1) if Δ is simply connected) which leads to the result stated in Theorem 19.1. Accordingly, we shall present an outline of this study.

It is easily seen that the equations (18.1), with $d_{ij}=e_{ij}=f_{ij}=l_i=0$ are the Haar equations for a function u which minimizes the integral

$$J(u, \Delta)=\iint_\Delta (a_{\alpha\beta}u^\alpha_x u^\beta_x+2b_{\alpha\beta}u^\alpha_x u^\beta_y+c_{\alpha\beta}u^\alpha_y u^\beta_y+2g_\alpha u^\alpha_x+2k_\alpha u^\alpha_y+H)dxdy$$

where H is chosen summable and just large enough to make the integrand non-negative. It is also easy to show that $J(u, \Delta)$ is lower semicontinuous with respect to weak convergence in $\mathfrak{P}_2$ and that minimizing functions exist which coincide on Δ^* with any given function of class $\mathfrak{P}_2$. Also if a function u_0 of class $\mathfrak{P}_2$ on Δ satisfies (18.1) with $d_{ij}=e_{ij}=f_{ij}=l_i=0$, then it can easily be shown that it minimizes $J(u, \Delta)$ among all functions u of class $\mathfrak{P}_2$ on Δ which coincide with u_0 on Δ^*. If $g_i=k_i=0$ $(i=1, \cdots, N)$, we may show as in (16.1) that a minimizing function u_0 satisfies (15.3) with $K=M/m$, so that the continuity restrictions obtained for z in §16 follow. If g and k are any functions of class L_2 (whether they satisfy (18.4) or not) the above minimizing argument is valid. If g and k satisfy

$$(20.1) \qquad \iint_{\Delta\cdot C(P,r)} (g^2 + k^2)dxdy \leqq Tr^\pi, \qquad 0 < \pi < \mu,$$

for all circles with center at a point P in Δ, we conclude as in §16 that a solution u_0 satisfies a condition (15.4) and hence a condition (15.1) for circles with centers at P and therefore obeys the conclusions of Theorem 15.2.

If l_i satisfy (18.4), we may find potential functions

$$(20.2) \qquad V^i(x, y) = \frac{1}{4\pi}\iint_\Delta \log\,[(\xi - x)^2 + (\eta - y)^2]l_i(\xi, \eta)d\xi d\eta$$

which can be shown to be of class $\mathfrak{P}_2''$ and to satisfy a condition (17.3) on any region in the plane and to satisfy

$$(20.3) \qquad \int_{R^*} V^i_x dy - V^i_y dx = \iint_{\Delta\cdot R} l_i dxdy$$

for almost all rectangles R in the plane. We then can solve (18.1) with given boundary values with only the d_{ij}, e_{ij}, and $f_{ij}=0$ by solving it with l_i also zero and g_i and k_i replaced by $g_i - V^i_x$ and $k_i - V^i_y$. The solution is easily shown to be unique.

21. We now let G be an arbitrary bounded region and let $\mathfrak{V}_2$ denote the linear subspace of $\mathfrak{P}_2$ consisting of functions u in $\mathfrak{P}_2$ which vanish on G^*. For such functions u, we define the linear operator Tu as the unique solution U in $\mathfrak{V}_2$ of

$$\int_{R^*} (a_{i\beta}U_x^\beta + b_{i\beta}U_y^\beta + d_{i\beta}u^\beta)dy - (b_{\beta i}U_x^\beta + c_{i\beta}U_y^\beta + e_{i\beta}u^\beta)dx$$
$$= \int\int_R (d_{\beta i}u_x^\beta + e_{\beta i}u_y^\beta + f_{i\beta}u^\beta)dxdy.$$

It can be shown that Tu is completely continuous over $\mathfrak{V}_2$ (not $\mathfrak{P}_2$ unless Δ is restricted) and even carries any function u in $\mathfrak{V}_2$ into a function U which is continuous on Δ and can be bounded as in (15.2), the λ there being any number less than $\frac{1}{2}\mu$ (μ in (18.4) and the M depending on the norm of u and of the operator T). Then from the Riesz theory [23] of generalized integral equations, we can conclude that the transformation

$$u - \rho Tu = \phi \tag{21.1}$$

has a linear inverse if ρ is not one of a set of isolated characteristic values. If ρ is not a characteristic value, we can solve the equations (18.1), with the parameter ρ introduced as a multiplier of all the d_{ij}, e_{ij}, or f_{ij}, as follows: First let ϕ be the solution of (18.1) with $\rho=0$ and then solve (21.1) for u in terms of ϕ; the function u so obtained is immediately seen to be the desired solution. In doing this, the g_i and k_i may be merely functions of class L_2 on Δ, but we assume that the d_{ij}, e_{ij}, f_{ij}, and l_i satisfy (18.4). As in §20, if g and k satisfy (20.1) at a point P, we may draw similar conclusions, where the bound for $|\bar{u}(P)|$ given in (15.2) depends only on the quantities m, M, P, μ, T, π, δ, a, the norm of T, and the norm of the inverse operator of (21.1), δ and a having their significance in §15.

22. Let us now suppose that ρ is not a characteristic value and let P_0 be any point in Δ. Let us consider the solution u of (18.1) where $l=0$ and g and k are in L_2 on $\Delta - C(P_0, \sigma)$ and zero in $C(P_0, \sigma)$; such functions satisfy any condition of the type (20.1) for circles $C(P_0, r)$. Clearly the value of $\bar{u}(P_0)$ is a linear functional defined on the space $L_2[\Delta - C(P_0, \sigma)]$ for each $\sigma > 0$. We therefore conclude [24] the existence of functions $\Delta_1^{ij}(x_0, y_0; x, y)$ and $\Delta_2^{ij}(x_0, y_0; x, y)$ such that

$$\bar{u}^i(x_0, y_0) = \int\int_\Delta (\Delta_1^{i\beta} g_\beta + \Delta_2^{i\beta} k_\beta)dxdy.$$

From the way these Δ_k^{ij} arise, we can conclude that there exist functions $G^{ij}(x_0, y_0; x, y)$ of class $\mathfrak{P}_1'$ over Δ and of class $\mathfrak{P}_2''$ on $\Delta - C(P_0, \sigma)$ for each $\sigma > 0$ such that $G_x^{ij} = \Delta_1^{ij}$, $G_y^{ij} = \Delta_2^{ij}$. It can also be shown that the G^{ij} possess other properties which lead us to call them the *Green's matrix* for (18.1). Moreover if the g and k satisfy (20.1) for every P and r, we conclude that the continuous representative $\bar{u}$ of the solution u of (18.1) in $\mathfrak{B}_2$ is represented everywhere by

$$(22.1)\quad \bar{u}^i(x_0, y_0) = \iint_\Delta (G_x^{i\beta} g_\beta + G_y^{i\beta} k_\beta + G^{i\beta} l_\beta)\,dx\,dy, \quad i = 1, \cdots, N,$$

and $\bar{u}^i$ satisfies a uniform Hölder condition on any region H with $\bar{H}$ interior to Δ, this Hölder condition depending only on the distance of $\bar{H}$ from Δ^*, the quantities m, M, P, μ, T, π, a, and any upper bounds for δ and the norms T and the inverse in (21.1).

23. Finally, if (u, v, V) is any solution of (18.6) on a circle $C(P_1, b)$ which satisfies (18.7) there, we can, if ρ is not characteristic, express u in a circle $C(P_1, c)$ with $c < b$ in the form

$$(23.1)\quad \bar{u}^i(x_0, y_0) = \iint_{C(P_1,b)-C(P_1,d)} (G_x^{i\beta} g_\beta + G_y^{i\beta} k_\beta + G^{i\beta} l_\beta)\,dx\,dy,$$
$$(x_0, y_0) \in C(P_1, c),\ c < d < b,$$

in which

$$g_i = \overset{1}{A}_{i\beta} u^\beta + \overset{1}{B}_{i\beta} v^\beta + \overset{1}{C}_{i\beta} V^\beta, \qquad k_i = \overset{2}{A}_{i\beta} u^\beta + \overset{2}{B}_{i\beta} v^\beta + \overset{2}{C}_{i\beta} V^\beta,$$
$$l_i = \overset{3}{A}_{i\beta} u^\beta + \overset{3}{B}_{i\beta} v^\beta + \overset{3}{C}_{i\beta} V^\beta$$

where the A_{ij}^k, B_{ij}^k, and C_{ij}^k are bounded and measurable and independent of (x_0, y_0). Also there exists a number $\delta_0 > 0$ which depends only on m, M, P, and μ such that if $\delta \leqq \delta_0$, the norm of T is less than or equal to $\frac{1}{2}$ and the norm of the inverse of (21.1) is less than or equal to 2 for all $\rho \leqq 1$. Thus $\rho = 1$ is surely not a characteristic value if $b \leqq \frac{1}{2}\delta_0$ and the representation (23.1) satisfies a uniform Hölder condition in $C(P_1, c)$ which depends only on m, M, P, μ, Q, ν (of (18.7)), and $b - c$, if $b \leqq \delta_0$. Thus Theorem 19.1 then follows, for, if the solution is given on Δ and H is any subregion with $\bar{H} \subset \Delta$, we may choose b and c so that $\bar{H}$ can be covered by a finite number of circles $C(P_i, c)$ where $C(P_i, b)$ belongs to Δ for each i.

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first two appear in the Duke Mathematical Journal, vol. 6 (1940), pp. 170–186, 187–215, and the other six will probably appear shortly in the University of California Mathematical Series.

(i) *Functions of several variables and absolute continuity* I, by J. W. Calkin.

(ii) and (iii) the same, *parts* II *and* III.

(iv) *The existence of functions minimizing certain multiple integrals.*

(v) *Lemmas in potential theory.*

(vi) *A general form of Haar's lemma.*

(vii) *Elliptic linear self adjoint systems of partial differential equations in two independent variables.*

(viii) *The differentiability of functions minimizing certain double integrals.*

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NULLIFYING FUNCTIONS[1]

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Introduction. A function $f(x)$ defined on the unit interval $(0, 1)$ will be called nullifying if we can find a set S of $(0, 1)$ for which $m(S)=1$, $m(\{f(x); x \varepsilon S\})=0$. Examples of homeomorphisms which are nullifying and hence termed singular are well known.[2] We shall however consider simply the nullifying property itself.

If $f(x)$ is nullifying and $\phi(x)$ is not, one might expect that $\phi(x)+f(x)$ shares with ϕ the property of being not nullifying. But this is not always true as the following example shows. Let $.\alpha_1\alpha_2 \cdots$ denote the dyadic expansion for x, that is, $x=\alpha_1/2+\alpha_2/2^2+ \cdots$ with $\alpha_i=0$ or 1. Let $v_1(x)=.\alpha_1 0 \alpha_3 0 \cdots$ and $v_2(x)=.0\alpha_2 0\alpha_4 \cdots$. It is easily verified that both v_1 and v_2 are nullifying. Hence $f(x)=1-v_1$ is also nullifying. Let $\phi=x$. Then $f+\phi=1-v_1+x=1+v_2$ is also nullifying.

But this suggests the question: Does there exist a nullifying function $f(x)$ such that $f(x)+\rho x$ is nullifying for every value of ρ? We construct such a function in the present note.

Our method of proof can be summarized as follows. Considering the set $\{f(x)+\rho x; x \varepsilon (0, 1)\}$, we let $\rho=\cot\theta$. (Note $\theta\neq 0$.) If this set has measure zero, this will still be true if we multiply by $\sin\theta$ and conversely. Thus we may consider the sets $\{x\cos\theta+f(x)\sin\theta; x \varepsilon (0, 1)\}$ for each $\theta\neq 0$ between $-\pi/2$ and $\pi/2$. If we consider the line through the origin of inclination θ, we can assign a coordinate to each of its points in the usual manner with positive direction to the right or, in the case of the y axis, upwards. The set $\{x\cos\theta+f(x)\sin\theta; x \varepsilon (0, 1)\}$ is the set of coordinates of the projection onto this line of the graph of $f(x)$. Thus it suffices to find a function $f(x)$ which is such that the projection of its graph onto any line not parallel to the x axis is of measure zero. We proceed to find the graph of such a function by an intersection process on sets in the plane. This process is described in detail in what follows.

A more general question is: Given $F(x, y, \rho)$, under what circumstances can we find a function $f(x)$ such that $F(x, f(x), \rho)$ is nullifying in x for every value of ρ? It is comparatively easy to abstract the properties of $F=y+\rho x$ which are essential to the present discussion, and these will prove sufficient to obtain an answer to the question.

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[1] Presented to the Society, December 29, 1939.

[2] Cf. E. R. van Kampen and Aurel Wintner, *On a singular monotone function*, Journal of the London Mathematical Society, vol. 12 (1937), pp. 243–244. References to preceding examples are given in this paper.

However the writer hopes to obtain a more penetrating analysis of this subject soon. The writer will also consider the more general question of obtaining an $f(x)$ the substitution of which will make nullifying not only one but a set of F's.

Definition of a G_s. Let us divide evenly the unit square of the plane in squares of side $1/2^s$. The coordinates of the vertices of these squares are dyadic rational, that is, of the form $a/2^s$ for some integer a. Two squares are said to be in the same column if they have the same projection on the x axis. A set of such squares with one and only one (closed) square in each column will be termed[3] a G_s.

Notation. If l_θ is the line through the origin with inclination θ and P is any point of the plane, we shall denote by $\pi_\theta(P)$ the projection of P onto l_θ. If S is a set of points, we denote by $\pi_\theta(S)$ the projection of S onto l_θ. The linear measure of a linear set S we denote by $\mu(S)$.

LEMMA 1. *Let m be such that $0 \leqq m \leqq 1$. Let $\epsilon > 0$ and a G_s be given. Let α be such that $\pi/2 \geqq \alpha \geqq \pi/4$ and* $\cot \alpha = m$. *Then we can find a $G_{s+t} \subset G_s$ such that for $0 \leqq \theta \leqq \pi/2$,*

$$\mu(\pi_\theta(G_{s+t})) \leqq (1 + m^2)^{1/2} \sin \left| \theta - \alpha \right| + \epsilon.$$

PROOF. Let us consider a square B of G_s. Then it is possible to take a line $l^{(1)}$ of slope $-m$, such that for every x in the projection of B on the x axis, we have a point (x, y) in B and on $l^{(1)}$. Let $l^{(1)}$ intersect the left-hand side of B at P_B and the right-hand side at Q_B.

Let us divide B into squares of side $1/2^{s+t}$. Now it is readily seen that for each column of smaller squares in B there is at least one square with an interior point on $l^{(1)}$. (If $m=0$, this is not true, but then we can substitute for "interior point," "interior point or point on the upper side.") Define G_{s+t} so as to contain for each column the lowest such square.

We also take a square of side $1/2^{s+t}$, whose upper right-hand vertex is at P_B. We denote the lower left-hand end point of this square by P_B'. Similarly we take a square of side $1/2^{s+t}$ whose lower left-hand vertex is Q_B, and we denote the upper right-hand vertex by Q_B'.

We are going to consider $\pi_\theta(G_{s+t})$ and we let $<$ and $\leqq$ refer to the order of the points on l_θ, the direction of the greater being to the right of the smaller.

We shall show that if $P \in G_{s+t}$, and $0 \leqq \theta \leqq \alpha$, then $\pi_\theta(P_B') \leqq \pi_\theta(P)$.

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[3] We shall ignore the logical distinction between a set of squares and the corresponding set of points. G_s denotes either a (closed) set of points or a set of squares according to the context.

For P_B' (x_B', y_B') is the lower left-hand vertex of a square whose upper right-hand vertex alone is on $l^{(1)}$. Now every square C of G_{s+t} contains an interior point which is on $l^{(1)}$ (if $m=0$, confer above). Thus if $l^{(1)\prime}$ is parallel to $l^{(1)}$ and through P_B', every point P (x, y) of G_{s+t} is above or on $l^{(1)\prime}$. Furthermore $x>x_B'$.

Let P' (x', y') be the intersection of $l^{(1)\prime}$ and the line through P parallel to the y axis. P is above P' and hence $\pi_\theta(P)\geqq\pi_\theta(P')$. On the other hand $x'=x>x_B'$. Let $\bar{l}$ denote the line through P_B' perpendicular to l_θ. The slope of $\bar{l}$ is more strongly negative than that of $l^{(1)\prime}$. It follows that for $x\geqq x_B'$, $l^{(1)\prime}$ is to the right of $\bar{l}$ and hence $\pi_\theta(P')\geqq\pi_\theta(P_B')$. This and the previous result yield $\pi_\theta(P)\geqq\pi_\theta(P_B')$.

A similar argument will show that $\pi_\theta(Q_B')\geqq\pi_\theta(P)$.

Thus $\pi_\theta(P_B')\leqq\pi_\theta(P)\leqq\pi_\theta(Q_B')$. This implies that the projection of that part of G_{s+1} which lies in B is contained in the interval $\pi_\theta(P_B')\pi_\theta(Q_B')$. Now if P_B has the coordinates (x_B, y_B), then Q_B has the coordinates $(x_B+1/2^s, y_B-m/2^s)$, P_B' has the coordinates $(x_B-1/2^{s+t}, y_B-1/2^{s+t})$, and Q_B' has the coordinates $(x_B+1/2^s+1/2^{s+t}, y_B-m/2^s+1/2^{s+t})$.

Consider $l_{\theta+\pi/2}$, the line through the origin perpendicular to l_θ. According to a formula of elementary analytic geometry, the directed distance of any point P (x, y) to $l_{\theta+\pi/2}$ is $x\cos\theta+y\sin\theta$, the directed distance being positive if (x, y) is to the right of $l_{\theta+\pi/2}$ and negative to the left. This directed distance is not changed if we project P onto l_θ, and so $x\cos\theta+y\sin\theta$ represents also the directed distance of $\pi_\theta(P)$ from $l_{\theta+\pi/2}$, or since l_θ is perpendicular to $l_{\theta+\pi/2}$, from the origin O along l_θ.

Thus $O\pi_\theta(P_B')$ has length $(x_B-1/2^{s+t})\cos\theta+(y_B-1/2^{s+t})\sin\theta$ and $O\pi_\theta(Q_B')$ has the length $(x_B+1/2^s+1/2^{s+t})\cos\theta+(y_B-m/2^s+1/2^{s+t})\sin\theta$. Hence $\pi_\theta(P_B')\pi_\theta(Q_B')$ has length

$$(1/2^s + 1/2^{s+t-1})\cos\theta - (m/2^s - 1/2^{s+t-1})\sin\theta$$

for $0\leqq\theta\leqq\alpha$. Thus the projection of that part of G_{s+1} which lies in B has measure not greater than

$$1/2^s(\cos\theta - m\sin\theta) + 1/2^s(1/2^{t-1}(\cos\theta + \sin\theta)$$
$$< 1/2^s(\cos\theta - m\sin\theta) + 1/2^s(1/2^{t-2}).$$

There are 2^s squares like B, and this implies that for $0<\theta\leqq\alpha$, the projection of G_{s+t} has measure not greater than

$$(\cos\theta - m\sin\theta) + 1/2^{t-2} = (1+m^2)^{1/2}\sin(\alpha-\theta) + 1/2^{t-2}.$$

A similar argument holds in the case $\alpha<\theta\leqq\pi/2$, with however the smaller added squares in different positions, and during the argument certain inequalities are reversed. The corresponding result is that, for $\alpha<\theta\leqq\pi/2$,

$$\begin{aligned}\mu(\pi_\theta(G_{s+t})) &< m\sin\theta-\cos\theta+1/2^{t-2}\\ &= (1+m^2)^{1/2}\sin(\theta-\alpha)+1/2^{t-2}.\end{aligned}$$

If t is taken large enough so that $1/2^{t-2}<\epsilon$, the final results of the preceding two paragraphs are sufficient to prove our lemma.

LEMMA 2. *Suppose* $m>1$, *and that a* G_s *and an* $\epsilon>0$ *are given. Let* α *be such that* $\pi/4>\alpha>0$, $\cot\alpha=m$. *Then we can find a* $G_{s+t}\subset G_s$ *such that for* $0\leqq\theta<\pi/2$,

$$\mu(\pi_\theta(G_{s+t}))\leqq(1+m^2)^{1/2}\sin|\theta-\alpha|+\epsilon.$$

PROOF. Let B be any square of G_s. Let $l^{(1)}, l^{(2)}, \cdots, l^{(k)}$ denote a set of lines of slope $-m$ with the following properties: (a) each $l^{(i)}$ contains an interior point of B; (b) if $P_B^{(i)}Q_B^{(i)}$ is the line segment $B\cdot l^{(i)}$ (B is closed), $P_B^{(1)}$ is on the left side of B, $Q_B^{(k)}$ is on the right side of B, and for $i=1, 2, \cdots, k-1$, $P_B^{(i+1)}$ is on the upper side of B, $Q_B^{(i)}$ is on the lower side of B, and $Q_B^{(i)}$ has the same x coordinate as $P_B^{(i+1)}$. The existence of such a set of lines is easily shown.

Let the coordinates of $P_B^{(i)}$ be $(x_B^{(i)}, y_B^{(i)})$, those of $Q_B^{(i)}$ be $(u_B^{(i)}, v_B^{(i)})$. From the above we see that $x_B^{(1)}=a/2^s$, $y_B^{(1)}=b/2^s+\rho/2^s$ where $0<\rho\leqq1$, $u_B^{(1)}=a/2^s+\rho/m\cdot2^s$, $u_B^{(2)}=a/2^s+(\rho+1)/m\cdot2^s$, and in general $u_B^{(i)}=a/2^s+(\rho+i-1)/2^s\cdot m$, for $i=1, 2, \cdots, k-1$, while $u_B^{(k)}=(a+1)/2^s$. From this it follows that

$$(\rho+k-2)/m\cdot2^s<1/2^s\leqq(\rho+k-1)/m\cdot2^s$$

or $(\rho+k-2)<m<\rho+k-1$ or $k-1<m+(1-\rho)\leqq k$. Hence if $[m]$ denotes the largest integer less than or equal to m, then $k=[m]$, $[m]+1$, or $[m]+2$. Thus $k\leqq m+2$.

We next divide the square B of S_s into smaller squares of side $1/2^{s+t}$. The lines $l^{(1)}, \cdots, l^{(k)}$ have been chosen so that for each x in the projection of B on the x axis, there is at least one $l^{(i)}$ which contains a point (x, y) in B. This can be used to show that for each column of smaller squares in B, there is at least one smaller square which contains a point on some $l^{(i)}$ and interior to B. Define G_{s+t} so as to contain the lowest such square in each column.

For each i, let us consider those squares of G_{s+t} which contain points of $l^{(i)}$. An argument similar to that used in Lemma 1 will show that the projection of this part of G_{s+t} on l_θ has measure not greater than

$$(\text{length of } l^{(i)}\cdot B)\sin|\theta-\alpha|+1/2^{s+t-2}.$$

The sum of the lengths of the $l^{(i)}\cdot B$ is $(1/2^s)(1+m^2)^{1/2}$ and there are at most $[m]+2$ of them. Thus

$$\mu(\pi_\theta(G_{s+t}\cdot B)) \leqq (1/2^s)((1+m^2)^{1/2}\sin|\theta-\alpha| + ([m]+2)/2^{t-2}).$$

Since there are 2^s such squares B, we have

$$\mu(\pi_\theta(G_{s+t})) \leqq (1+m^2)^{1/2}\sin|\theta-\alpha| + ([m]+2)/2^{t-2}.$$

We can take t sufficiently large to obtain our result.

This result is interesting only in the range $0<\theta<2\alpha$, for outside this range other methods give more effective inequalities.

The argument of Lemmas 1 and 2 can be modified to apply to the case in which m is negative, with the following result.

LEMMA 3. *Suppose an $m<0$, a G_s and an $\epsilon>0$ are given. Let α be such that* $\cot\alpha=m$ *and* $-\pi/2\leqq\alpha<0$. *Then we can find a $G_{s+t}\subset G_s$ such that for* $-\pi/2\leqq\theta<0$,

$$\mu(\pi_\theta(G_{s+t})) \leqq (1+m^2)^{1/2}\sin|\theta-\alpha|+\epsilon.$$

LEMMA 4. *Let m_1 and m_2 be given with $m_2>m_1\geqq 0$. Let α_i be such that* $\pi/2\geqq\alpha_i>0$, $\cot\alpha_i=m_i$ for $i=1, 2$. *Consider the function*

$$F(\theta) = \min\left((1+m_2^2)^{1/2}\sin(\theta-\alpha_2),\ (1+m_1^2)^{1/2}\sin(\alpha_1-\theta)\right)$$

for $\alpha_2\leqq\theta\leqq\alpha_1$. Then for this range of θ,

$$F(\theta) \leqq (m_2-m_1)/2(m_1^2+1)^{1/2}.$$

PROOF. For $\alpha_2\leqq\theta\leqq\alpha_1$, $(1+m_2^2)^{1/2}\sin(\theta-\alpha_2)$ is increasing while $(1+m_1)^{1/2}\sin(\alpha_1-\theta)$ is decreasing. Also we readily see that if η is such that

$$(1+m_2^2)^{1/2}\sin(\eta-\alpha_2) = (1+m_1^2)^{1/2}\sin(\alpha_1-\eta)$$

then $F(\theta)$ is equal to the first expression for $\alpha_2\leqq\theta\leqq\eta$ and to the second expression for $\eta\leqq\theta\leqq\alpha_1$. Hence $F(\eta)$ is the maximum value of $F(\theta)$ for the given range of θ.

Expanding $\sin(\eta-\alpha_2)$ and $\sin(\alpha_1-\eta)$, using the definition of α_i, collecting terms in $\sin\eta$ and $\cos\eta$ and dividing by $\cos\eta$ yields

$$\tan\eta = 2/(m_1+m_2).$$

The value of $F(\eta)$ is then seen to be

$$(m_2-m_1)/2(1+(m_1+m_2)^2/4)^{1/2} \leqq (m_2-m_1)/2(1+m_2^2)^{1/2}.$$

This and the result of the preceding paragraph prove the lemma.

Consider the sequence 0, 1, 0, -1, 2, 3/2, 1, 1/2, 0, $-1/2$, -1, $-3/2$, -2, 4, 15/4, $\cdots$, 1/4, 0, $-1/4$, $\cdots$, $-15/4$, -4, $\cdots$. We denote the ith term in this sequence by m_i. Notice that the terms of this sequence can be grouped so that the first group contains 1, the second group 3, the third group 9, and the kth group $2^{2k-3}+1$. The maximum value in the kth group is 2^{k-2} and the difference between any two adjacent terms in this group is $1/2^{k-2}$.

Let $\{\epsilon_i\}$ denote the sequence such that if i is a subscript of the kth group of the preceding paragraph, then $\epsilon_i=1/2^k$. Let $G^{(0)}=G_0$, the unit square itself, and $G^{(i)}$ be defined as the G_{s+t}, which results when either Lemma 1, 2, or 3 (depending on m_i) is applied to $G^{(i-1)}$, m_i and ϵ_i.

LEMMA 5. *Let $G^{(i)}$ be as above. Let θ be such that $\pi/2\geqq\theta>0$. Then* $\lim_{i\to\infty}\mu(\pi_\theta(G^{(i)}))=0$.

PROOF. Let $\epsilon>0$ be given. Take k such that $\epsilon/2>1/2^{k-1}$ and $2^{k-2}>\cot\theta$. Then we can find an m_i in the kth group such that $m_i>0$ and $m_i\geqq\cot\theta>m_{i+1}$.

Now since $G^{(i)}\supset G^{(i+1)}$, we have $\pi_\theta(G^{(i)})\supset\pi_\theta(G^{(i+1)})$ and $\mu(\pi_\theta(G^{(i)}))\geqq\mu(\pi_\theta(G^{(i+1)}))$. Thus $\mu(\pi_\theta(G^{(i+1)}))$ is subject to the inequality which Lemmas 1 and 2 impose upon $\mu(\pi_\theta(G^{(i)}))$. Thus

$$\mu(\pi_\theta(G^{(i+1)}))\leqq\min\left[(1+m_i^2)^{1/2}\sin\left|\theta-\alpha_i\right|,\ (1+m_{i+1}^2)^{1/2}\cdot\sin\left|\theta-\alpha_{i+1}\right|\right]+\epsilon_i,\qquad(\cot\alpha_i=m_i).$$

Since $\alpha_i\leqq\theta\leqq\alpha_{i+1}$, Lemma 4 yields

$$\mu(\pi_\theta(G^{(i+1)}))\leqq(m_i-m_{i+1})/2(1+m_{i+1}^2)^{1/2}+\epsilon_i<1/2^{k-1}+1/2^k<\epsilon.$$

Also if $j\geqq i+1$, $G^{(j)}\subset G^{(i+1)}$, and we obtain

$$0<\mu(\pi_\theta(G^{(j)}))\leqq\mu(\pi_\theta(G^{(i+1)}))<\epsilon.$$

Thus we have shown that given an $\epsilon>0$, we can find an i such that for $j>i$ this equation holds. The lemma is now proved.

LEMMA 6. *Let $G^{(i)}$ be as in Lemma 5. Let $G=\prod G^{(i)}$. Then G is a non-empty closed set such that if $0<\theta\leqq\pi/2$, then $\mu(\pi_\theta(G))=0$.*

PROOF. Since the $G^{(i)}$'s form a decreasing sequence of closed sets, their intersection is a non-empty closed set. Since $G\subset G^{(i)}$, $0\leqq\mu(\pi_\theta(G))\leqq\mu(\pi_\theta(G^{(i)}))$. Thus Lemma 5 now implies $\mu(\pi_\theta(G))=0$.

Results similar to those of Lemmas 5 and 6 hold for $0>\theta>-\pi/2$. The method of obtaining them should be clear from the preceding discussion and we merely state the final result as follows.

LEMMA 7. *If G is as in Lemma 6, then for $0>\theta>-\pi/2$, $\mu(\pi_\theta(G))=0$.*

LEMMA 8. *G is the graph of a function $y=f(x)$.*

PROOF. For $0 \leqq a \leqq 1$, let p_a denote the line $x = a$. Then the sets $p_a \cdot G^{(i)}$ form a decreasing sequence of closed intervals on p_a, with one and only one point (a, b) in common. Thus $p_a \cdot G$ consists of one and only one point (a, b) and b is a function of a.

Now, as we have pointed out in the proof of Lemma 1, for any point P (x, y), $x \cos \theta + y \sin \theta$ is the directed distance of $\pi_\theta(P)$ along l_θ from the origin. Using Lemmas 6, 7, and 8, we obtain that for $\pi/2 \geqq \theta > 0$ or $0 > \theta > -\pi/2$,

$$\begin{aligned} 0 = \mu(\pi_\theta(G)) &= \mu(\{x \cos \theta + y \sin \theta;\ (x, y) \,\varepsilon\, G\}) \\ &= \mu(\{x \cos \theta + f(x) \sin \theta;\ 0 \leqq x \leqq 1\}) \\ &= |\sin \theta|\, \mu(\{x \cot \theta + f(x);\ 0 \leqq x \leqq 1\}). \end{aligned}$$

Letting $\rho = \cot \theta$ we obtain that for every value of ρ,

$$\mu(\{f(x) + \rho x;\ 0 \leqq x \leqq 1\}) = 0.$$

Since $f(x)$ is the limit of step functions, it is measurable, and we have proved the following theorem.

THEOREM. *There exists a measurable function* $f(x)$ *defined for* $0 \leqq x \leqq 1$, *such that for every value of* ρ, $f(x) + \rho x$ *is nullifying.*

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CERTAIN SELF-RECIPROCAL FUNCTIONS

BRIJ MOHAN

In 1932 and 1933 I [5, 6] gave some rules connecting different classes of self-reciprocal functions. The object of this note is to derive some new self-reciprocal functions with the help of those rules.

I will say that a function is R_ν if it is self-reciprocal for J_ν transforms, where $\nu > -1$.

I will make use of the following results given in the papers referred to:

If $f(x)$ is R_μ, the functions $g(x)$ given by the following integral formulas are all R_ν:

(i) $$g(x) = x^{(\nu-\mu+1)/2}\int_0^\infty y^{(\nu-\mu+1)/2}J_{(\mu+\nu)/2}(xy)f(y)dy,$$

(ii) $$g(x) = x^{(\mu-\nu+1)/2}\int_0^\infty y^{(\mu-\nu+1)/2}J_{(\mu+\nu)/2}(xy)f(y)dy,$$

(iii) $$g(x) = \int_0^\infty \frac{y^{\mu+1/2}f(xy)}{(1+y^2)^{1+\mu/2+\nu/2}}dy,$$

(iv) $$g(x) = \int_1^\infty \frac{y^{1/2-\mu}f(xy)}{(y^2-1)^{1-\mu/2+\nu/2}}dy,$$

(v) $$g(x) = \int_0^1 \frac{y^{1/2+\mu}f(xy)}{(1-y^2)^{1+\mu/2-\nu/2}}dy.$$

If, in (ii) we take the familiar R_μ function

(1) $$x^{\mu+1/2}e^{-x^2/2}$$

for $f(x)$, we get

$$g(x) = x^{(\mu-\nu+1)/2}\int_0^\infty y^{(\mu-\nu+1)/2}J_{(\mu+\nu)/2}(xy)\cdot y^{\mu+1/2}e^{-y^2/2}dy$$

$$= x^{(\mu-\nu+1)/2}\int_0^\infty y^{3\mu/2-\nu/2+1}e^{-y^2/2}J_{\mu/2+\nu/2}(xy)dy.$$

Evaluating this integral by Hankel's formula [7], we get

$$g(x) = x^{(\mu-\nu+1)/2}\frac{\Gamma(\mu+1)(x/2^{1/2})^{\mu/2+\nu/2}}{2^{-3\mu/4+\nu/4}\Gamma(1+\mu/2+\nu/2)}e^{-x^2/2}$$

$$\cdot\ {}_1F_1(\nu/2-\mu/2;\ 1+\mu/2+\nu/2;\ x^2/2),\qquad \mu > -1.$$

This shows that the function $x^{\mu+1/2}e^{-x^2/2}\,{}_1F_1(\nu/2-\mu/2;1+\mu/2+\nu/2;x^2/2)$, $(\mu>-1)$, which is the same as

$$x^{\nu+2n+1/2}e^{-x^2/2}\,{}_1F_1(-n;n+\nu+1;x^2/2),\qquad \nu+2n>-1, \tag{2}$$

is R_ν.

Particular cases. (α) Since, when n is a positive integer,

$$L_n^{(\alpha)}(x)=\frac{\Gamma(1+\alpha+n)}{n!\Gamma(\alpha+1)}\,{}_1F_1(-n;\alpha+1;x),$$

where $L_n^{(\alpha)}(x)$ denotes the generalized Laguerre polynomial of order n, it follows that the function $x^{\nu+2n+1/2}e^{-x^2/2}L_n^{(n+\nu)}(x^2/2)$, $(\nu+2n>-1)$, is R_ν for positive integral values of n.

This result was given by Howell [4].

(β) Since, when n is a positive integer,

$$T_\mu^n(x)=\frac{(-1)^n}{n!\Gamma(1+\mu)}\,{}_1F_1(-n;1+\mu;x),$$

where $T_\mu^n(x)$ is Sonine's polynomial of order n, it follows that the function

$$x^{\nu+2n+1/2}e^{-x^2/2}T_{n+\nu}^n(x^2/2),\qquad \nu+2n>-1,$$

is R_ν for positive integral values of n.

(γ) If we put $n=-\nu-1/2$, our function becomes $x^{-\nu-1/2}e^{-x^2/2}\cdot{}_1F_1(\nu+1/2;1/2;x^2/2)$.

As a particular case, when ν is an integer, this becomes a constant multiple of

$$x^{-\nu-1/2}e^{-x^2/4}D_{-2\nu-1}(x). \tag{3}$$

(δ) When $n=1/2-\nu$, the function is $x^{-\nu+3/2}e^{-x^2/2}\,{}_1F_1(\nu-1/2;3/2;x^2/2)$, $(\nu<2)$.

If ν is an integer, this is a constant multiple of

$$x^{-\nu+1/2}e^{-x^2/4}D_{-2\nu+1}(x). \tag{4}$$

The functions (3) and (4) were given by Mitra [2].

Taking the same function (1) for $f(x)$, we have, from (iii),

$$g(x)=\int_0^\infty\frac{y^{\mu+1/2}e^{-x^2y^2/2}(xy)^{\mu+1/2}}{(1+y^2)^{1+\mu/2+\nu/2}}\,dy=x^{\mu+1/2}\int_0^\infty\frac{y^{2\mu+1}e^{-x^2y^2/2}}{(1+y^2)^{1+\mu/2+\nu/2}}\,dy.$$

Evaluating this integral by a formula given by Whittaker [8] we find that $g(x)$ is a constant multiple of

$$x^{(\mu+\nu-1)/2}e^{x^2/4}W_{-1/2-3\mu/4-\nu/4,(\nu-\mu)/4}(x^2/2),$$

where $\mu > -1$.

Thus, we see that the function

$$x^{\nu-2m-1/2}e^{x^2/4}W_{3m-\nu-1/2,m}(x^2/2), \tag{5}$$

where $\nu-4m > -1$, is R_ν.

This function was given by Bailey [1].[1]

If we put the same function (1) for $f(x)$ in (i) and evaluate the corresponding integral by Hankel's formula [7], we arrive back at the R_ν function

$$x^{\nu+1/2}e^{-x^2/2}. \tag{6}$$

If, in (iv), we take the same function for $f(x)$ we arrive at the function (6) again.

If we take the same function for $f(x)$ in (v), we again arrive at the function (2).

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[1] Recently, Dhar [3] has given another proof of the fact that this function is R_ν. The other R_ν function $x^{\nu+2m-1/2}e^{x^2/4}W_{-3m-\nu-1/2,m}(x^2/2)$ that he gives, is the same as (5) with "$-m$" written for "m."

A REMARK ON THE SUM AND THE INTERSECTION OF TWO NORMAL IDEALS IN AN ALGEBRA

TADASI NAKAYAMA

Let F be a quotient field of a commutative domain of integrity o in which the usual arithmetic holds.[1] Consider an algebra $\mathfrak{A}$ with a unit element over F. Let $\mathfrak{J}_1$, $\mathfrak{J}_2$, $\mathfrak{J}_3$, $\mathfrak{J}_4$ be four arbitrary maximal orders in $\mathfrak{A}$ and $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$ be three arbitrary normal ideals. We prove the following theorems.

THEOREM 1. *If* $\mathfrak{J}_1 \cap \mathfrak{J}_2 = \mathfrak{J}_3 \cap \mathfrak{J}_4$ [*or* $(\mathfrak{J}_1, \mathfrak{J}_2) = (\mathfrak{J}_3, \mathfrak{J}_4)$], *then either* $\mathfrak{J}_1 = \mathfrak{J}_3$, $\mathfrak{J}_2 = \mathfrak{J}_4$ *or* $\mathfrak{J}_1 = \mathfrak{J}_4$, $\mathfrak{J}_2 = \mathfrak{J}_3$.

THEOREM 2. *Both the left and the right orders of* $(\mathfrak{J}_1, \mathfrak{J}_2)$ *are* $\mathfrak{J}_1 \cap \mathfrak{J}_2$. *Also* $\mathfrak{J}_1 \cap \mathfrak{J}_2 \subseteqq \mathfrak{J}_3$ *if and only if* $(\mathfrak{J}_1, \mathfrak{J}_2) \supseteqq \mathfrak{J}_3$; *if this is the case the distance ideal* $\mathfrak{d}_{21}$ *of* $\mathfrak{J}_2$ *to* $\mathfrak{J}_1$ *is divisible by the distance ideal*[2] $\mathfrak{d}_{31}$ *of* $\mathfrak{J}_3$ *to* $\mathfrak{J}_1$.

THEOREM 3. *The left, say, order* $\mathfrak{o}$ *of the intersection* $\mathfrak{a} \cap \mathfrak{b}$ [*the sum* $(\mathfrak{a}, \mathfrak{b})$] *is an intersection of two suitable maximal orders.*

More precisely, if $\mathfrak{r}$ and $\mathfrak{s}$ are normal ideals such that $\mathfrak{b} = \mathfrak{r}\mathfrak{a}\mathfrak{s}$ in the sense of proper multiplication and if $\mathfrak{t}$ is the smallest two-sided ideal of the right order of $\mathfrak{a}$ which divides $\mathfrak{s}$ while $\mathfrak{t}'$ is the largest two-sided ideal of the same maximal order which is divisible by $\mathfrak{s}$, then $\mathfrak{o}$ is the intersection of the left orders of the two normal ideals $\mathfrak{a} \cap \mathfrak{r}\mathfrak{a}\mathfrak{t}$ and $\mathfrak{a} \cap \mathfrak{r}\mathfrak{a}\mathfrak{t}'$ [$(\mathfrak{a}, \mathfrak{r}\mathfrak{a}\mathfrak{t})$ and $(\mathfrak{a}, \mathfrak{r}\mathfrak{a}\mathfrak{t}')$].[3] *The left order of* $\mathfrak{a} \cap \mathfrak{b}$ *coincides with the right order of* $(\mathfrak{a}^{-1}, \mathfrak{b}^{-1})$.

THEOREM 4. $\mathfrak{a} \cap \mathfrak{b} \subseteqq \mathfrak{c}$ *implies* $(\mathfrak{a}^{-1}, \mathfrak{b}^{-1}) \supseteqq \mathfrak{c}^{-1}$ *and conversely.*

For the proof we have, according to the well known reduction, only to treat the case where F is a p-adic field $F = F_p$ and $\mathfrak{A}$ is a normal simple algebra over F. Then $\mathfrak{A}$ is a (complete) matric ring $D_r = \sum_{i,k=1}^{r} \epsilon_{ik} D$ over a division algebra D, where ϵ_{ik} is a system of matric units commutative with every element of D. D possesses a unique maximal order I, and I has a unique prime ideal P.

Notation. If a_{ik}, $(i, k = 1, 2, \cdots, r)$, is a system of rational integers, we denote by $M(a_{ik})$ the ideal $\sum_{i,k} \epsilon_{ik} P^{a_{ik}}$ in $\mathfrak{A}$.

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[1] In the following we shall adopt the terminologies used in M. Deuring, *Algebren*, Ergebnisse der Mathematik, vol. 4, no. 1, 1935.

[2] If the algebra is a quaternion algebra, then the converse is also valid. Cf. M. Eichler, Journal für die reine und angewandte Mathematik, vol. 174 (1936), §7.

[3] Thus the intersection and the sum are no more normal ideals except for trivial cases; cf. Nakayama, Proceedings of the Imperial Academy of Japan, vol. 12 (1936).

$M(a_{ik})$ is an order if and only if $a_{ii}=0$, $a_{il}+a_{lk}\geqq a_{ik}$ for all i, k, l. On assuming this condition it is a maximal order if and only if $\sum a_{ik}=0$. By a simple calculation we then have the following lemma.[4]

LEMMA 1. *A necessary and sufficient condition that $M(a_{ik})$ be a maximal order is that there should exist r rational integers c_i such that $a_{ik}=c_k-c_i$. Every normal ideal whose left and right orders are $M(c_k-c_i)$ and $M(d_k-d_i)$ respectively has the form $P^aM(d_k-c_i)=M(d_k-c_i+a)$.*

It follows from a lemma of Chevalley[5] that a maximal order in $\mathfrak{A}$ has really the form $M(a_{ik})$ (whence the form $M(c_k-c_i)$) whenever it contains all diagonal ϵ_{11}, ϵ_{22}, $\cdots$, ϵ_{rr}.

LEMMA 2. *There exists a regular element α in $\mathfrak{A}$ such that*

$$\alpha^{-1}\mathfrak{J}_1\alpha = M(0), \quad \alpha^{-1}\mathfrak{J}_2\alpha = M(c_k - c_i); \qquad c_1 \geqq c_2 \geqq \cdots \geqq c_r.$$

PROOF. There is, as is well known, a regular element β such that $\beta^{-1}\mathfrak{J}_1\beta=M(0)$. Consider the distance ideal $\mathfrak{d}_{12}=(\mathfrak{J}_2\mathfrak{J}_1)^{-1}=\mathfrak{J}_1\delta$ of $\mathfrak{J}_1$ to $\mathfrak{J}_2$. The theory of elementary divisors tells the existence of two units ξ, η in $M(0)$ such that $\gamma=\xi\beta^{-1}\delta\beta\eta$ is a diagonal matrix with diagonal elements P^{c_i}, $(c_1\geqq\cdots\geqq c_r)$, $\gamma=\sum\epsilon_{ii}P^{c_i}$, where we denote, for the sake of convenience, a prime element of the prime ideal P by the same letter P. Put $\alpha=\beta\eta$. Then this α possesses the required property: $\alpha^{-1}\mathfrak{J}_1\alpha=\eta^{-1}\beta^{-1}\mathfrak{J}_1\beta\eta=M(0)$, $\alpha^{-1}\mathfrak{J}_2\alpha=\gamma^{-1}M(0)\gamma=M(c_k-c_i)$.

LEMMA 3. *There exist two regular elements α, β in $\mathfrak{A}$ such that*

$$\alpha\mathfrak{a}\beta = M(0), \qquad \alpha\mathfrak{b}\beta = M(d_k - c_i);$$
$$c_1 \geqq c_2 \geqq \cdots \geqq c_r, \ d_1 \geqq d_2 \geqq \cdots \geqq d_r.$$

PROOF. Let $\mathfrak{J}_1'$, $\mathfrak{J}_2'$ $[\mathfrak{J}_3', \mathfrak{J}_4']$ be the left and the right orders of $\mathfrak{a}$ $[\mathfrak{b}]$. According to the above lemma there exist γ, β such that $\gamma^{-1}\mathfrak{J}_1'\gamma=\beta^{-1}\mathfrak{J}_2'\beta=M(0)$, $\gamma^{-1}\mathfrak{J}_3'\gamma=M(c_k-c_i)$, $\beta^{-1}\mathfrak{J}_4'\beta=M(d_k'-d_i')$. $\gamma^{-1}\mathfrak{a}\beta$ is a two-sided ideal of $M(0)$ and has a form $P^aM(0)$. Put $\alpha=(\gamma P^a)^{-1}$. Then $\alpha\mathfrak{a}\beta=M(0)$. Moreover, $\alpha\mathfrak{b}\beta$ is of a form $M(d_k'-c_i+b)$ (Lemma 1). We put $d_k=d_k'+b$, and this completes the proof.

We note further that the left order of an ideal $M(a_{ik})$ is $M(b_{ik})$ where $b_{ik}=\max_j(a_{ij}-a_{kj})$.

After these preliminaries our theorems are easy to prove. In Theorem 1 we may, according to Lemma 2, assume that $\mathfrak{J}_1=M(0)$, $\mathfrak{J}_2=M(c_k-c_i)$, $(c_1\geqq\cdots\geqq c_r)$. Suppose $\mathfrak{J}_1\cap\mathfrak{J}_2=\mathfrak{J}_3\cap\mathfrak{J}_4$. Since

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[4] Cf. Nakayama, Japanese Journal of Mathematics, vol. 13 (1937), p. 339.

[5] Chevalley, Abhandlungen aus dem mathematischen Seminar der Hamburgischen Universität, vol. 10 (1934), p. 87.

$\epsilon_{ii} \in \mathfrak{J}_1 \cap \mathfrak{J}_2 \subseteqq \mathfrak{J}_3, \mathfrak{J}_4$, it follows that $\mathfrak{J}_3, \mathfrak{J}_4$ have the form $\mathfrak{J}_3 = M(d_k - d_i)$; $\mathfrak{J}_4 = M(f_k - f_i)$. Moreover $\max(d_k - d_i, f_k - f_i) = \max(0, c_k - c_i)$, $(i, k = 1, 2, \cdots, r)$. This implies $\max(d_k - d_i, f_k - f_i) = 0$ if $i \geqq k$, whence $d_1 \geqq \cdots \geqq d_r$ and $f_1 \geqq \cdots \geqq f_r$. On applying the same relation to $i = 1$, $k = r$, we find that either $d_1 = d_r$ or $f_1 = f_r$. In the first case we have $d_1 = \cdots = d_r$, $f_1 - f_i = c_1 - c_i$, $(i = 1, 2, \cdots, r)$, whence $\mathfrak{J}_1 = \mathfrak{J}_3$, $\mathfrak{J}_2 = \mathfrak{J}_4$. The second case gives of course $\mathfrak{J}_1 = \mathfrak{J}_4$, $\mathfrak{J}_2 = \mathfrak{J}_3$.

The assertion about the sum follows now from Theorem 2, which is in turn contained in Theorems 3 and 4.

As to Theorem 3 we notice first that if α, β are two regular elements, the ideals $\alpha\mathfrak{r}\alpha^{-1}$, $\beta^{-1}\mathfrak{s}\beta$, $\beta^{-1}\mathfrak{t}\beta$, $\beta^{-1}\mathfrak{t}'\beta$ have the same significance for $\alpha\mathfrak{a}\beta$ and $\alpha\mathfrak{b}\beta$ as the ideals $\mathfrak{r}$, $\mathfrak{s}$, $\mathfrak{t}$, $\mathfrak{t}'$ have for $\mathfrak{a}$ and $\mathfrak{b}$. Hence it is sufficient, by Lemma 3, to consider the case where

$$\mathfrak{a} = M(0), \qquad \mathfrak{b} = M(d_k - c_i); \qquad c_1 \geqq c_2 \geqq \cdots \geqq c_r, \quad d_1 \geqq d_2 \geqq \cdots \geqq d_r. \tag{1}$$

Then $\mathfrak{a} \cap \mathfrak{b} = M(\max(0, d_k - c_i))$ and $\mathfrak{o} = M(a_{ik})$ with

$$\begin{aligned} a_{ik} &= \max_j (\max(0, d_j - c_i) - \max(0, d_j - c_k)) \\ &= \begin{cases} \max(0, d_1 - c_i) - \max(0, d_1 - c_k) = f_k - f_i & \text{for } i \geqq k, \\ \max(0, d_r - c_i) - \max(0, d_r - c_i) = g_k - g_i & \text{for } i \leqq k, \end{cases} \end{aligned}$$

where $f_i = -\max(0, d_1 - c_i)$, $g_i = -\max(0, d_r - c_i)$. Since $f_k - f_i \geqq$ or $\leqq g_k - g_i$ according as $i \geqq k$ or $i \leqq k$, we find that $\mathfrak{o}$ is the intersection of the two maximal orders $M(f_k - f_i)$ and $M(g_k - g_i)$. Further, if we put $\gamma = \sum \epsilon_{ii} P^{-c_i}$, $\delta = \sum \epsilon_{ii} P^{-d_i}$, then $\mathfrak{r} = \gamma P^a M(0)$ and $\mathfrak{s} = M(0) P^{-a} \delta$, whence $\mathfrak{t} = P^{a - d_r} M(0)$, $\mathfrak{t}' = P^{d_1 - a} M(0)$. From this we can easily verify the precise characterization of $\mathfrak{o}$ given in the theorem.

The part on the sum $(\mathfrak{a}, \mathfrak{b})$ can be shown by a similar computation. And indeed from that computation we obtain the last assertion in the theorem.

Finally, to prove Theorem 4 we observe again that we have only to consider the case where $\mathfrak{a}$, $\mathfrak{b}$ have the form (1). $\mathfrak{a} \cap \mathfrak{b} = M(\max(0, d_k - c_i))$, $(\mathfrak{a}^{-1}, \mathfrak{b}^{-1}) = M(\min(0, c_k - d_i))$ because $\mathfrak{b}^{-1} = M(c_k - d_i)$, and here we notice that $\max(0, d_k - c_i) = -\min(0, c_i - d_k)$. The third normal ideal $\mathfrak{c}$ can be expressed as $\mathfrak{c} = \tau^{-1} M(0) \sigma^{-1}$ with regular elements $\sigma = \sum \epsilon_{ik} s_{ik}$, $\tau = \sum \epsilon_{ik} t_{ik}$. Let $P^{c_{ik}}$ be the exact power of P which divides s_{ik}, $P^{c_{ik}} \| s_{ik}$; if $s_{ik} = 0$ we put $c_{ik} = \infty$. Let similarly $P^{d_{ik}} \| t_{ik}$. It is evident that $M(a_{ik})$, with a system of rational integers a_{ik}, contains $\mathfrak{c}^{-1} = \sigma M(0) \tau$ if and only if

$$c_{ij} + d_{lk} \geqq a_{ik}, \qquad \text{for all } i, j, k, l. \tag{2}$$

Hence, if we show that the same condition is also necessary and sufficient in order that $M(-a_{ki}) \subseteqq \mathfrak{c}$, then we will be through. But this is also easy to see. For, $\mathfrak{c}=\tau^{-1}M(0)\sigma^{-1}$ consists of all $\eta=\sum \epsilon_{ik}y_{ik} =\tau^{-1}(\sum \epsilon_{ik}x_{ik})\sigma^{-1}$ with $x_{ik} \varepsilon I$. On taking a pair (j, l) of indices, let us consider those η such that $y_{ik}=0$ for $(i, k)\neq(j, l)$. In other words, we consider the equation $\tau^{-1}(\sum \epsilon_{ik}x_{ik})\sigma^{-1}=\epsilon_{jl}y_{jl}$. But this is equivalent to $\sum \epsilon_{ik}x_{ik}=\tau\epsilon_{jl}y_{jl}\sigma$, or

$$(3) \qquad x_{ik} = t_{ij}y_{jl}s_{lk}, \qquad i, k = 1, 2, \cdots, r.$$

Suppose now $M(-a_{ki}) \subseteqq \mathfrak{c}$. Then (3) with $y_{jl}=P^{-a_{lj}}$ must have a solution $x_{ik} \varepsilon I$. Hence $0 \leqq d_{ij}-a_{lj}+c_{lk}$ (for all i, k). Since (j, l) was an arbitrary pair of indices, we have thus established (2). Assume conversely (2). Then obviously $x_{ik}=t_{ij}P^{-a_{lj}}s_{lk} \varepsilon I$ whence $\epsilon_{jl}P^{-a_{lj}} \varepsilon \mathfrak{c}$ and $M(-a_{ki}) \subseteqq \mathfrak{c}$.

A second proof of the last part of Theorem 3 is as follows: We observe first that every ideal $\mathfrak{m}$ in $\mathfrak{A}$ is additively generated by regular elements contained in $\mathfrak{m}$.[6] For, if $\xi \varepsilon \mathfrak{m}$ we take a scalar element a (εF) in $\mathfrak{m}$ different from all the characteristic roots of the matrix which represents ξ in a faithful representation of $\mathfrak{A}$. Then $\xi-a$ ($\varepsilon \mathfrak{m}$) is evidently a regular element and $\xi=(\xi-a)+a$. Now, let α be any regular element from the left order of $\mathfrak{a} \cap \mathfrak{b}$; $\alpha(\mathfrak{a} \cap \mathfrak{b}) \subseteqq \mathfrak{a} \cap \mathfrak{b}$. Since $\alpha\mathfrak{a}$ and $\alpha\mathfrak{b}$ are normal ideals, we have, from Theorem 4, $(\mathfrak{a}^{-1}\alpha^{-1}, \mathfrak{b}^{-1}\alpha^{-1}) \supseteqq \mathfrak{a}^{-1}, \mathfrak{b}^{-1}$ whence $(\mathfrak{a}^{-1}, \mathfrak{b}^{-1})\alpha^{-1} \supseteqq (\mathfrak{a}^{-1}, \mathfrak{b}^{-1})$, $(\mathfrak{a}^{-1}, \mathfrak{b}^{-1}) \supseteqq (\mathfrak{a}^{-1}, \mathfrak{b}^{-1})\alpha$. This shows that the left order of $\mathfrak{a} \cap \mathfrak{b}$ is contained in the right order of $(\mathfrak{a}^{-1}, \mathfrak{b}^{-1})$. But the converse can be seen in quite a similar manner.

Remark. The structure of the residue class algebra $\mathfrak{J}_1 \cap \mathfrak{J}_2/p(\mathfrak{J}_1 \cap \mathfrak{J}_2)$ is easy to analyze, but perhaps does not deserve a detailed discussion. We merely note that the algebra is not symmetric, in fact is not weakly symmetric,[7] except for the trivial case $(\mathfrak{J}_1)_p=(\mathfrak{J}_2)_p$; this remark may be of some interest in view of a recent paper by R. Brauer.[8]

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[6] We exclude here the trivial case of a finite underlying field F.

[7] See Brauer-Nesbitt, Proceedings of the National Academy of Sciences, vol. 23 (1937); Nakayama-Nesbitt, Annals of Mathematics, (2), vol. 39 (1938).

[8] Brauer, *On modular and p-adic representations of algebras*, Proceedings of the National Academy of Sciences, vol. 25 (1939).

A NOTE ON CONVEX FUNCTIONS[1]

BERNARD FRIEDMAN

It is well known that, just from the definition of convexity, a function of one variable which is convex must also be absolutely continuous. It was not known whether this property would generalize to convex functions of more than one variable. In this note there is constructed a convex function of two variables which is not absolutely continuous.

Carathéodory's definition of absolute continuity [1] for functions of two variables may be stated as follows:

Let $f(x, y)$ be any functions of two variables and S the square with the vertices

$$(x-h, y-h), \quad (x-h, y+h), \quad (x+h, y-h), \quad (x+h, y+h).$$

Let $I(S)$ be the following interval function

$$\begin{aligned} I(S) = {} & f(x+h, y+h) + f(x-h, y-h) \\ & - f(x+h, y-h) - f(x-h, y+h). \end{aligned}$$

Then $f(x, y)$ is said to be absolutely continuous in a two-dimensional region R if $f(x, y)$ is absolutely continuous in each variable separately and if also $I(S)$ is an absolutely continuous function of S. Carathéodory [2] proves the absolute continuity of $f(x, y)$ is a necessary and sufficient condition that there exist functions $g(x)$, $h(y)$ and $k(x, y)$ such that

$$f(x, y) = \int_0^x g(x)dx + \int_0^y h(y)dy + \int_0^x \int_0^y k(x, y)dxdy + C.$$

The counterexample of this note will be constructed from the integral of Cantor's middle third function which can be defined as follows:

$$\begin{aligned} m(0) &= 0, & m(1) &= 1, \\ m(x) &= 1/2, & 1/3 &\leqq x \leqq 2/3, \\ m(x) &= 1/4, & 1/9 &\leqq x \leqq 2/9, \\ m(x) &= 3/4, & 7/8 &\leqq x \leqq 8/9, \cdots . \end{aligned}$$

It is well known that $m(x)$ is a continuous monotonically increasing function whose derivative is zero almost everywhere. Obviously $m(x)$

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[1] Presented to the Society, April 8, 1938.

is not absolutely continuous and not convex, but $f(x)=\int_0^x m(t)dt$ will be both convex and absolutely continuous. $f(x)$ is convex since its derivative $m(x)$ is everywhere non-decreasing.

It is to be noticed that if $f(x)$ is a convex function of x, then $f(x, y)\equiv f(x+y)$ is a convex function of two variables x and y.

However if $f(x, y)$ were absolutely continuous in the two variables x and y, by Carathéodory's theorem it could be represented as

$$f(x+y)=\int_0^x g(x)dx+\int_0^y h(y)dy+\int_0^x\int_0^y k(x, y)dxdy+C.$$

But differentiating with respect to x it follows that

$$f'(x+y)=m(x+y)=g(x)+\int_0^y k(x, y)dy$$

for almost all values of x. Let x_0 be some value of x for which the equality holds. Then

$$m(x_0+y)=g(x_0)+\int_0^y k(x_0, y)dy$$

or $m(x_0+y)$ would be absolutely continuous, which is impossible.

Therefore $f(x+y)$ is a convex function of two variables which is not absolutely continuous.

It is quite easy to prove directly that $I(S)$ is not absolutely continuous. The direct proof, however, is uninteresting and since it requires some tedious computations, it will be omitted.

BIBLIOGRAPHY

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2. Carathéodory, loc. cit., p. 654.

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REAL ASYMPTOTIC SOLUTIONS OF REAL DIFFERENTIAL EQUATIONS[1]

MONROE H. MARTIN

Let us consider the system of differential equations

$$(1) \quad \dot{x}_k = \sum_{j=1}^{n} a_{kj}x_j + \phi_k(x_1, \cdots, x_n), \qquad \dot{x}_k = \frac{dx_k}{dt}, \qquad k = 1, \cdots, n,$$

in which the a_{kj} denote constants and the $\phi_k(x_1, \cdots, x_n)$ are power series in $x_1, \cdots, x_n$ converging in a given neighborhood of the origin $x_1 = \cdots = x_n = 0$, and containing neither constant nor linear terms. The differential equations in the system are supposed to be real and we shall investigate real solutions

$$(2) \qquad x_k = x_k(t), \qquad k = 1, \cdots, n,$$

which are asymptotic to the origin as t tends to $+\infty$ through real values, that is, real solutions for which

$$(3) \qquad \lim_{t \to +\infty} (x_1^2 + \cdots + x_n^2) = 0.$$

When $n = 2$ the results to which we shall be led are well known[2] and may be formulated as follows:

If the characteristic constants of the matrix $\|a_{kj}\|$ *in* (1) *are both negative and one is not an integral multiple of the other, or are conjugate complex numbers with negative real parts, all the points in a suitably restricted neighborhood of the origin lie on real solutions of* (1) *which are asymptotic to the origin as t tends to* $+\infty$. *If the characteristic constants are real and of opposite signs, a point in a suitably restricted neighborhood of the origin lies on a real solution of* (1) *asymptotic to the origin as t tends to* $+\infty$ *if, and only if, it is a point of a certain real analytic arc containing the origin.*

Recent dynamical investigations[3] by the author require an extension of this theorem to the cases $n>2$. That the extension is not obvious has been maintained by Bieberbach.[4] The difficulty in generalizing the theorem arises when some of the characteristic constants of the matrix $\|a_{kj}\|$ are complex. The need for a proof of the existence of real solutions asymptotic to the origin when some of the characteristic constants are complex has been emphasized by Wintner.[5]

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[1] Presented to the Society, December 29, 1938, under the title *Restricted problems in three bodies.*

[2] See, for example, E. Picard, *Traité d'Analyse*, vol. 3, 1928, pp. 206–213. In connection with the case where the characteristic constants have opposite signs, cf. P. Painlevé, *Gewöhnliche Differentialgleichungen*; *Existenz der Lösungen*, Encyklopädie der mathematischen Wissenschaften, vol. 2, pt. 1^1, p. 221 (footnote 116), where it is pointed out that Poincaré failed to recognize the necessity of proving what is equivalent to showing that a point not on the stated analytic arc cannot lie on a solution of (1) asymptotic to the origin as t tends to $+\infty$. See also E. Picard, loc. cit., p. 207.

[3] Not yet published.

The general theorem to be established follows.

THEOREM. *If the first m of the characteristic constants* λ_k *of the matrix* $\|a_{kj}\|$ *in the system* (1) *are negative or have negative real parts, while the remaining* $n-m$ *are positive or have positive real parts, if the elementary divisors of the characteristic matrix are linear, and if none of the linear commensurability relations*

$$\lambda_k = p_1\lambda_1 + \cdots + p_m\lambda_m,$$
$$k = 1, \cdots, m;\ p_k = 0, 1, \cdots;\ p_1 + \cdots + p_m \geqq 2,$$

holds, a point in a suitably restricted neighborhood of the origin lies on a real solution of (1) *asymptotic to the origin as t tends to* $+\infty$ *if, and only if, it is a point of a certain real, analytic, m-dimensional manifold regular at the origin.*

A companion theorem corresponding to the remaining $n-m$ characteristic constants and dealing with real solutions asymptotic to the origin as t tends to $-\infty$ may obviously be formulated.

Our investigation is based on the following results[6] due to Poincaré, Picard and Horn.

Assume given the system of differential equations

$$\dot z_k = \lambda_k z_k + \phi_k(z_1, \cdots, z_n), \qquad k = 1, \cdots, n,$$

in which z_k *denote real or complex variables,* $\phi_k(z_1, \cdots, z_n)$ *denote real or complex power series in* $z_1, \cdots, z_n$, *which converge in a given neighborhood of* $z_1 = \cdots = z_n = 0$ *and begin with terms of at least the second degree in* $z_1, \cdots, z_n$. *The constants* λ_k *may be real or complex and it is assumed that* $\lambda_1, \cdots, \lambda_m$ *have negative real parts while* $\lambda_{m+1}, \cdots, \lambda_n$ *have positive real parts. In addition the constants* λ_k *are subject to the hypothesis that none of the linear commensurability relations*

$$\lambda_k = p_1\lambda_1 + \cdots + p_m\lambda_m, \qquad k = 1, \cdots, m,$$

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[4] L. Bieberbach, *Differentialgleichungen*, 3d edition, 1930, p. 113.

[5] A. Wintner, *Librationstheorie des restringierten Dreikörperproblems*, Mathematische Zeitschrift, vol. 32 (1930), pp. 661–662.

[6] E. Picard, op. cit., pp. 17–21, and pp. 187–188; J. Horn, *Gewöhnliche Differentialgleichungen beliebiger Ordnung*, 1905, pp. 319–325. For (5) in the text above, see page 321 of Horn's book.

holds, where $p_1, \cdots, p_m$ *denote non-negative integers for which* $p_1+\cdots+p_m \geqq 2$.

If $C_1, \cdots, C_m$ *denote m arbitrary real or complex constants, it is possible to find power series for* $z_1, \cdots, z_n$ *in the arguments*

$$u_1 = C_1 e^{\lambda_1 t}, \cdots, u_m = C_m e^{\lambda_m t}, \tag{4}$$

which converge for sufficiently small absolute values of these arguments and which satisfy the differential equations. These power series have the form

$$\begin{aligned} z_j &= u_j + P_j(u_1, \cdots, u_m), & j &= 1, \cdots, m, \\ z_k &= P_k(u_1, \cdots, u_m), & k &= m+1, \cdots, n, \end{aligned} \tag{5}$$

where the power series $P_1, \cdots, P_n$ *begin with terms of at least the second degree in* $u_1, \cdots, u_m$.

The complex characteristic constants of the real matrix $\|a_{kj}\|$ necessarily occur in conjugate pairs and we arrange the notation so that

$$\begin{aligned} \bar{\lambda}_{2\nu_1-1} &= \lambda_{2\nu_1}, & \nu_1 &= 1, \cdots, j_0, \\ \bar{\lambda}_{\nu_2} &= \lambda_{\nu_2} < 0, & \nu_2 &= 2j_0+1, \cdots, m, \\ \bar{\lambda}_{m+2\nu_3-1} &= \lambda_{m+2\nu_3}, & \nu_3 &= 1, \cdots, k_0, \\ \bar{\lambda}_{\nu_4} &= \lambda_{\nu_4} > 0, & \nu_4 &= m+2k_0+1, \cdots, n, \end{aligned} \tag{6}$$

where j_0, k_0 denote integers such that $0 \leqq 2j_0 \leqq m$, $0 \leqq 2k_0 \leqq n-m$. The $\nu_1, \nu_2, \nu_3, \nu_4$ will always be used in the sequel in the same sense as employed here. Thus, to indicate the real and imaginary parts of the complex characteristic constants, we shall write

$$\begin{aligned} \lambda_{2\nu_1-1} &= \alpha_{\nu_1} + i\beta_{\nu_1}, & \lambda_{2\nu_1} &= \alpha_{\nu_1} - i\beta_{\nu_1}, \\ \lambda_{m+2\nu_3-1} &= \alpha_{m+\nu_3} + i\beta_{m+\nu_3}, & \lambda_{m+2\nu_3} &= \alpha_{m+\nu_3} - i\beta_{m+\nu_3}. \end{aligned}$$

From the theory of linear transformations it follows that there exists a real linear transformation with nonvanishing determinant transforming the variables $x_1, \cdots, x_n$ into the variables $y_1, \cdots, y_n$ and the system (1) of differential equations into the system

$$\begin{aligned} \dot{y}_{2\nu_1-1} &= \alpha_{\nu_1} y_{2\nu_1-1} - \beta_{\nu_1} y_{2\nu_1} + \Phi_{2\nu_1-1}(y_1, \cdots, y_n), \\ \dot{y}_{2\nu_1} &= \beta_{\nu_1} y_{2\nu_1-1} + \alpha_{\nu_1} y_{2\nu_1} + \Phi_{2\nu_1}(y_1, \cdots, y_n), \\ \dot{y}_{\nu_2} &= \lambda_{\nu_2} y_{\nu_2} + \Phi_{\nu_2}(y_1, \cdots, y_n), \\ \dot{y}_{m+2\nu_3-1} &= \alpha_{m+\nu_3} y_{m+2\nu_3-1} - \beta_{m+\nu_3} y_{m+2\nu_3} + \Phi_{m+2\nu_3-1}(y_1, \cdots, y_n), \\ \dot{y}_{m+2\nu_3} &= \beta_{m+\nu_3} y_{m+2\nu_3-1} + \alpha_{m+\nu_3} y_{m+2\nu_3} + \Phi_{m+2\nu_3}(y_1, \cdots, y_n), \\ \dot{y}_{\nu_4} &= \lambda_{\nu_4} y_{\nu_4} + \Phi_{\nu_4}(y_1, \cdots, y_n), \end{aligned} \tag{7}$$

where $\Phi_1, \cdots, \Phi_n$ denote real power series in $y_1, \cdots, y_n$ beginning with terms of at least the second degree in these variables.

If complex variables $z_1, \cdots, z_n$ be introduced by means of the equations

$$(8)\quad \begin{array}{lll} z_{2\nu_1-1} = y_{2\nu_1-1} + iy_{2\nu_1}, & z_{2\nu_1} = y_{2\nu_1-1} - iy_{2\nu_1}, & z_{\nu_2} = y_{\nu_2}, \\ z_{m+2\nu_3-1} = y_{m+2\nu_3-1} + iy_{m+2\nu_3}, & z_{m+2\nu_3} = y_{m+2\nu_3-1} - iy_{m+2\nu_3}, & z_{\nu_4} = y_{\nu_4}, \end{array}$$

the differential equations (7) take the simple form

$$(9)\qquad \dot{z}_k = \lambda_k z_k + \sum_{p_1+\cdots+p_n\geqq 2} A^{(k)}_{p_1\cdots p_n} z_1^{p_1} \cdots z_n^{p_n}, \qquad k = 1, \cdots, n,$$

where

$$(10)\qquad \overline{A}^{(k)}_{p_1\cdots p_n} = A^{(j)}_{q_1\cdots q_n} \qquad \text{whenever } \overline{\lambda}_k = \lambda_j,$$

and $q_1, \cdots, q_n$ is the permutation of $p_1, \cdots, p_n$ obtained by interchanging every pair p_k, p_l for which k and l are such that $\overline{\lambda}_k=\lambda_l$.

It is apparent from (8) that a solution of (7) will be real if, and only if, the corresponding solution of (9) satisfies the condition

$$(11)\qquad \bar{z}_k = z_j \qquad \text{whenever } \overline{\lambda}_k = \lambda_j,$$

and we shall now construct solutions of (9) satisfying these conditions.

The following method was suggested by my colleague E. Titt and I have adopted it in preference to my own as less cumbersome.

Referring to the definition of $\nu_1, \nu_2, \nu_3, \nu_4$ in (6), it follows from (10) that if

$$(12)\qquad z_{2\nu_1-1},\ z_{2\nu_1},\ z_{\nu_2},\ z_{m+2\nu_3-1},\ z_{m+2\nu_3},\ z_{\nu_4}$$

denotes a set of n real or complex functions of t which is a solution of (9), then the set

$$(13)\qquad \bar{z}_{2\nu_1},\ \bar{z}_{2\nu_1-1},\ \bar{z}_{\nu_2},\ \bar{z}_{m+2\nu_3},\ \bar{z}_{m+2\nu_3-1},\ \bar{z}_{\nu_4},$$

obtained from (12) by replacing $z_k(t)$ by $\bar{z}_j(t)$ whenever $\overline{\lambda}_k=\lambda_j$ $(1\leqq k\leqq j\leqq n)$, also represents a solution of (9). Accordingly a solution of (7) obtained from (8), will be real if, and only if, the two solutions (12) and (13) of (9) are identical.

The results of Poincaré and Picard apply to the system (9) and we write the solution (5) in the form

$$(14)\qquad z_k = \sum_{k_1+\cdots+k_m\geqq 1} C^{(k)}_{k_1\cdots k_m} u_1^{k_1} \cdots u_m^{k_m},$$

where we have, of course, when $k_1+\cdots+k_m=1$,

$$(15)\quad C^{(j)}_{k_1\cdots k_m}=1,\quad k_1=0,\cdots,k_j=1,\cdots,k_m=0;\, j=1,\cdots,m,$$

the $C^{(k)}_{k_1\cdots k_m}$ remaining, for which $k_1+\cdots+k_n=1$, $k=1,\cdots,n$, being zero. Choose the arbitrary constants $C_1,\cdots,C_m$ in (4) so that

$$(16)\qquad \bar{C}_p = C_q \qquad \text{whenever } \bar{\lambda}_p = \lambda_q,\ (p \leqq q \leqq m),$$

and therefore

$$(17)\qquad \bar{u}_p = u_q \qquad \text{whenever } \bar{\lambda}_p = \lambda_q,\ (p \leqq q \leqq m).$$

Selecting the solution (14) for the solution in (12), we shall prove that it is identical with the solution (13). Upon substituting (14) in (9) and equating coefficients of $u_1^{k_1}\cdots u_m^{k_m}$ it may be verified that $C^{(k)}_{k_1\cdots k_m}$, for $k_1+\cdots+k_m=r$ $(r\geqq 2)$ and k fixed, is expressible as a polynomial in the $C^{(k)}_{k_1\cdots k_m}$ with $k_1+\cdots+k_m<r$ and $k=1,2,\cdots,n$. These recurrence equations determine the $C^{(k)}_{k_1\cdots k_m}$ uniquely, once the coefficients of the linear terms in $u_1,\cdots,u_n$ of (14) are given. Taking the coefficients of the linear terms in $u_1,\cdots,u_m$ of (12) as given by (15), it follows from (17) that they are identical with the coefficients of the linear terms in $u_1,\cdots,u_m$ of (13), and the two solutions are therefore identical.

Accordingly, the constants C_j in (4) being chosen subject to (16) and the complex solution (14) of (9) inserted in (8), a real solution of (7) is obtained when equations (8) are solved for $y_1,\cdots,y_n$.

If we write

$$(18)\qquad u_{2r_1-1}=v_{2r_1-1}+iv_{2r_1},\qquad u_{2r_1}=v_{2r_1-1}-iv_{2r_1},\qquad u_{r_2}=v_{r_2},$$

the afore-mentioned real solution of (7) takes the form

$$(19)\qquad \begin{aligned} y_j &= v_j + Q_j(v_1,\cdots,v_m), &\qquad j&=1,\cdots,m,\\ y_k &= Q_k(v_1,\cdots,v_m), &\qquad k&=m+1,\cdots,n, \end{aligned}$$

where $Q_1,\cdots,Q_n$ denote real power series beginning with terms of at least the second degree in the real variables $v_1,\cdots,v_m$, and converging for sufficiently small $|v_1|,\cdots,|v_m|$. If we let

$$C_{2r_1-1}=a_{r_1}+ib_{r_1},\qquad C_{2r_1}=a_{r_1}-ib_{r_1},\qquad C_{r_2}=a_{r_2},$$

it is readily seen that

$$(20)\qquad \begin{aligned} v_{2r_1-1} &= e^{\alpha_{r_1}t}(a_{r_1}\cos\beta_{r_1}t - b_{r_1}\sin\beta_{r_1}t),\\ v_{2r_1} &= e^{\alpha_{r_1}t}(a_{r_1}\sin\beta_{r_1}t + b_{r_1}\cos\beta_{r_1}t), \qquad v_{r_2}=a_{r_2}e^{\lambda_{r_2}t}. \end{aligned}$$

The constants a_{ν_1}, b_{ν_1}, a_{ν_2} are real and arbitrary. It therefore follows from (19) and (20) that there exists a real, analytic, m-dimensional manifold of solutions of (7) in a sufficiently small neighborhood of the origin $y_1 = \cdots = y_n = 0$. This m-dimensional manifold is regular at the origin, and, since α_{ν_1}, λ_{ν_2} are negative, any one of its points lies on a real solution of (1) asymptotic to the origin as t tends to $+\infty$.

We shall now show that a sufficiently small neighborhood of the origin $y_1 = \cdots = y_n = 0$ contains no points on real solutions of (7) asymptotic to the origin as t tends to $+\infty$ other than the points of the above manifold.

Solving the first m equations in (19) for $v_1, \cdots, v_m$ and substituting these expressions in the remaining $n-m$ equations, we find the equations of the m-dimensional manifold in non-parametric form, namely

$$y_k = Q_k^*(y_1, \cdots, y_m), \qquad k = m+1, \cdots, n, \tag{21}$$

where Q_k^* denote power series beginning with terms of the second degree at least in $y_1, \cdots, y_m$ and converging for sufficiently small $|y_1|, \cdots, |y_m|$.

Let us introduce new coordinates $\eta_1, \cdots, \eta_n$ by means of the equations

$$\begin{aligned} \eta_j &= y_j, & j &= 1, \cdots, m, \\ \eta_k &= y_k - Q_k^*(y_1, \cdots, y_m), & k &= m+1, \cdots, n. \end{aligned}$$

The differential equations (7) retain their form after transformation, the y's being replaced, of course, by η's. The coefficients of the linear terms are unaltered and the power series $\Phi_1, \cdots, \Phi_n$ become power series in $\eta_1, \cdots, \eta_n$ which will be denoted by $\Phi_1^*, \cdots, \Phi_n^*$ respectively.

The last $n-m$ of the transformed equations must be satisfied when we put $\eta_{m+1} = \cdots = \eta_n = 0$, and therefore

$$\Phi_k^* = \sum_{l=m+1}^{n} \Phi_{kl}\eta_l, \qquad k = m+1, \cdots, n, \tag{22}$$

where Φ_{kl} denote power series in $\eta_1, \cdots, \eta_n$ without constant terms. Multiplying the last $n-m$ transformed equations by $\eta_{m+1}, \cdots, \eta_n$, respectively and adding, there results

$$\sum_{k=m+1}^{n} \eta_k\dot{\eta}_k = \sum_{k=m+1}^{n} \alpha_k\eta_k^2 + \sum_{k=m+1}^{n}\sum_{l=m+1}^{n} \Phi_{kl}\eta_k\eta_l,$$

where we have employed (22) to substitute for Φ_k^*. Writing

$$\rho^2 = \eta_{m+1}^2 + \cdots + \eta_n^2, \qquad \mu_k = \eta_k/\rho, \qquad k = m+1, \cdots, n,$$

in the above equation and dividing by ρ^2, we find

$$\dot{\rho}/\rho = \sum_{k=m+1}^{n} \alpha_k \mu_k^2 + \sum_{k=m+1}^{n} \sum_{l=m+1}^{n} \Phi_{kl}\mu_k\mu_l. \tag{23}$$

Let $\alpha^* = \min\{\alpha_{m+1}, \cdots, \alpha_n\}$ and $\Phi^* = \max\{|\Phi_{kl}|\}$, where $\Phi^* = \Phi^*(\eta_1, \cdots, \eta_n)$ tends to zero as the point $(\eta_1, \cdots, \eta_n)$ tends towards the origin $\eta_1 = \cdots = \eta_n = 0$. Now $\mu_{m+1}^2 + \cdots + \mu_n^2 = 1$, so that we have

$$\left| \sum_{k=m+1}^{n} \sum_{l=m+1}^{n} \Phi_{kl}\mu_k\mu_l \right| \leqq (n-m)\Phi^*,$$

and hence from (23) we see that

$$\dot{\rho}/\rho \geqq \alpha^* - (n-m)\Phi^*, \qquad \alpha^* > 0. \tag{24}$$

If possible let $y_k = y_k(t)$ be a real solution of (7) which is asymptotic to the origin $y_1 = \cdots = y_n = 0$ as t tends to $+\infty$, yet for which there is no $\bar{t}$ such that the solution lies in the m-dimensional manifold (21) for $t > \bar{t}$.

Employing $\eta_1, \cdots, \eta_n$ as coordinates, we should then have a real solution $\eta_k = \eta_k(t)$ of the transformed differential equations with the property that to any positive number ϵ there corresponds a $t_\epsilon > 0$ such that we have

$$\eta_1^2 + \cdots + \eta_n^2 < \epsilon \tag{25}$$

for all $t > t_\epsilon$. In addition there must exist a $t_0 > t_\epsilon$ which is such that on writing $\eta_k^0 = \eta_k(t_0)$, $\rho_0^2 = \eta_{m+1}^{0^2} + \cdots + \eta_n^{0^2}$ $(k = 1, \cdots, n)$, we have $\rho_0 > 0$. Since Φ^* is arbitrarily small in the neighborhood of the origin of $\eta_1 = \cdots = \eta_n = 0$, it is possible to choose ϵ at the outset so small that

$$\Phi^* = \Phi^*(\eta_1(t), \cdots, \eta_n(t)) < \alpha^*/2(n-m) \qquad \text{for } t > t_\epsilon.$$

Inequality (24) then becomes $\dot{\rho}/\rho > \alpha^*/2$ for $t > t_\epsilon$. Integrating from t_0 to t $(t > t_0)$, we find

$$\rho > \rho_0 e^{(\alpha^*/2)(t-t_0)},$$

which, since $\alpha^* > 0$, implies a contradiction to (25).

Since the transformation from the system (1) to the system (7) was a real linear transformation with nonvanishing determinant the theorem has now been established.

University of Maryland

ON THE SUPPORTING-PLANE PROPERTY OF A CONVEX BODY[1]

DAVID MOSKOVITZ AND L. L. DINES

In an earlier paper,[2] the authors have shown that in a linear space $\mathfrak{S}$ with an inner product, a set $\mathfrak{M}$ which is closed and linearly connected is supported at a set of boundary points which is everywhere dense on the boundary of $\mathfrak{M}$, and an example is given to show that such a set $\mathfrak{M}$ may have boundary points through which no supporting plane exists. The purpose of this paper is to show that if a set, in addition to being linearly connected and closed, also possesses inner points, then it is completely supported at its boundary points. In (I), reference was made to a paper by Ascoli in which such a result was obtained in a separable space. We do not assume our space $\mathfrak{S}$ to be separable. The definitions and results of (I) will be used in this paper.

A set $\mathfrak{K}$, which is a proper subset of the space $\mathfrak{S}$, will be called a *convex body* if it is linearly connected, closed, and possesses inner points. In the sequel $\mathfrak{K}$ will always denote a convex body.

With reference to the set $\mathfrak{K}$, there is associated with each point x of the space $\mathfrak{S}$ a nonnegative number $r(x)$: if x is an inner point of $\mathfrak{K}$, $r(x)$ is defined as the least upper bound of the radii of spheres about x which do not contain points exterior to $\mathfrak{K}$; for other points of $\mathfrak{S}$, $r(x)$ is defined to be zero. We will call $r(x)$ *the radius at the point* x.

If x_1 is a point of $\mathfrak{K}$, all points x of the sphere $\|x-x_1\| \leqq r(x_1)$ are points of $\mathfrak{K}$.

THEOREM 1. *Let r_1 and r_2 be the radii at the points x_1 and x_2, respectively, of the convex body $\mathfrak{K}$. Then the radius r at the point*

$$x = x_1 + k(x_2 - x_1), \qquad 0 \leqq k \leqq 1,$$

satisfies

$$r \geqq r_1 + k(r_2 - r_1).$$

PROOF. Let $y = x + \rho u$, where $\rho = r_1 + k(r_2 - r_1)$ and $\|u\| = 1$. The points $y_1 = x_1 + r_1 u$ and $y_2 = x_2 + r_2 u$ are points of $\mathfrak{K}$. But from the definitions of x, ρ, and y, it follows that $y = y_1 + k(y_2 - y_1)$. Hence y, being on the segment joining y_1 and y_2, is also a point of $\mathfrak{K}$. Consequently all points on the boundary of the sphere with radius ρ and center x are in $\mathfrak{K}$. Since $\mathfrak{K}$ is linearly connected, also all points within this sphere are in $\mathfrak{K}$. Therefore $r \geqq \rho$ and the theorem is established.

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[1] Presented to the Society, September 5, 1939.

[2] *On convexity in a linear space with an inner product*, Duke Mathematical Journal, vol. 5 (1939), pp. 520–534. Hereafter, this paper will be referred to by (I).

The following corollaries, which appear self-evident in ordinary space, can be shown to be direct consequences of the preceding theorem.

COROLLARY 1. *Each point of the segment joining two inner points of* $\mathfrak{K}$ *is an inner point of* $\mathfrak{K}$.

COROLLARY 2. *If* x_0 *is a boundary point and* x_1 *an inner point of* $\mathfrak{K}$, *then the points* $x = x_1 + k(x_0 - x_1)$ *are inner points of* $\mathfrak{K}$ *for* $0 \leqq k < 1$, *and exterior points for* $k > 1$.

With reference to a given boundary point x_0 of the set $\mathfrak{K}$, there is associated with each point x, other than x_0, of the space $\mathfrak{S}$ a non-negative number $s(x)$, defined by[8]

$$s(x) = r(x)/\|x - x_0\|.$$

If x is an exterior point or a boundary point of $\mathfrak{K}$, other than x_0, $s(x)$ is equal to zero; if x is an inner point of $\mathfrak{K}$, $s(x)$ is positive; $s(x_0)$ is not defined.

It is also obvious that $s(x) \leqq 1$, since $r(x) \leqq \|x - x_0\|$.

THEOREM 2. *Let* x_0 *be a given boundary point of the convex body* $\mathfrak{K}$, *and let* x_t *be given by*

$$x_t = x_0 + tu, \qquad \text{where } t > 0,\ \|u\| = 1. \tag{1}$$

Then, for fixed u,

(a) $s(x_t)$ *is a non-decreasing function as* $t \to 0$; *and*

(b) $\lim_{t \to 0} s(x_t)$ *exists.*

PROOF. In case there are no points of $\mathfrak{K}$ given by (1), the theorem is obviously true, for then

$$s(x_t) = 0 \quad \text{for} \quad t > 0, \qquad \lim_{t \to 0} s(x_t) = 0.$$

In case there are points of $\mathfrak{K}$ given by (1), let x_1 and x_2 be two points of $\mathfrak{K}$ on (1) for parameter values t_1 and t_2, where $t_1 < t_2$; then we have

$$\begin{aligned} x_1 &= x_0 + t_1 u, & x_2 &= x_0 + t_2 u; \\ s(x_1) &= r(x_1)/t_1, & s(x_2) &= r(x_2)/t_2. \end{aligned} \tag{2}$$

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[8] Since s is a function of x_0 as well as x, a more explicit notation would be $s(x_0, x)$; but the simpler notation will suffice, inasmuch as the function is to be used in the sequel only with reference to a fixed boundary point x_0.

But $x_1 = x_0 + (t_1/t_2)(x_2 - x_0)$, and hence, by Theorem 1, we have

$$r(x_1) \geqq \frac{t_1}{t_2} r(x_2),$$

since $r(x_0) = 0$. Therefore, by (2), $s(x_1) \geqq s(x_2)$.

This result establishes part (a) of the theorem. Since $s(x_t)$ cannot exceed one, obviously part (b) of the theorem is true.

Let Σ be the unit sphere about x_0; and let p_u be the point on Σ given by $p_u = x_0 + u$, $\|u\| = 1$. Let $x_t = x_0 + tu$, $(0 < t < 1)$, be the segment joining x_0 to p_u; and let[4]

$$\sigma(u) = \lim_{t \to 0} s(x_t).$$

We thus have a function $\sigma(u)$ uniquely defined at each point p_u on the sphere Σ. Obviously, by its definition, we have

$$0 \leqq \sigma(u) \leqq 1.$$

Also $\sigma(u) = 0$ only if the segment joining x_0 to p_u does not contain any inner points of $\mathfrak{K}$. If the segment joining x_0 to p_u contains inner points of $\mathfrak{K}$, we have $\sigma(u) > 0$.

LEMMA 1. *Let p_u and p_v be two points on Σ, such that*

$$p_u = x_0 + u, \qquad p_v = x_0 + v, \qquad v = -u.$$

Then at least one of the numbers $\sigma(u)$ or $\sigma(v)$ is equal to zero.

PROOF. Assume $\sigma(u) > 0$; then the segment joining x_0 to p_u contains inner points. Consequently, by Corollary 2, the segment joining x_0 to p_v does not contain any inner points. Therefore, $\sigma(v) = 0$.

THEOREM 3. *Let x_0 be a given boundary point of the convex body $\mathfrak{K}$, and let Σ be the unit sphere about x_0. Let p_u and p_v given by*

$$p_u = x_0 + u, \qquad \|u\| = 1, \qquad p_v = x_0 + v, \qquad \|v\| = 1$$

be two distinct points on Σ, for which $\sigma(u)$ and $\sigma(v)$ are both positive. Then there exists a point p_w distinct from p_u and p_v for which

$$\sigma(w) > \tfrac{1}{2}[\sigma(u) + \sigma(v)].$$

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[4] The limit was shown to exist in Theorem 2; we are denoting the value of this limit by $\sigma(u)$. It may be of interest to note that

$$\sigma(u) = \lim_{x \to x_0,\ \text{along}\ x_t = x_0 + tu} s(x) = \lim_{x \to x_0} \frac{r(x) - r(x_0)}{\|x - x_0\|} = r'_u(x_0)$$

is the *directional derivative* of $r(x)$ at x_0 in the direction u.

PROOF. Let x_t and y_t be points of $\mathfrak{K}$ given by

$$x_t = x_0 + tu, \qquad y_t = x_0 + tv, \qquad 0 < t < 1,$$

and let $((u, v)) = \lambda$. Then, certainly $|\lambda| \leqq 1$. But if $\lambda = 1$, $u = v$, and p_u and p_v are not distinct. If $\lambda = -1$, $u = -v$, in which case not both of the numbers $\sigma(u)$ and $\sigma(v)$ can be positive, because of Lemma 1. Consequently, we have

$$-1 < \lambda < 1.$$

Let $z_t = \frac{1}{2}(x_t + y_t)$; then $z_t = x_0 + \xi t w$, where $\|w\| = 1$ and $\xi = \frac{1}{2}(1+\lambda)^{1/2}$. Thus

$$0 < \xi < 1.$$

We thus have a point p_w on the sphere Σ defined by $p_w = x_0 + w$. Now, $r(z_t) \geqq \frac{1}{2}[r(x_t) + r(y_t)]$, by Theorem 1. Hence

$$s(z_t) = \frac{r(z_t)}{\xi t} \geqq \frac{1}{2\xi}\left[\frac{r(x_t)}{t} + \frac{r(y_t)}{t}\right] = \frac{1}{2\xi}\left[s(x_t) + s(y_t)\right],$$

and

$$\lim_{t \to 0} s(z_t) \geqq \frac{1}{2\xi} \lim_{t \to 0} \left[s(x_t) + s(y_t)\right],$$

from which

$$\sigma(w) \geqq \frac{1}{2\xi}\left[\sigma(u) + \sigma(v)\right] > \frac{1}{2}\left[\sigma(u) + \sigma(v)\right].$$

Thus the theorem is established.

Let $\bar{\sigma}$ denote the least upper bound of the function $\sigma(u)$ as p_u varies over the sphere Σ. Then, also $0 \leqq \bar{\sigma} \leqq 1$; and $\bar{\sigma} = 0$ is possible only for sets which do not have any inner points. For a convex body $\mathfrak{K}$, we have $0 < \bar{\sigma} \leqq 1$.

In the material which follows, it is to be understood that x_0 is a *fixed* boundary point of the convex body $\mathfrak{K}$, $s(x)$ is defined relative to x_0, Σ is the unit sphere about x_0, $\sigma(u)$ is the function defined above on the boundary of Σ, and $\bar{\sigma}$ the least upper bound of $\sigma(u)$ on Σ.

THEOREM 4. *If there is a point p_u on Σ for which $\sigma(u) = \bar{\sigma}$, this point is unique.*

PROOF. Suppose, if possible, that there were a second point p_v for which $\sigma(v) = \bar{\sigma}$. Then, by Theorem 3, since $\bar{\sigma} > 0$, there would be a point p_w for which

$$\sigma(w) > \tfrac{1}{2}[\sigma(u) + \sigma(v)] = \bar{\sigma}.$$

But since no $\sigma(w)$ can exceed $\bar{\sigma}$, there cannot be a second point p_v of the type described.

THEOREM 5. *Let p_u be a point on Σ for which $\sigma(u)=\bar{\sigma}$. If v satisfies the conditions $\|v\|=1$ and $((u, v))<0$, then the points $z_t=x_0+tv$, $t>0$, are exterior points of $\mathfrak{K}$.*

PROOF. Let $p_u=x_0+u$, $\|u\|=1$, and $p_v=x_0+v$, $\|v\|=1$; and let $((u, v))=-\lambda$, where $\lambda>0$. Assume, if possible, that there is a point $z=x_0+dv$, $d>0$, belonging to $\mathfrak{K}$. Let w be the projection (defined in (I)) of z on the line through x_0 and p_u. Then

$$w = p_u + c(x_0 - p_u),$$

where

$$\begin{aligned} c &= \frac{((z - p_u, x_0 - p_u))}{\|p_u - x_0\|^2} = ((z - x_0 - u, - u)) \\ &= ((dv - u, - u)) = 1 + \lambda d. \end{aligned}$$

Hence,

$$w = p_u - (1 + \lambda d)u = x_0 - \lambda du.$$

On the segment joining x_0 to p_u, let $x_t=x_0+tu$ be an inner point of $\mathfrak{K}$. Let y_t be the projection of x_0 on the line through x_t and z. Then

$$y_t = x_t + k(z - x_t)$$

where

$$k = \frac{((x_0 - x_t, z - x_t))}{\|z - x_t\|^2} = \frac{((- tu, dv - tu))}{\|z - x_t\|^2} = \frac{\lambda td + t^2}{d^2 + 2\lambda td + t^2},$$

since $z-x_t=z-x_0+x_0-x_t=dv-tu$ and

$$\|z - x_t\|^2 = d^2 - 2td((u, v)) + t^2 = d^2 + 2\lambda td + t^2.$$

From the above value of k, it is easily seen that $0<k<1$, which means that y_t is a point of $\mathfrak{K}$. The following are easily established:

$$\|y_t - x_0\|^2 = \frac{t^2d^2(1 - \lambda^2)}{d^2 + 2\lambda td + t^2} \neq 0,$$

since $\lambda\neq\pm 1$, and

$$\|z - w\|^2 = d^2(1 - \lambda^2).$$

From these, and previous results, we obtain

$$\frac{\|x_t - x_0\|^2}{\|y_t - x_0\|^2} = \frac{t^2(d^2 + 2\lambda td + t^2)}{t^2 d^2(1 - \lambda^2)} = \frac{d^2 + 2\lambda td + t^2}{d^2(1 - \lambda^2)} = \frac{\|z - x_t\|^2}{\|z - w\|^2}.$$

Therefore, we have

$$(3) \qquad \frac{\|x_t - x_0\|}{\|y_t - x_0\|} = \frac{\|z - x_t\|}{\|z - w\|}.$$

Now $s(y_t) = r(y_t)/\|y_t - x_0\|$ and $s(x_t) = r(x_t)/\|x_t - x_0\|$, where $r(y_t)$ and $r(x_t)$ denote the radii at the points y_t and x_t, respectively. Also $r(y_t) \geqq (1-k)r(x_t)$, by Theorem 1 and the definition of y_t. Hence

$$(4) \qquad \begin{aligned} \frac{s(y_t)}{s(x_t)} = \frac{r(y_t)}{\|y_t - x_0\|}\cdot\frac{\|x_t - x_0\|}{r(x_t)} &\geqq (1 - k)\frac{\|x_t - x_0\|}{\|y_t - x_0\|} \\ &= (1 - k)\frac{\|z - x_t\|}{\|z - w\|}, \end{aligned}$$

the last equality being a consequence of (3).

But $k = \|y_t - x_t\|/\|z - x_t\|$ and $1 - k = \|z - y_t\|/\|z - x_t\|$. Therefore, from (4), we have

$$(5) \qquad s(y_t) \geqq \frac{\|z - y_t\|}{\|z - w\|} s(x_t).$$

Now,

$$\lim_{t\to 0} \frac{\|z - y_t\|}{\|z - w\|} = \frac{d}{d(1 - \lambda^2)^{1/2}} = \frac{1}{(1 - \lambda^2)^{1/2}} > 1,$$

since z and w are independent of t, while $y_t \to x_0$ as $t \to 0$. Therefore, from (5),

$$\lim_{t\to 0} s(y_t) \geqq \frac{1}{(1 - \lambda^2)^{1/2}} \sigma(u) > \sigma(u) = \bar{\sigma}.$$

But this is impossible; hence the assumption that z was a point of $\mathfrak{K}$ is untenable.

THEOREM 6. *Let p_u be a point on Σ for which $\sigma(u) = \bar{\sigma}$. Then the plane*

$$(6) \qquad \pi(x) \equiv ((u, x - x_0)) = 0$$

is a supporting plane of $\mathfrak{K}$ through the boundary point x_0.

PROOF. If the plane (6) were not a supporting plane, there would be a point z of $\mathfrak{K}$ for which $\pi(z)<0$. Let $v=(z-x_0)/\|z-x_0\|$; then

$$((u, v)) = \frac{\pi(z)}{\|z - x_0\|} < 0, \qquad z = x_0 + \|z - x_0\|v.$$

But, v satisfies the conditions of Theorem 5; therefore z must be an exterior point of $\mathfrak{K}$. Consequently, there cannot be a point z of $\mathfrak{K}$ for which $\pi(z)<0$; and (6) is indeed a supporting plane.

THEOREM 7. *Let x_0 be a given boundary point of the convex body $\mathfrak{K}$, and let Σ be the unit sphere about x_0. There is a unique point $p_{\bar{u}}$ on Σ for which $\sigma(\bar{u})=\bar{\sigma}$.*

PROOF. We have only to show the existence of one point $p_{\bar{u}}$ for which $\sigma(\bar{u})=\bar{\sigma}$. The uniqueness of this point will follow from Theorem 4.

From the definition of $\bar{\sigma}$ it follows that for any preassigned $\epsilon>0$, there exists a point on Σ for which the value of σ is greater than $\bar{\sigma}-\epsilon$. Choose a monotone decreasing sequence of positive numbers $\{\epsilon_n\}$ with limit zero. Corresponding to each ϵ_n there exists a point p_{u_n} on Σ for which $\sigma(u_n)>\bar{\sigma}-\epsilon_n$. We wish to show that the sequence of points $\{p_{u_n}\}$ on Σ converges.

Let $p_{u_n}=x_0+u_n$, $\|u_n\|=1$, and $p_{u_m}=x_0+u_m$, $\|u_m\|=1$. Then

$$\|p_{u_n} - p_{u_m}\|^2 = 2 - 2((u_n, u_m)). \tag{7}$$

Let $w=\frac{1}{2}(1/\xi)(u_n+u_m)$, where ξ is so chosen that $\|w\|=1$. Then we have

$$\xi^2 = \tfrac{1}{2}[1 + ((u_n, u_m))]. \tag{8}$$

Let $p_w=x_0+w$; from the proof of Theorem 3, we know that

$$\sigma(w) \geqq \frac{1}{2\xi}[\sigma(u_n) + \sigma(u_m)] > \frac{1}{2\xi}[2\bar{\sigma} - \epsilon_n - \epsilon_m].$$

But $\bar{\sigma}\geqq\sigma(w)$; hence $\bar{\sigma}>(1/\xi)[\bar{\sigma}-(\epsilon_n+\epsilon_m)/2]$, from which

$$\xi^2 > \left[1 - \frac{1}{2\bar{\sigma}}(\epsilon_n + \epsilon_m)\right]^2.$$

Using the value of ξ^2 from (8) we easily find that

$$((u_n, u_m)) > 2\left[1 - \frac{1}{2\bar{\sigma}}(\epsilon_n + \epsilon_m)\right]^2 - 1.$$

Then using (7), we obtain

$$\|p_{u_n} - p_{u_m}\|^2 < \frac{4}{\bar{\sigma}}(\epsilon_n + \epsilon_m) - \frac{1}{\bar{\sigma}^2}(\epsilon_n + \epsilon_m)^2. \tag{9}$$

Since $\lim_{n,m\to\infty} \|p_{u_n}-p_{u_m}\| = 0$ and the space $\mathfrak{S}$ is complete, as was assumed in (I) and throughout this paper, the sequence $\{p_{u_n}\}$ converges to a point $p_{\bar{u}}$. This point $p_{\bar{u}}$ is on Σ, and moreover $\sigma(\bar{u}) = \bar{\sigma}$, since it is easily shown that $\sigma(\bar{u})$ is greater than $\bar{\sigma}-\epsilon$ for any preassigned positive ϵ.

THEOREM 8. *A convex body $\mathfrak{K}$ is completely supported at its boundary points.*

PROOF. Let x_0 be a boundary point of $\mathfrak{K}$. There exists a point p_u on the unit sphere Σ about x_0 for which $\sigma(u) = \bar{\sigma}$, by Theorem 7. Hence the plane $((u, x-x_0)) = 0$ is a supporting plane of $\mathfrak{K}$ through x_0, by Theorem 6. Since similar statements can be made for each boundary point, $\mathfrak{K}$ is completely supported at its boundary points.

From the material above, the following additional result may be established without much difficulty:

COROLLARY 3. *There exists a unique supporting plane through the boundary point x_0 of the convex body $\mathfrak{K}$ only if $\bar{\sigma} = 1$; for $\bar{\sigma} < 1$, there is an infinite number of supporting planes through x_0.*

A primary classification of boundary points of a convex body may thus be made in terms of $\bar{\sigma}$, which is a function defined over the boundary of the convex body.

CARNEGIE INSTITUTE OF TECHNOLOGY

A GENERALIZATION OF OSTROWSKI'S THEOREM ON MATRIC IDENTITIES[1]

NEAL H. McCOY

The purpose of this note is to generalize a recent theorem due to Ostrowski[2] which is itself a generalization of a theorem proved by Phillips in 1919.[3] We shall first indicate the nature of Ostrowski's result.

Let $A_1=I, A_2, \cdots, A_m$ be square matrices of order n, I being the unit matrix, and let $x_1, \cdots, x_m$ be numerical parameters. Denote by $F(x_1, \cdots, x_m)$ the determinant of the matrix

$$x_1A_1 + x_2A_2 + \cdots + x_mA_m. \tag{1}$$

Let $\Phi(x_1, \cdots, x_m)$ be the greatest common divisor of the n^2 minors of order $n-1$ of the matrix (1), and set $F/\Phi = F^*(x_1, \cdots, x_m)$. We may now state the theorem of Ostrowski in the following form:[4]

Theorem 1. *If $B_1, \cdots, B_m$ are matrices of order n, commutative with each other and satisfying the equation*

$$A_1B_1 + A_2B_2 + \cdots + A_mB_m = 0, \tag{2}$$

then

$$F^*(B_1, \cdots, B_m) = 0.$$

Further, if $\Psi(x_1, \cdots, x_m)$ is any polynomial with the property that $\Psi(B_1, \cdots, B_m)=0$ for every set of commutative matrices satisfying (2), *then $\Psi(x_1, \cdots, x_m)$ is divisible by $F^*(x_1, \cdots, x_m)$.*

In this theorem it is tacitly assumed that the elements of the matrices as well as the coefficients of the polynomials are real or complex numbers. In Theorem 3 below we find an extension of the first part of Theorem 1, valid if the elements and coefficients are in an arbitrary commutative ring R with unit element 1. To generalize the second part of the theorem, we find it necessary to make an additional restriction on R, namely, that there exists no nonzero polynomial $\phi(\lambda)$, with coefficients in R, such that $\phi(a)=0$ for all elements a of R. The result obtained under this restriction is stated as Theorem 4.

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[1] Presented to the Society, September 8, 1939.

[2] A. Ostrowski, *On a theorem concerning identical relations between matrices*, Quarterly Journal of Mathematics, vol. 9 (1938), pp. 241–245.

[3] H. B. Phillips, *Functions of matrices*, American Journal of Mathematics, vol. 41 (1919), pp. 266–278.

[4] The assumption that $A_1=I$ is not strictly necessary but assures us that $F(x_1, \cdots, x_m)$ does not vanish identically. For the generalization below, we wish to have $A_1=I$ and so we state the theorem at once in this form.

The form of our theorems is suggested by a recent generalization, in another direction, of Frobenius' theorem concerning the minimum equation of a matrix.[5] Since this plays an important part in the proof of Theorem 4, we state it explicitly before proceeding.

THEOREM 2. *Let R be an arbitrary commutative ring with unit element* 1, *and A a matrix of order n with elements in R. Let λ be an indeterminate, denote by $f(\lambda)$ the determinant of the matrix $\lambda I-A$, and let $h_{ij}(\lambda)$ be the minors of $\lambda I-A$ of order $n-1$. Then, an element $g(\lambda)$ of $R[\lambda]$ has the property that $g(A)=0$, if and only if $g(\lambda)h_{ij}(\lambda)\equiv 0\ (f(\lambda))$, $(i, j=1, 2, \cdots, n)$.*

It will be seen that Theorems 3 and 4 bear roughly the same relation to Ostrowski's theorem that Theorem 2 does to Frobenius'.

Henceforth we shall let $A_1=I, A_2, \cdots, A_m$ be matrices of order n with elements in a commutative ring R with unit element 1, and let $x_1, \cdots, x_m$ be indeterminates. Denote by $F(x_1, \cdots, x_m)$ the determinant of the matrix

$$x_1A_1 + x_2A_2 + \cdots + x_mA_m. \tag{3}$$

Let $F_{ij}(x_1, \cdots, x_m)$ be the elements of the adjoint of this matrix, and denote by $\mathfrak{m}$ the ideal of those elements $f(x_1, \cdots, x_m)$ of the ring $R[x_1, \cdots, x_m]$ such that

$$f(x_1, \cdots, x_m)F_{ij}(x_1, \cdots, x_m) \equiv 0\ (F(x_1, \cdots, x_m)),\quad i, j = 1, 2, \cdots, n.$$

We may now state the following theorem:

THEOREM 3. *If $f(x_1, \cdots, x_m)\equiv 0$ ($\mathfrak{m}$), and $B_1, \cdots, B_m$ are commutative matrices of order n, with elements in R, such that*

$$A_1B_1 + A_2B_2 + \cdots + A_mB_m = 0, \tag{4}$$

then $f(B_1, \cdots, B_m)=0$.

The proof is a simple modification of Ostrowski's, and will be only briefly indicated, using his notation so far as possible. Set $A_\mu=(a_{ik}^{(\mu)})$. Now, by hypothesis, we have equations of the form

$$f(x_1, \cdots, x_m)F_{ij}(x_1, \cdots, x_m) = h_{ij}(x_1, \cdots, x_m)F(x_1, \cdots, x_m). \tag{5}$$

But

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[5] N. H. McCoy, *Concerning matrices with elements in a commutative ring*, this Bulletin, vol. 45 (1939), pp. 280–284.

$$\sum_{j=1}^{n}\left(\sum_{\mu=1}^{m} x_\mu a_{jk}^{(\mu)}\right)F_{ji}(x_1, \cdots, x_m) = \delta_i^k F(x_1, \cdots, x_m).$$

Multiply this last equation by $f(x_1, \cdots, x_m)$ and use (5), getting

$$F(x_1, \cdots, x_m)\sum_{j=1}^{n}\left(\sum_{\mu=1}^{m} x_\mu a_{jk}^{(\mu)}\right) h_{ji}(x_1, \cdots, x_m) \tag{6}$$
$$= \delta_i^k F(x_1, \cdots, x_m)f(x_1, \cdots, x_m).$$

Now, if $F(x_1, \cdots, x_m)$ is arranged in terms of decreasing powers of x_1, it is clear that the first term is x_1^n. It follows readily that, in the ring $R[x_1, \cdots, x_m]$, $F(x_1, \cdots, x_m)$ is not a divisor of zero, and can therefore by divided out of equation (6), yielding

$$\sum_{j=1}^{n}\left(\sum_{\mu=1}^{m} x_\mu a_{jk}^{(\mu)}\right) h_{ji}(x_1, \cdots, x_m) = \delta_i^k f(x_1, \cdots, x_m).$$

Let $C_{jk}=\sum_{\mu=1}^{m} B_\mu a_{jk}^{(\mu)}$. Then we have from the preceding equation, and the fact that the B's are commutative,

$$\sum_{j=1}^{n} C_{jk}h_{ji}(B_1, \cdots, B_m) = \delta_i^k f(B_1, \cdots, B_m).$$

Now this equation corresponds to Ostrowski's equation (2.3), and the remainder of the proof will be omitted as from this point the proof coincides with his.

If R is a field, or more generally, a domain of integrity with unique factorization into primes, then it follows readily that $\mathfrak{m}$ is the principal ideal $(F(x_1, \cdots, x_m)/D(x_1, \cdots, x_m))$, where $D(x_1, \cdots, x_m)$ is the greatest common divisor of the $F_{ij}(x_1, \cdots, x_m)$. In this case, our Theorem 3 can be expressed in the same form as the first part of Theorem 1.

Before proceeding, we pause to make a remark which will indicate how, in another way, the ideal $\mathfrak{m}$ has properties generalizing the familiar properties of the minimum function of a single matrix with elements in a field. From the definition of $\mathfrak{m}$, it follows that if $g(x_1, \cdots, x_m) \equiv 0$ $(\mathfrak{m})$, then

$$g(x_1, \cdots, x_m) \text{ adj } (x_1A_1 + \cdots + x_mA_m) = KF(x_1, \cdots, x_m),$$

where K is a matrix with elements in $R[x_1, \cdots, x_m]$. By taking determinants and dividing by F^{n-1}, we see that

$$[g(x_1, \cdots, x_m)]^n \equiv 0 \ (F(x_1, \cdots, x_m)).$$

In particular, it follows at once that in $R[x_1, \cdots, x_m]$ the ideals $\mathfrak{m}$

and $(F(x_1, \cdots, x_m))$ have precisely the same prime ideal divisors.

We now turn to the problem of generalizing the second part of Theorem 1, and we find it necessary to make the following additional assumption concerning R. *If $f(\lambda)$ is an element of $R[\lambda]$ such that for every element a of R, $f(a)=0$, then $f(\lambda)=0$.* It follows readily by induction on the number of indeterminates that if in $R[x_1, \cdots, x_k]$, $f(x_1, \cdots, x_k)$ has the property that $f(a_1, \cdots, a_k)=0$ for all choices of $a_1, \cdots, a_k$ in R, then $f(x_1, \cdots, x_k)=0$, $(k=1, 2, \cdots)$. We now prove the following lemma:

Lemma. *Let $x_1, \cdots, x_m$ be indeterminates, and consider an element of $R[x_1, \cdots, x_m]$ of the form*

$$g(x_1, \cdots, x_m) = x_1^n + g_1(x_2, \cdots, x_m)x_1^{n-1} + \cdots + g_n(x_2, \cdots, x_m).$$

If $f(x_1, \cdots, x_m)$ is an element of $R[x_1, \cdots, x_m]$ such that for every choice of $a_2, \cdots, a_m$ in R, $f(x_1, a_2, \cdots, a_m)$ is divisible by $g(x_1, a_2, \cdots, a_m)$ in $R[x_1]$, then $f(x_1, \cdots, x_m)$ is divisible by $g(x_1, \cdots, x_m)$ in the ring $R[x_1, \cdots, x_m]$.

Let x represent the composite set $x_2, \cdots, x_m$; and a the set of elements $a_2, \cdots, a_m$ of R. Then we may write

$$g(x_1, \cdots, x_m) = x_1^n + g_1(x)x_1^{n-1} + \cdots + g_n(x),$$

and

$$f(x_1, \cdots, x_m) = f_0(x)x_1^p + f_1(x)x_1^{p-1} + \cdots + f_p(x).$$

Now, by hypothesis, we have for arbitrary but fixed a a relation of the form

$$\begin{aligned} f_0(a)x_1^p + \cdots + f_p(a) = {} & [x_1^n + g_1(a)x_1^{n-1} + \cdots + g_n(a)] \\ & \cdot [h_0x_1^{p-n} + h_1x_1^{p-n-1} + \cdots + h_{p-n}], \end{aligned} \tag{7}$$

where the h_i are elements of R. But equation (7) is equivalent to a set of $p+1$ equations in R, obtained by equating the coefficients of the different powers of x_1 on each side of (7). These equations take the form

$$\begin{aligned} f_0(a) &= h_0, \\ f_1(a) &= h_1 + h_0g_1(a), \\ &\cdots\cdots\cdots\cdots, \\ f_{p-n}(a) &= h_{p-n} + h_{p-n-1}g_1(a) + \cdots + h_0g_{p-n}(a), \end{aligned} \tag{8}$$

together with the set of equations

$$(9) \qquad f_t(a) = \sum_{i+j=t} h_i g_j(a), \qquad t = p - n + 1, \cdots, p.$$

The equations (8) can be solved in turn for the h_i, and these unique solutions take the form

$$h_i = G_i[f_1(a), \cdots, f_{p-n}(a); g_1(a), \cdots, g_{p-n}(a)], \quad i = 0, 1, \cdots, p - n,$$

where the G_i are polynomials with integral coefficients. Thus, in (7), the coefficients h_i are uniquely determined by the choice of a. Now let us set

$$(10) \qquad h_i(x) = G_i[f_1(x), \cdots, f_{p-n}(x); g_1(x), \cdots, g_{p-n}(x)], \qquad i = 0, 1, \cdots, p - n,$$

so that our original h_i is $h_i(a)$. Then equations (9) state that for *every* a in R,

$$f_t(a) - \sum_{i+j=t} h_i(a) g_j(a) = 0,$$

and therefore, by hypothesis on R, it follows that

$$(11) \qquad f_t(x) = \sum_{i+j=t} h_i(x) g_j(x), \qquad t = p - n + 1, \cdots, p.$$

But equations (10) and (11) are precisely the set of equations which state that

$$f(x_1, \cdots, x_m) = g(x_1, \cdots, x_m)[h_0(x)x_1^{p-n} + \cdots + h_{p-n}(x)],$$

and the lemma is established.

We shall now prove the following theorem under the assumption on R which we have made throughout this section:

THEOREM 4. *If* $\phi(x_1, \cdots, x_m)$ *is an element of* $R[x_1, \cdots, x_m]$ *with the property that* $\phi(B_1, \cdots, B_m) = 0$ *for every* $B_1, \cdots, B_m$ *which are commutative and satisfy* (4), *then* $\phi(x_1, \cdots, x_m) \equiv 0$ $(\mathfrak{m})$.

In the proof of this theorem we shall not distinguish between the ring R and the ring of matrices of the form aI, where a is in R. Accordingly, we identify I with 1, the unit element of R.

Let $a_2, \cdots, a_m$ be arbitrary elements of R, and let us choose

$$(12) \qquad B_1 = -a_2A_2 - \cdots - a_mA_m, \quad B_2 = a_2, \cdots, \quad B_m = a_m.$$

Then clearly condition (4) is satisfied, and by hypothesis we have

$$\phi(B_1, B_2, \cdots, B_m) = \phi(B_1, a_2, \cdots, a_m) = 0.$$

Now if $F(x_1, \cdots, x_m)$ and $F_{ij}(x_1, \cdots, x_m)$ have the same meaning as above, then clearly the determinant of x_1-B_1 is $F(x_1, a_2, \cdots, a_m)$, and the first minors of the matrix x_1-B_1 are precisely the $F_{ij}(x_1, a_2, \cdots, a_m)$, except possibly for sign.

Since $\phi(B_1, a_2, \cdots, a_m)=0$, it follows by Theorem 2 that, in the ring $R[x_1]$, $\phi(x_1, a_2, \cdots, a_m)F_{ij}(x_1, a_2, \cdots, a_m)$ is divisible by $F(x_1, a_2, \cdots, a_m)$, $(i, j=1, 2, \cdots, n)$. But $F(x_1, \cdots, x_m)$ is of the form of the $g(x_1, \cdots, x_m)$ in the lemma, and hence the lemma can be applied as the above holds for all choices of $a_2, \cdots, a_m$ in R. Thus

$$\phi(x_1, \cdots, x_m)F_{ij}(x_1, \cdots, x_m) \equiv 0\ (F(x_1, \cdots, x_m)),\quad i, j = 1, 2, \cdots, n,$$

that is,

$$\phi(x_1, \cdots, x_m) \equiv 0\ (\mathfrak{m}).$$

This completes the proof of the theorem.

We have established Theorem 4, following Ostrowski, not in fact under the assumption that $\phi(B_1, \cdots, B_m)=0$ for *every* $B_1, \cdots, B_m$ which are commutative and satisfy (4), but under the weaker assumption that the B's may be restricted to be in the special form (12). Because of the homogeneity of the polynomials considered, it is not difficult to show that we can further restrict our hypothesis by assuming always that $a_m=1$. If, in the lemma, we consider only homogeneous polynomials, we can also in it assume that $a_m=1$. Thus, for the case $m=2$, the lemma and Theorem 4 are true for *any* commutative ring R with unit element. Theorem 4, so interpreted, then yields an actual generalization of Theorem 2.

Note added in proof: The assumption that $A_1=I$ is used, so far as Theorem 3 is concerned, only to make sure that $F(x_1, \cdots, x_m)$ is not a divisor of zero in $R[x_1, \cdots, x_m]$. The remarks to follow will show that it is certainly sufficient, although by no means necessary, to assume that for some i, $|A_i|$ is not a divisor of zero in R.

If $ab=0$, $a\neq 0$, a is called an *annihilator* of b. It is quite easy to prove, although this fact does not seem to be in the literature, that *if* $g(x_1, \cdots, x_m)$ *is an element of* $R[x_1, \cdots, x_m]$ *with an annihilator* $h(x_1, \cdots, x_m)$ *in* $R[x_1, \cdots, x_m]$, *then* $g(x_1, \cdots, x_m)$ *has an annihilator* a *in* R. Otherwise expressed, $g(x_1, \cdots, x_m)$ is a divisor of zero in $R[x_1, \cdots, x_m]$ if and only if all coefficients in g are annihilated by the *same* element a of R.

SMITH COLLEGE

THE RADIUS AND MODULUS OF n-VALENCE FOR ANALYTIC FUNCTIONS WHOSE FIRST $n-1$ DERIVATIVES VANISH AT A POINT

LYNN H. LOOMIS

The principal result of this note is the determination of the precise radius and modulus of n-valence for the class of functions $f(z)=z^n+a_{n+1}z^{n+1}+\cdots$ analytic and less than or equal to M in modulus in $|z|\leqq 1$. This result readily leads to the radius and modulus of n-valence for the more general class of functions $f(z)=az^n+a_{n+1}z^{n+1}+\cdots$ analytic and less than or equal to M in modulus in $|z|\leqq R$. Finally, we note certain approximations which rather naturally suggest themselves in a search for more easily calculable constants.

We consider only expansions about the origin of functions $f(z)$ with $f(0)=0$, the generalization to expansions about a of functions $f(z)$ with $f(a)=b$ being obvious. Each circle mentioned will be understood to have the origin ($w=0$ or $z=0$) as center. The phrases *radius of n-valence* and *modulus of n-valence*, which usually refer to a class of functions, will also be used with reference to a single function. The radius of n-valence of the function $f(z)$ is the radius of the largest circle within which $f(z)$ assumes no value more than n times, and assumes at least one value n times. The modulus of n-valence of $f(z)$ is the radius of the largest circle of which the interior is covered exactly n times by the map under $f(z)$ of $|z|<\rho$, where ρ is the above radius of n-valence. Consider now one of the classes defined above. It is obvious that for each function $w=f(z)$ of the class there is a neighborhood of $z=0$ in which the function assumes no value more than n times, and assumes exactly n times every value in a sufficiently small neighborhood of $w=0$. The radius of n-valence ρ_n of the class is the radius of the largest circle within which *no* function of the class assumes a value more than n times. The modulus of n-valence m_n of the class is the radius of the largest circle of which the interior is covered exactly n times by the map of $|z|<\rho_n$ under *every* function of the class.

THEOREM. *Consider the class of functions* $f(z)=z^n+a_{n+1}z^{n+1}+\cdots$ *analytic and less than or equal to* M $(M>1)$[1] *in modulus in* $|z|\leqq 1$,

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[1] The restriction to $M>1$ is necessary. By the Cauchy coefficient inequality, $M\geqq 1$, and if $M=1$ the class consists of the single function $f(z)=z^n$ for which the theorem is false.

where n and M are the constants of the class. The radius ρ_n and the modulus m_n of n-valence of the class are given by

$$m_n = \frac{M\rho_n^n(1 - M\rho_n)}{M - \rho_n}, \qquad \rho_n = M_n - (M_n^2 - 1)^{1/2},$$

where

$$M_n = \frac{1}{2}\left[\left(1 + \frac{1}{n}\right)M + \left(1 - \frac{1}{n}\right)\frac{1}{M}\right].$$

The case for $n=1$ is completely treated in the literature. The results are due to Landau and Dieudonné.[2] The values of ρ_1 and m_1 are

$$\rho_1 = M - (M^2 - 1)^{1/2}, \qquad m_1 = M\rho_1^2 = M\rho_1\frac{(1 - M\rho_1)}{M - \rho_1}.$$

The following proof for general n consists of

(a) a proof that every function of the class is n-valent and covers $|w| < m_n$ exactly n times in $|z| < \rho_n$, and

(b) the exhibition of a single function of the class for which ρ_n and m_n are respectively the radius and modulus of n-valence.

We employ the device which Dieudonné used to find the radius of star-shapedness in the case $n=1$ (Montel, loc. cit., p. 94), and then apply a theorem due to S. Ozaki,[3] which states that if $f(z)$ is analytic in $|z| \leqq r$ and has n zeros there, none on the circumference, and if for some real α, $\Re[e^{i\alpha}zf'(z)/f(z)] > 0$ on $|z| = r$, then $f(z)$ is n-valent in the circle $|z| < r$.

Consider

$$g(z) = f(z)/z^n = 1 + a_{n+1}z + \cdots.$$

Since $g(z)$ is analytic and less than or equal to M in modulus in $|z| \leqq 1$ with $g(0) = 1$, the following inequalities, results of the Schwarz lemma, are valid (Montel, loc. cit., p. 91):

$$(1) \qquad |g(z)| \geqq \frac{M(1 - Mr)}{M - r},$$

$$(2) \qquad |g'(z)| \leqq \frac{M^2 - |g(z)|^2}{M(1 - r^2)},$$

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[2] See Montel, *Leçons sur les Fonctions Univalentes ou Multivalentes*, 1933, pp. 90–95. This book contains a convenient collection of material relating to this paper, and we shall often refer to it.

[3] S. Ozaki, *Some remarks on the univalency and multivalency of functions*, Science Reports of Tokyo Bunrika Daigaku, section A, 2, no. 32, 1934, pp. 41–55.

where $r=|z|<1/M$. From (1) and (2),

$$(3)\qquad \left|\frac{zg'(z)}{g(z)}\right| \leqq \frac{r(M^2-1)}{(M-r)(1-Mr)}$$

if $r<1/M$. But we have

$$g'(z)=\frac{z^nf'(z)-nz^{n-1}f(z)}{z^{2n}},$$

$$(4)\qquad \frac{zf'(z)}{f(z)}=\frac{zg'(z)}{g(z)}+n.$$

By (3) and (4) $\Re[zf'(z)/f(z)]>0$ whenever

$$n>\frac{r(M^2-1)}{(M-r)(1-Mr)}.$$

The latter inequality reduces, for $r<1/M$, to

$$r^2-\left[\left(1+\frac{1}{n}\right)M+\left(1-\frac{1}{n}\right)\frac{1}{M}\right]r+1>0,$$

which is satisfied (factoring the left member) if

$$r<M_n-(M_n^2-1)^{1/2}=\rho_n,$$

where

$$M_n=\frac{1}{2}\left[\left(1+\frac{1}{n}\right)M+\left(1-\frac{1}{n}\right)\frac{1}{M}\right].$$

Now $\rho_n<1/M$, for we shall see that ρ_n is the smallest positive zero of the derivative of

$$Mx^n(1-Mx)/(M-x),$$

and this lies between 0 and $1/M$. But from (1)

$$(5)\qquad |f(z)|\geqq\frac{Mr^n(1-Mr)}{M-r}.$$

Hence $f(z)$ has precisely n zeros in $|z|<r$ and none on the boundary $|z|=r$, for every $r\leqq\rho_n$. Then:

(1) By (5), $|f(z)|\geqq m_n$ on $|z|=\rho_n$. Thus, by Rouché's theorem, $f(z)$ assumes exactly n times in $|z|<\rho_n$ every value w with $|w|<m_n$.

(2) We have seen that $\Re[zf'(z)/f(z)]>0$ on $|z|=r<\rho_n$.

Thus, by the theorem of Ozaki noted above, $f(z)$ is n-valent in the region $|z| < r$ for every $r < \rho_n$, and hence n-valent in $|z| < \rho_n$. This completes the proof of (a).

It follows from the Schwarz lemma that equality can occur in (1) for a point in $0 < |z| < 1$ only if $g(z)$ is of the form

$$g(z) = \frac{M(1 - Me^{i\theta}z)}{M - e^{i\theta}z}\cdot$$

We obtain the simplest function with $e^{i\theta} = 1$, giving

$$f_0(z) = \frac{Mz^n(1 - Mz)}{M - z}\cdot$$

This function is of the class considered, and has ρ_n and m_n for its radius and modulus of n-valence. For,

$$f_0'(z) = \frac{M^2nz^{n-1}}{(M - z)^2}\left\{z^2 - \left[\left(1 + \frac{1}{n}\right)M + \left(1 - \frac{1}{n}\right)\frac{1}{M}\right]z + 1\right\},$$

and it is evident that the zero of $f_0'(z)$ nearest the origin (except $z = 0$ itself) is precisely ρ_n. Also

$$f_0(\rho_n) = \frac{M\rho_n^n(1 - M\rho_n)}{M - \rho_n} = m_n.$$

Thus these constants are respectively the radius and modulus of n-valence for this function, and hence by (a) for the class considered in the theorem.

COROLLARY 1. *Consider the class of functions* $f(z) = az^n + a_{n+1}z^{n+1} + \cdots$ *analytic and less than or equal to* M $(M > |a|R^n)$ *in modulus in* $|z| \leq R$, *the constants of the class being* $|a|$ $(\neq 0)$, n, M *and* R. *The radius* ρ_n *and the modulus* m_n *of* n*-valence of the class are given by*

$$\rho_n = R\sigma, \qquad m_n = \frac{M\sigma^n(|a|R^n - M\sigma)}{M - |a|R^n\sigma},$$

where σ *is defined by*

$$\sigma = M_n - (M_n^2 - 1)^{1/2},$$

$$M_n = \frac{1}{2}\left[\left(1 + \frac{1}{n}\right)\frac{M}{|a|R^n} + \left(1 - \frac{1}{n}\right)\frac{|a|R^n}{M}\right].$$

This corollary follows immediately from the theorem upon considering the function $g(z) = f(Rz)/aR^n$.

The results on the modulus of n-valence relate to a result due to Walsh and Seidel,[4] who do not assume $f'(0) = \cdots = f^{(n-1)}(0) = 0$, but who, on the other hand, do not obtain the sharp inequality.

COROLLARY 2. *For the class of functions of Corollary* 1

$$\rho_n > \frac{|a| R^{n+1}}{2M}, \qquad m_n > M\left(\frac{|a| R^n}{2M}\right)^{n+1}.$$

For, $M/|a|R^n > 1$, from which it follows that $M/|a|R^n \geqq M_n > 1$. Therefore, since

$$M_n - (M_n^2 - 1)^{1/2} = \frac{1}{M_n + (M_n^2 - 1)^{1/2}} > \frac{1}{2M_n},$$

we find that

$$\frac{|a| R^{n+1}}{2M} < \rho_n. \tag{6}$$

Moreover, if $f(z) = Mz^n(|a|R^{n+1} - Mz)/(MR^{n+1} - |a|R^{2n}z)$, then $f(\rho_n) = m_n$, $f'(\rho_n) = 0$, $f(|a|R^{n+1}/2M) < m_n$ by (6), and the last inequality easily gives

$$M\left(\frac{|a| R^n}{2M}\right)^{n+1} < m_n. \tag{7}$$

But (6) and (7) constitute the corollary.

The inequality (7) is an improvement over an approximation due to Privaloff,[5] which states that the image of the circle $|z| < R$ under the function $w = f(z)$ of the class of Corollary 1 covers at least n times the circle

$$|w| < \frac{8}{3} M\left(\frac{R^n |a|}{4M}\right)^{n+1}.$$

For,

$$M\left(\frac{|a| R^n}{2M}\right)^{n+1} > \frac{8}{3} M\left(\frac{|a| R^n}{4M}\right)^{n+1},$$

and by (7), the circle $|w| < M(|a|R^n/2M)^{n+1}$ is covered exactly n times by the image of $|z| < \rho_n$ and hence at least n times by that of $|z| < R$.

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[4] Walsh and Seidel, *On the derivatives of functions analytic in the unit circle*, Proceedings of the National Academy of Sciences, vol. 24 (1938), pp. 337–340.

[5] J. Privaloff, *Sur un théorème de M. Bloch*, Recueil Mathématique de Moscou, vol. 35 (1928), pp. 111–121.

We conclude with a few miscellaneous remarks.

1. The modulus of n-valence for the class of the theorem is also the radius of the largest circle within which converge all the power series (in $w^{1/n}$) of the inverse functions.

2. It is evident from the proof of part (b) of the theorem that the radius and modulus of n-valence for the function $f(z)$ are ρ_n and m_n only if

$$f(z) = \frac{Mz^n(1 - Me^{i\theta}z)}{M - e^{i\theta}z}$$

for some real θ. For any other function of the class the radius and modulus of n-valence are greater, respectively, than ρ_n and m_n.

3. The inequalities $\leqq M$, $\leqq 1$ of the theorem may be replaced by $<M$, <1 without affecting the validity of the work.

4. It is easily seen that the equations of Corollary 1 give, for instance, the radius of n-valence for the class of functions of the form $f(z) = a_f z^n + a_{n+1}z^{n+1} + \cdots$ analytic and less than or equal to M_f in modulus in $|z| \leqq R$, including just those functions for which $M_f/|a_f|$ is less than some preassigned bound ($M/|a|$ of the equations of the corollary). No modulus of n-valence exists for this class.

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INVERSE PROBLEMS OF THE CALCULUS OF VARIATIONS FOR MULTIPLE INTEGRALS[1]

WILLIAM A. PATTERSON

1. **Introduction.** The simplest case of the inverse problem of Darboux is that in which an ordinary differential equation in the normal form $y''=\phi(x, y, y')$ is assigned with the requirement that we ascertain, first, under what conditions $\phi(x, y, y')$ is the solution for y'' of the Euler equation of a variation problem of the form $\int_{x_1}^{x_2} f(x, y, y')dx = \min$ and then that we determine the most general integrand function f corresponding to an admissible function[2] $\phi(x, y, y')$.

For partial differential equations the simplest analogous problem is that of finding the most general *first order* multiple integral variation problem *associated* with an assigned partial differential equation, that is, the most general integrand function f of a variation problem of the form

$$\text{(I)} \qquad \int_{(n)} f(x_1, \cdots, x_n, z, p_1, \cdots, p_n)dx_1 \cdots dx_n = \min, \quad p_i = \partial z/\partial x_i,$$

of which the extremal hypersurfaces are the integral hypersurfaces $z=z(x_1, \cdots, x_n)$ of a prescribed partial differential equation.

A systematic study of such inverse problems of Darboux type for certain important classes of partial differential equations is made in this paper.

2. **A uniqueness theorem.** Consider a partial differential equation of the form

$$\begin{aligned} F \equiv A_{\alpha\beta}(x_1, \cdots, x_n, z, p_1, \cdots, p_n)p_{\alpha\beta} \\ + B(x_1, \cdots, x_n, z, p_1, \cdots, p_n) = 0, \end{aligned} \tag{2.1}$$

where $p_{ij}=\partial^2 z/\partial x_i\partial x_j$, and $A_{ij}=A_{ji}$ $(i, j=1, \cdots, n)$ and B are arbitrary analytic functions of $x_1, \cdots, x_n, z, p_1, \cdots, p_n$. In (2.1) as elsewhere in this paper, a repeated Greek letter is an umbral index indicating a summation with range 1 to n, unless otherwise indicated.

Equation (2.1), as it stands, may have an equation of variation which is self-adjoint on every hypersurface $z=z(x_1, \cdots, x_n)$. If so there is always a multiple integral of the form (I) having $F=0$ as its

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[1] Presented to the Society, December 29, 1939.

[2] Cf. G. Darboux, *Théorie des Surfaces*, vol. 3, 1887, p. 53. For the case of $n \geqq 2$ dependent variables $y_1, \cdots, y_n$ see L. LaPaz, Proceedings of the National Academy of Sciences, vol. 17 (1931), pp. 459–463.

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Euler-Lagrange equation. The most general such integral is known to have an integrand function of the form[3]

$$(2.2)\qquad f = G + C + D_\alpha p_\alpha + \partial\omega_\alpha/\partial x_\alpha,$$

where $G(x_1, \cdots, x_n, z, p_1, \cdots, p_n)$ is a particular solution of

$$(2.3)\qquad g_{p_i p_j} = A_{ij}, \qquad i, j = 1, \cdots, n,$$

where the set $\{C(x_1, \cdots, x_n, z), D_i(x_1, \cdots, x_n, z)\}$ is a particular solution of

$$(2.4)\qquad C_z - D_{\alpha x_\alpha} = G_{p_\alpha x_\alpha} + G_{p_\alpha z} p_\alpha - G_z - B,$$

and where the ω_i occurring in the partial derivatives with respect to x_i are arbitrary functions of $x_1, \cdots, x_n, z$ alone. Whether (2.1), as it stands, has a self-adjoint equation of variation or not, there may exist equivalent equations which do have self-adjoint equations of variation. We proceed to the question of the existence of such equivalent equations.

For a problem of minimizing the multiple integral (I), the Lagrange partial differential equation must necessarily be a second order partial differential equation[4] linear in the p_{ij} and with a self-adjoint equation of variation. Thus, if the integral hypersurfaces $z = z(x_1, \cdots, x_n)$ of $F = 0$ are to be the extremal hypersurfaces of such a problem, the partial differential equation $F = 0$ must be equivalent to a partial differential equation of the second order with the properties of linearity and self-adjointness of the equation of variation just described. Since the analytic partial differential equation $F = 0$ is itself linear in the p_{ij}, it follows that the most general linear equation

$$a_{\mu\nu}(x_1, \cdots, x_n, z, p_1, \cdots, p_n) p_{\mu\nu} + b(x_1, \cdots, x_n, z, p_1, \cdots, p_n) = 0$$

equivalent to $F = 0$ is of the form $M \cdot F = 0$, where $M \neq 0$ is a function of $x_1, \cdots, x_n, z, p_1, \cdots, p_n$ alone. A function $M(x_1, \cdots, x_n, z, p_1, \cdots, p_n) \neq 0$ and such that the equation $M \cdot F = 0$ has a self-adjoint equation of variation will be called a ***multiplier*** of the partial differential equation $F = 0$. Since multipliers which differ only by a nonzero constant factor are not regarded as distinct, it is permissible to restrict attention to multipliers $M > 0$.

If the equation $M \cdot F = 0$ is to have a self-adjoint equation of variation, then the multiplier M must satisfy the following relations[5] identically in the variables $x_1, \cdots, x_n, z, p_1, \cdots, p_n$:

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[3] Cf. L. LaPaz, Transactions of this Society, vol. 32 (1930), p. 513.

[4] Integrand functions linear in the p_i are excluded from consideration.

[5] L. LaPaz, loc. cit., p. 512.

$$(2.5)\qquad M\cdot A_{ij} = M\cdot A_{ji}, \qquad i, j = 1, 2, \cdots, n,$$

$$(2.6)\qquad \begin{aligned} &\partial(M\cdot A_{ij})/\partial p_k = \partial(M\cdot A_{kj})/\partial p_i, \qquad i, j, k = 1, 2, \cdots, n;\ k \neq i,\\ &\partial(M\cdot A_{i\alpha})/\partial x_\alpha + p_\alpha\partial(M\cdot A_{i\alpha})/\partial z - \partial(M\cdot B)/\partial p_i = 0,\\ &\qquad\qquad\qquad\qquad i = 1, 2, \cdots, n.\end{aligned}$$

It is evident that the conditions (2.5) are automatically satisfied. Hence if a multiplier $M(x_1, \cdots, x_n, z, p_1, \cdots, p_n)$ exists, it must satisfy the relations (2.6) which may be written in the form

$$(2.7)\qquad \begin{aligned} &A_{ij}\partial M/\partial p_k - A_{kj}\partial M/\partial p_i + (\partial A_{ij}/\partial p_k - \partial A_{kj}/\partial p_i)\cdot M = 0,\\ &A_{i\alpha}\partial M/\partial x_\alpha + p_\alpha\cdot A_{i\alpha}\partial M/\partial z - B\cdot\partial M/\partial p_i\\ &\qquad + (\partial A_{i\alpha}/\partial x_\alpha + p_\alpha\partial A_{i\alpha}/\partial z - \partial B/\partial p_i)\cdot M = 0.\end{aligned}$$

If $F=0$, as it stands, has a self-adjoint equation of variation, then (2.7) reduces to

$$(2.8)\qquad \begin{aligned} A_{ij}\partial M/\partial p_k - A_{kj}\partial M/\partial p_i &= 0,\\ A_{i\alpha}\partial M/\partial x_\alpha + p_\alpha\cdot A_{i\alpha}\partial M/\partial z - B\partial M/\partial p_i &= 0.\end{aligned}$$

If $\mathfrak{a}$, the determinant of the matrix (A_{ij}), is different from zero, it follows from (2.8_1) that $\partial M/\partial p_i=0$ $(i=1, 2, \cdots, n)$. But then (2.8_2) is seen to imply that $\partial M/\partial x_i=\partial M/\partial z=0$ $(i=1, 2, \cdots, n)$; that is to say, M is a constant. This completes the proof of the following result.

THEOREM 2.1. *If $F=0$, as it stands, has a self-adjoint equation of variation in a neighborhood in which $\mathfrak{a}\neq 0$, then $F=0$ admits a constant as its most general multiplier in this neighborhood.*

As an immediate corollary to Theorem 2.1 we have the following important uniqueness theorem.

THEOREM 2.2. *In a neighborhood in which $\mathfrak{a}\neq 0$, the equation $F=0$ admits at most one multiplier M.*

3. **Variation problems in $(n+1)$-space of which the extremals are minimal hypersurfaces.** As an interesting application of Theorem 2.2, let us consider the inverse problem of Darboux for minimal hypersurfaces[6] in $(n+1)$-space $(n>2)$. The corresponding inverse problem for straight lines in $(n+1)$-space $(n>2)$, that is, for solution curves of the system of differential equations $y_i''=0$ $(i=1, 2, \cdots, n)$ has recently been treated by D. R. Davis[7] who has found that the most general integrand function $f(x, y_1, \cdots, y_n, y_1', \cdots, y_n')$ of a varia-

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[6] For the case $n=2$ see L. LaPaz, Acta Szeged, vol. 5 (1932), pp. 199–207.

[7] D. R. Davis, Transactions of this Society, vol. 33 (1931), pp. 244–251.

tion problem of the form $\int_{x_1}^{x_2} f dx = \min$ for which the extremals are the solution curves of the system of differential equations $y_i'' = 0$ $(i=1, 2, \cdots, n)$ is of the form $f = g(x, y_1, \cdots, y_n, y_1', \cdots, y_n') + dt(x, y_1, \cdots, y_n)/dx$, where g is a suitably chosen particular solution of a (compatible) system of second order partial differential equations, with $i<j=1, 2, \cdots, n$, of the form

$$\partial^2 g/\partial y_i' \partial y_j' = P_{ij}(y_1', \cdots, y_n', y_1 - y_1' x, \cdots, y_n - y_n' x),$$

and where t is an arbitrary function of its arguments. The functions P_{ij} are arbitrary solutions of a certain system of partial differential equations considered by Davis. In sharp contrast to this result we shall show that the most general variation problem (I) for which the extremals are minimal hypersurfaces in the sense defined below is uniquely determined up to an additive function of the form $\partial\omega_\alpha/\partial x_\alpha$ where the ω_i $(i=1, 2, \cdots, n)$ are arbitrary functions of $x_1, \cdots, x_n, z$.

A surface $z = z(x_1, \cdots, x_n)$ defined and of class C'' in a region R of $(x_1, \cdots, x_n)$-space will be called a minimal hypersurface if z and its partial derivatives of the first and second orders satisfy in R the partial differential equation

$$(3.1) \qquad F_1 \equiv [(1 + p_\gamma p_\gamma)\delta_\alpha^\beta - p_\alpha p_\beta]\cdot p_{\alpha\beta} = 0,$$

where δ_j^i is Kronecker's delta. $F_1=0$ is of the form (2.1) with $\mathfrak{a} \equiv (1+p_\gamma p_\gamma)^{n-1} \neq 0$ and therefore by Theorem 2.2 there exists at most one multiplier M for $F_1=0$. But the Lagrange partial differential equation for the problem $\int_{(n)} (1+p_\gamma p_\gamma)^{1/2} dx_1 \cdots dx_n = \min$ is of the form $M \cdot F_1 = 0$ where $M = (1+p_\gamma p_\gamma)^{-3/2}$. Hence, calculating the integrand function f of (2.2), we reach the following conclusion.

THEOREM 3.1. *The partial differential equation* $F_1=0$ *of minimal hypersurfaces admits the unique multiplier* $M = (1+p_\gamma p_\gamma)^{-3/2}$. *The most general variation problem* (I) *for which the extremal hypersurfaces are minimal hypersurfaces has an integrand function* f *of the form* $f = (1+p_\gamma p_\gamma)^{1/2} + \partial\omega_\alpha/\partial x_\alpha$, *where the* ω_i *are arbitrary functions of* $x_1, \cdots, x_n, z$ *alone.*

4. Variation problems associated with the equation $A_{\alpha\beta} p_{\alpha\beta} + B(x_1, \cdots, x_n, z) = 0$.

In this section we shall consider the problem of finding the most general variation problem (I) associated with the partial differential equation

$$(4.1) \qquad F_2 \equiv A_{\alpha\beta} p_{\alpha\beta} + B(x_1, \cdots, x_n, z) = 0,$$

where $A_{ij} = A_{ji}$ $(i, j = 1, 2, \cdots, n)$ are real constants not all zero and B is an arbitrary analytic function of $x_1, \cdots, x_n, z$.

Partial differential equations of the form (4.1) are of much importance in pure and applied mathematics. For example if we choose $n=2$, $A_{ij}=\delta^i_j$ and $B=8z/(1+x_1^2+x_2^2)^2$, (4.1) reduces to the partial differential equation studied by Schwarz in his researches on minimal surfaces. For the same value of n and suitable choice of A_{ij} and B, the equation (4.1), with appropriate interpretation of the variables x_1, x_2 and z, includes many of the most important partial differential equations of mathematical physics.

It is easy to verify that all equations (4.1) have self-adjoint equations of variation and hence, as they stand, are Euler-Lagrange equations. We therefore proceed immediately to the problem of determining the most general integrand function f of the problem (I) known to be associated with an equation of the form $F_2=0$. Application of Theorem 2.1 and calculation of the function f of (2.2) leads to the following theorem.

THEOREM 4.1. *If $\mathfrak{a}\neq 0$, then the partial differential equation $F_2=0$ admits a constant as its most general multiplier. The most general variation problem* (I) *associated with $F_2=0$ has an integrand function f of the form $f=\frac{1}{2}A_{\alpha\beta}p_\alpha p_\beta-\int_{z_0}^{z}Bdz+\partial\omega_\alpha/\partial x_\alpha$, where the ω_i are arbitrary functions of $x_1, \cdots, x_n, z$.*

As an application of this case we find that the most general variation problem (I) associated with $s=\partial^2 z/\partial x_1\partial x_2=0$, which is of the type $F_2=0$ with $n=2$ and $B=0$, has an integrand function of the form $f=p_1p_2/2+\partial\omega_1/\partial x_1+\partial\omega_2/\partial x_2$ where ω_1, ω_2 are arbitrary functions of x_1, x_2, z.

If $\mathfrak{a}=0$ our discussion proceeds on the basis of the system (2.8) written for the equation $F_2=0$. It is evident that in every case the resulting system of equations S can be replaced by an equivalent system S^* of $R\leqq 2n+1$ independent partial differential equations.[8] The number R of equations in S^* is equal to the rank of (A), the matrix of coefficients of the system S. Inspection of (A) discloses that its last column is a linear combination of the n preceding columns. Thus R is always less than $2n+1$. Actually, when $\mathfrak{a}=0$, R is at most $2n-1$ and its precise value is given by the following lemmas which we state without proof.[9]

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[8] Cf. E. Goursat, *Leçons sur l'Intégration des Équations aux Dérivées Partielles du Premier Ordre*, 1921, p. 66.

[9] These proofs and other details omitted in this paper are to be found in the writer's doctoral dissertation, *Inverse Problems of the Calculus of Variations for Multiple Integrals*, The Ohio State University, 1936. For assistance in the preparation of this dissertation and the present paper, the writer wishes to express his indebtedness to Professor Lincoln LaPaz of The Ohio State University.

LEMMA 1. *If* (A_{ij}) *is of rank* r $(r=2, 3, \cdots, n-1)$, *the matrix* (A) *is of rank* $n+r$.

LEMMA 2. *If* (A_{ij}) *is of rank* 1, *the matrix* (A) *is of rank* n.

Where it is desirable to retain the original variables x_i, a straightforward but somewhat involved discussion based on Lemmas 1 and 2 enables us to treat[10] both the case $1<r<n$ and the case $r=1$. We give a simpler treatment of the cases $1\leqq r<n$ using another lemma.

LEMMA 3. *If* (A_{ij}) *is of rank* r, *then by means of a nonsingular linear transformation* $X_i=c_{i\alpha}x_\alpha$ $(i=1, 2, \cdots, n)$, *where the* c_{ij} *are constants, the partial differential equation* $F_2=0$ *can be transformed into one such that*[11] $A_{ij}=\delta_j^i$, $A_{ii}=\pm 1$ $(i\leqq r)$, $A_{ii}=0$ $(i>r)$.

By use of Lemma 3, (2.8) can be replaced by the equivalent system

$$(4.2)\quad A_{ii}\partial M/\partial p_k - A_{ki}\partial M/\partial p_i = 0,\quad i = 1, 2, \cdots, r;\ k = 1, 2, \cdots, n,$$

$$(4.3)\quad A_{ii}(\partial M/\partial x_i + p_i\partial M/\partial z) - B\partial M/\partial p_i = 0, \qquad i = 1, 2, \cdots, n.$$

From (4.2) with $i=1$ and $k>1$ we infer $\partial M/\partial p_k=0$ $(k>1)$. If now $1<r<n$, the choice $i=2$, $k=1$ shows $\partial M/\partial p_1=0$. Hence for $1<r<n$ it follows from $\partial M/\partial p_i=0$ $(i=1, 2, \cdots, n)$ and (4.3) that $\partial M/\partial x_i+p_i\partial M/\partial z=0$ $(i\leqq r)$ and therefore that $\partial M/\partial x_i=\partial M/\partial z=0$ $(i\leqq r)$. This completes the proof of the following theorem.

THEOREM 4.2. *If the rank of* (A_{ij}) *is* r $(1<r<n)$, *then the partial differential equation* $F_2=0$, *when reduced to the normal form of Lemma* 3, *admits as its most general multiplier the function* $A(x_{r+1}, x_{r+2}, \cdots, x_n)$ *where* A *is different from zero but is otherwise an arbitrary function of its arguments.*

We now proceed to calculate the function f of (2.2). A particular solution of (2.3) formed for the equation $A(x_{r+1}, x_{r+2}, \cdots, x_n)\cdot F_2=0$ is $G=\frac{1}{2}A(x_{r+1}, x_{r+2}, \cdots, x_n)A_{\mu\mu}p_\mu p_\mu$. Substituting this value of G in (2.4), we have

$$C_z - D_{\alpha x_\alpha} = - B\cdot A(x_{r+1}, x_{r+2}, \cdots, x_n).$$

A particular solution of this equation is $D_i=0$ $(i=1, 2, \cdots, n)$, $C=-A(x_{r+1}, x_{r+2}, \cdots, x_n)\cdot\int_{z_0}^{z}Bdz$. We thus get the following result.

THEOREM 4.3. *If the rank of* A_{ij} *is* r $(1<r<n)$, *then the most general variation problem* (I) *associated with the equation* $F_2=0$, *when reduced to the normal form of Lemma* 3, *has an integrand function* f *of the form*

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[10] See the dissertation cited in footnote 9.

[11] Cf. L. E. Dickson, *Modern Algebraic Theories*, 1926, pp. 71–72.

$$f = A(x_{r+1}, x_{r+2}, \cdots, x_n)\cdot\left[\frac{A_{\mu\mu}p_\mu p_\mu}{2} - \int_{z_0}^{z} B\,dz\right] + \partial\omega_\alpha/\partial x_\alpha$$

where the ω_i *are arbitrary functions of* $x_1, x_2, \cdots, x_n, z$.

Finally if the rank of (A_{ij}) is $r=1$, then $\partial M/\partial p_k=0$ for all $k>1$ and (4.3) reduces to the single equation $A_{11}(\partial M/\partial x_1+p_1\partial M/\partial z)=B\cdot\partial M/\partial p_1$, which obviously has two independent solutions $M_1(x_1, z, p_1)$ and $M_2(x_1, z, p_1)$ for M. In this case we are thus led to a somewhat different result.

THEOREM 4.4. *If the rank of* (A_{ij}) *is* 1, *then the partial differential equation* $F_2=0$, *when reduced to the normal form of Lemma* 3, *admits as its most general multiplier the functions* $A(M_1, M_2, x_2, x_3, \cdots, x_n)$ *where* A *is different from zero but is otherwise an arbitrary function.*

The determination of the most general integrand function f when $r=1$ may be left to the reader.

As an application of this case we find that the most general multiplier for $r=\partial^2z/\partial x_1^2=0$, which is of type $F_2=0$ with $n=2$ and $B=0$, is $A(p_1, z-p_1x_1, x_2)$. The most general variation problem (I) associated with $r=0$ has an integrand function f of the form

$$f = \int_{p_{1_0}}^{p_1}\int_{p_{1_0}}^{p_1} A(p_1, z - p_1x_1, x_2)dp_1dp_1 + \partial\omega_1/\partial x_1 + \partial\omega_2/\partial x_2$$

where ω_1, ω_2 are arbitrary functions of x_1, x_2, z.

5. Variation problems associated with the equation $R(p, q)r+2S(p, q)s+T(p, q)t=0$.

Consider the partial differential equation

$$F_3 \equiv R(p, q)r + 2S(p, q)s + T(p, q)t = 0, \tag{5.1}$$

where $z=z(x, y)$, $p=\partial z/\partial x$, $q=\partial z/\partial y$, $r=\partial^2z/\partial x^2$, $s=\partial^2z/\partial x\partial y$, $t=\partial^2z/\partial y^2$, and the coefficients R, S, T are analytic functions of p and q. We shall exclude equations for which two of the coefficients R, S, T vanish identically, since such vanishing would reduce the equation $F_3=0$ to one of the special equations $r=0$, $s=0$ treated in the previous section. The system (2.7) with $n=2$ and suitable identification of the variables now has the form

$$\begin{aligned} S\cdot\partial M/\partial p - R\cdot\partial M/\partial q + M\cdot[\partial S/\partial p - \partial R/\partial q] &= 0,\\ T\cdot\partial M/\partial p - S\cdot\partial M/\partial q + M\cdot[\partial T/\partial p - \partial S/\partial q] &= 0,\\ R\cdot[\partial M/\partial x + p\partial M/\partial z] + S\cdot[\partial M/\partial y + q\partial M/\partial z] &= 0,\\ S\cdot[\partial M/\partial x + p\partial M/\partial z] + T\cdot[\partial M/\partial y + q\partial M/\partial z] &= 0. \end{aligned} \tag{5.2}$$

In contradistinction to (4.1), (5.1) may (Case I) or may not (Case II) have a self-adjoint equation of variation. In Case I, since $\partial S/\partial p-\partial R/\partial q=\partial T/\partial p-\partial S/\partial q=0$, (5.2) becomes the homogeneous system

$$\begin{aligned} &U_1M \equiv S\cdot\partial M/\partial p - R\cdot\partial M/\partial q = 0,\\ &U_2M \equiv T\cdot\partial M/\partial p - S\cdot\partial M/\partial q = 0,\\ &U_3M \equiv R\cdot[\partial M/\partial x + p\cdot\partial M/\partial z] + S\cdot[\partial M/\partial y + q\cdot\partial M/\partial z] = 0,\\ &U_4M \equiv S\cdot[\partial M/\partial x + p\cdot\partial M/\partial z] + T\cdot[\partial M/\partial y + q\cdot\partial M/\partial z] = 0. \end{aligned} \tag{5.3}$$

In either case the equations of (5.2) and (5.3) are found to be independent if and only if $\delta\equiv RT-S^2\neq 0$. Each of the Cases I, II may therefore be further subdivided according as (1) $\delta\not\equiv 0$, (2) $\delta\equiv 0$. We proceed to a discussion of these cases.

Case I_1. When (5.1) has a self-adjoint equation of variation and $\delta\neq 0$, it follows from Theorem 2.1 that $F_3=0$ has a constant as its most general multiplier. Hence we have proved the following.

THEOREM 5.1. *If the equation* $F_3=0$ *has a self-adjoint equation of variation in a neighborhood in which* $\delta\neq 0$, *then the most general variation problem* (I) *associated with* $F_3=0$ *in this neighborhood has an integrand function f of the form* $f=G+\partial\omega_1/\partial x+\partial\omega_2/\partial y$, *where* $G(x, y, z, p, q)$ *is a particular solution of* $g_{pp}=R$, $g_{pq}=S$, $g_{qq}=T$, *satisfying* $G_{px}+G_{qy}+pG_{pz}+qG_{qz}-G_z=0$ *and the* ω_i *are arbitrary functions of* x, y, z.

As an illustration we note that the most general variation problem (I) associated with the equation $q^2r+4pqs+p^2t=0$ has an integrand function of the form $f=p^2q^2/2+\partial\omega_1/\partial x+\partial\omega_2/\partial y$, where the ω_i are arbitrary functions of x, y, z.

Case I_2. When $\delta\equiv 0$ it follows that $R\cdot S\cdot T\not\equiv 0$ and hence, in any neighborhood in which $R\cdot S\cdot T\neq 0$, $U_{k+1}M\equiv(T/S)\cdot U_kM$ $(k=1, 3)$. The system $U_iM=0$ $(i=1, 3)$ is therefore equivalent to (5.3) and moreover is found to be complete. Hence we have the following result.

THEOREM 5.2. *If* $F_3=0$ *has a self-adjoint equation of variation in a neighborhood in which* $R\cdot S\cdot T\neq 0$ *and* $\delta\equiv 0$, *then it admits as its most general multiplier in this neighborhood the function* $M(x, y, z, p, q)=A(M_1, M_2, M_3)$ *where* M_i $(i=1, 2, 3)$ *are any three independent integrals of* $U_iM=0$ $(i=1, 3)$ *and* $A\neq 0$ *is an arbitrary function.*

The most general variation problem (I) can now be obtained in the usual manner. The equation $\phi(p+q)\cdot[r+2s+t]=0$, where ϕ is an arbitrary analytic function of its argument, illustrates this case.

We now take up Case II in which $F_3=0$, as it stands, does not have a self-adjoint equation of variation. If in addition $\delta\neq 0$, then we can infer from Theorem 2.2 that at most one multiplier $M(x, y, z, p, q)$ exists. To secure more precise information in this case we must discuss the nonhomogeneous system (5.2). To integrate (5.2) we transform it into a system linear and homogeneous in the first partial derivatives of a function $m(x, y, z, p, q, M)$ with $\partial m/\partial M \neq 0$, which defines M by means of the relation $m=\text{const}$. The resulting system

$$(5.4)\quad \begin{aligned} U_1 m &\equiv S\partial m/\partial p - R\partial m/\partial q - M\cdot[\partial S/\partial p - \partial R/\partial q]\cdot\partial m/\partial M = 0,\\ U_2 m &\equiv T\partial m/\partial p - S\partial m/\partial q - M\cdot[\partial T/\partial p - \partial S/\partial q]\cdot\partial m/\partial M = 0,\\ U_3 m &\equiv R(\partial m/\partial x + p\partial m/\partial z) + S(\partial m/\partial y + q\partial m/\partial z) = 0,\\ U_4 m &\equiv S(\partial m/\partial x + p\partial m/\partial z) + T(\partial m/\partial y + q\partial m/\partial z) = 0 \end{aligned}$$

is independent since $\delta \neq 0$. It is easy to verify that the commutator $(U_1 U_4)m \equiv U_5 m$ is not a linear combination of $U_i m$ $(i=1, 2, 3, 4)$ and we accordingly adjoin $U_5 m=0$ to the system (5.4). It is next found that the commutator $(U_1 U_2)m$ is a linear combination of $U_i m$ $(i=1, 2, 3, 4, 5)$ if and only if the following determinant D vanishes:

$$\begin{vmatrix} S & T & S\cdot[\partial T/\partial p+\partial S/\partial q]-R\cdot\partial T/\partial q-T\cdot\partial S/\partial p \\ R & S & S\cdot[\partial S/\partial p+\partial R/\partial q]-R\cdot\partial S/\partial q-T\cdot\partial R/\partial p \\ \partial S/\partial p-\partial R/\partial q & \partial T/\partial p-\partial S/\partial q & V \end{vmatrix}$$

where

$$\begin{aligned} V = S\cdot[\partial^2 T/\partial p^2 - \partial^2 R/\partial q^2] &- R\cdot[\partial^2 T/\partial p\partial q - \partial^2 S/\partial q^2] \\ &- T\cdot[\partial^2 S/\partial p^2 - \partial^2 R/\partial q\partial p]. \end{aligned}$$

We are thus led to distinguish two subcases II_{11} and II_{12} according as $D \not\equiv 0$ or $D \equiv 0$.

Case II_{11}. In a neighborhood in which $\delta \neq 0$ and $D \neq 0$, $(U_1 U_2)m \equiv U_6 m$ is not a linear combination of $U_i m$ $(i=1, 2, 3, 4, 5)$, and upon adjoining $U_6 m=0$ to our system we have a system of six independent equations in the partial derivatives of $m(x, y, z, p, q, M)$. Hence the most general integral of the system is a constant. Thus there exists no nonsingular solution of the system (5.2) and we have the following.

THEOREM 5.3. *If $F_3=0$ does not have a self-adjoint equation of variation in a neighborhood in which $\delta \neq 0$ and $D \neq 0$, it admits no multiplier in this neighborhood.*

The equation $p^2 r + 2q^2 s = 0$ is an illustration of this case.

Case II_{12}. In a neighborhood in which $\delta \neq 0$ and $D \equiv 0$ the system $U_i m=0$ $(i=1, 2, 3, 4, 5)$ is complete and has $m=A(m_1)$ for its most general integral where $m_1(p, q, M)$ is a particular solution of $U_i m=0$

$(i=1,\ 2)$ and A is an arbitrary function of its argument. The most general nonsingular solution M of the system (5.2) is obtained by solving for M the relation $A(m_1)=\text{const}$. The equation $M\cdot F_3=0$ now has a self-adjoint equation of variation and $M^2\cdot\delta\neq 0$. Hence, since multipliers which differ by a nonzero constant factor are not regarded as distinct, we have from Theorem 2.1 the following.

THEOREM 5.4. *If* $F_3=0$ *does not have a self-adjoint equation of variation in a neighborhood in which* $\delta\neq 0$ *and* $D\equiv 0$, *then it admits as its most general multiplier in this neighborhood the unique solution for* M *of the relation* $A(m_1)=\text{const}$.

The most general variation problem (I) can now be obtained in the usual manner. As an illustration of this case we cite the partial differential equation of minimal surfaces[12] $(1+q^2)r-2pqs+(1+p^2)t=0$.

Case II_2. If $\delta\equiv 0$ it follows that $R\cdot S\cdot T\not\equiv 0$, and hence in any neighborhood in which $R\cdot S\cdot T\neq 0$ we find that $U_4m\equiv(T/S)\cdot U_3m$. However, $U_2m\equiv(T/S)\cdot U_1m$ if and only if $\Delta\equiv\partial(R/S)^2/\partial q-2\partial(R/S)/\partial p$ vanishes. We therefore consider the two subcases II_{21} and II_{22} according as $\Delta\equiv 0$ or $\Delta\not\equiv 0$.

Case II_{21}. In a neighborhood in which $R\cdot S\cdot T\neq 0$ and $\Delta\equiv 0$, the system $U_im=0$ $(i=1,\ 3)$ is equivalent to (5.4) and furthermore is complete. Hence we have the following result in this case.

THEOREM 5.5. *If* $F_3=0$ *does not have a self-adjoint equation of variation in a neighborhood in which* $R\cdot S\cdot T\neq 0$, $\delta\equiv 0$ *and* $\Delta\equiv 0$, *then in this neighborhood* $F_3=0$ *admits as its most general multiplier the solution for* M *of the relation* $A(m_1,\ m_2,\ m_3,\ m_4)=\text{const.}$, *where* m_i $(i=1,\ 2,\ 3,\ 4)$ *are any four particular independent solutions of* $U_im=0$ $(i=1,\ 3)$ *and where* A *is different from zero but is otherwise an arbitrary function.*

The equation $q^2r-2pqs+p^2t=0$ is an illustration of this case.

Case II_{22}. In a neighborhood in which $R\cdot S\cdot T\neq 0$ and $\Delta\neq 0$ we find it more perspicuous to consider the system (5.2). If in (5.2) we multiply the first equation by $S(p,\ q)$, the second by $R(p,\ q)$ and subtract, we obtain $M\cdot[S\cdot\partial S/\partial p-S\cdot\partial R/\partial q+R\cdot\partial S/\partial q-R\cdot\partial T/\partial p]=0$. Since $\Delta\neq 0$ we must have $M=0$, and hence the following is valid.

THEOREM 5.6. *If* $F_3=0$ *does not have a self-adjoint equation of variation in a neighborhood in which* $R\cdot S\cdot T\neq 0$, $\delta\equiv 0$ *and* $\Delta\neq 0$, *then it admits no multiplier in this neighborhood.*

The equation $p^2r+2pqs+q^2t=0$ illustrates this case.

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[12] Cf. L. LaPaz, Acta Szeged, vol. 5 (1932), pp. 199–207.

COPELAND'S DEFINITION OF A STIELTJES INTEGRAL

R. L. JEFFERY

Our present interest in Copeland's definition[1] of the Stieltjes integral of $g(x)$ with respect to the monotone function $f(x)$ is due to a remark by T. H. Hildebrandt[2] to the effect that in the proof of the formula for integration by parts it is required that $f(x)=\frac{1}{2}\{f(x+0)+f(x-0)\}$. In looking over Copeland's paper it was found that it is only in the proof of this formula that $f(x)$ is so restricted. Furthermore, it became clear that the definition possesses a considerable degree of generality. In the present note we simplify the definition, and compare it with that of the Riemann-Stieltjes integral and the Lebesgue-Stieltjes integral.

The classical definition of a Riemann-Stieltjes integral is

$$\text{RS}\int_{\alpha}^{\beta} g\,df = \lim_{n\to\infty} \sum_{k=1}^{n} g(\xi_k)\{f(x_k) - f(x_{k-1})\}, \tag{1}$$

where (x_{k-1}, x_k) is a finite subdivision of the interval $\alpha \leqq x \leqq \beta$, with $x_i - x_{i-1} \to 0$ and ξ_k any point on (x_{k-1}, x_k). The limit (1) exists when g is continuous, but may fail to exist even for functions g of bounded variation unless further restrictions[3] are placed on the subdivision (x_{k-1}, x_k) or on the choice of ξ_k. Copeland's definition is likewise based on what can be interpreted as a sequence of finite sets $\{(x_k)\}$, $k=1, 2, \cdots, n$, of (α, β), and the integral is given by

$$\text{CS}\int_{\alpha}^{\beta} g\,df = \lim_{n\to\infty} \frac{g(x_1) + \cdots + g(x_n)}{n}. \tag{2}$$

The sequence $\{x_k\}$ is defined wholly in terms of f, and the limit (2) exists for a wide class of functions including functions of bounded variation.

The set $\{x_k\}$, $k=1, 2, \cdots, n$; $n=1, 2, \cdots$, on which (2) is based is denumerable. Consequently the value of the integral depends only on the values of g over this denumerable set, which permits g an undesirable amount of freedom. To obviate this defect, and to bring the definition more in line with that of the Riemann-Stieltjes integral, we introduce some changes in its formulation.

On the interval $\alpha \leqq x \leqq \beta$ let $g(x)$ be bounded, and $f(x)$ be bounded and non-decreasing. In order to keep the initial stages as simple as possible, we shall assume that $f(\alpha+0)=0$, $f(\beta-0)=1$. This restriction will be removed later. For a given n and for $k=1, 2, \cdots, n-1$ let x_k be the greatest lower bound of numbers x for which

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[1] This Bulletin, vol. 43 (1937), pp. 581–588.

[2] American Mathematical Monthly, vol. 45 (1938), p. 277.

[3] This point has been thoroughly covered by Hildebrandt, loc. cit., §§6, 7, 8, 9.

$$(3) \qquad f(x-0) \leqq k/n \leqq f(x+0).$$

The numbers x_k, $(k=1, 2, \cdots, n-1)$, all lie on the open interval $\alpha<x<\beta$. Let $a<x<b$ be an open interval on $\alpha<x<\beta$, and for any n let p be the number of points of the set $\{x_k\}$ on $a<x<b$. Then

$$f(b-0)-f(a+0)-1/n<p/n<f(b-0)-f(a+0)+1/n,$$

from which it follows that

$$(4) \qquad \lim_{n\to\infty} p/n = f(b-0)-f(a+0).$$

Also if x_0 is a point of discontinuity of f on $\alpha<x<\beta$ and q is the number of times x_0 is repeated in the set $\{x_k\}$, then

$$f(x_0-0)-1/n<q/n<f(x_0+0)+1/n,$$

which gives

$$(5) \qquad \lim_{n\to\infty} q/n = f(x_0+0)-f(x_0-0).$$

The set $\{x_k\}$, $k=1, 2, \cdots, n-1$, is non-decreasing on $\alpha<x<\beta$. Let ξ_1 be a point on $\alpha<x<x_1$, ξ_k a point on $x_{k-1}\leqq x\leqq x_k$ if $x_k\neq x_{k-1}$, $\xi_k=x_k$ if $x_k=x_{k-1}$, $k=2, 3, \cdots, n-1$, and ξ_n a point on $x_{n-1}<x<\beta$. Form the sum

$$G_n=\frac{g(\xi_1)+\cdots+g(\xi_n)}{n},$$

and over the open interval $\alpha<x<\beta$ define

$$(6) \qquad \text{CS}\int_{\alpha<x<\beta} g\,df = \lim_{n\to\infty} G_n$$

provided this limit exists. If x_0 is a point of discontinuity of f on $\alpha<x<\beta$, it follows from (5) that the part of G_n arising from x_0 tends to $g(x_0)\{f(x_0+0)-f(x_0-0)\}$. This leads us to define

$$(7) \qquad \text{CS}\int_{x_0} g\,df = g(x_0)\{f(x_0+0)-f(x_0-0)\},$$

which may be retained when x_0 is a point of continuity of f. If $a<x<b$

is an open interval on $\alpha<x<\beta$, set $g'=g$ on $a<x<b$, $g'=0$ elsewhere on $\alpha<x<\beta$, and define

$$\text{CS}\int_{a<x<b} gdf = \text{CS}\int_{\alpha<x<\beta} g'df. \tag{8}$$

We further define

$$\text{CS}\int_a^b gdf = \text{CS}\int_{a<x<b} gdf + g(a)\{f(a+0)-f(a)\} + g(b)\{f(b)-f(b-0)\}.$$

From (7) and (8) we see that

$$\text{CS}\int_a^b gdf = \int_{a<x<b} gdf + \int_a gdf + \int_b gdf$$

only if $f(a)=f(a-0)$, and $f(b)=f(b+0)$. It is true, however, that if c is a point on $a<x<b$, then

$$\text{CS}\int_a^b gdf = \text{CS}\int_a^c gdf + \text{CS}\int_c^b gdf.$$

With the restrictions $f(\alpha+0)=0$, $f(\beta-0)=1$ still holding, the Copeland-Stieltjes integral, $\text{CS}\int_a^b gdf$, has now been defined for every interval (a, b) on $\alpha\leqq x\leqq\beta$. We shall prove the following theorems.

THEOREM I. *The necessary and sufficient condition for the existence of the* CS-*integral is that the common part of the discontinuities of g and the continuities of f have zero measure with respect to f.*

THEOREM II. *If the* CS-*integral exists, the* LS-*integral (Lebesgue-Stieltjes integral) exists, and the two are equal.*

THEOREM III. *If* $F(x)=\int_a^x gdf$, *then* dF/df *exists and is equal to g except for at most a set of zero measure with respect to f.*

It is thus seen that CS-integration with respect to the monotone function $f(x)$ has the same degree of generality as ordinary Riemann integration with respect to the variable x. In other words: If Copeland's definition replaces that of Riemann, it makes no difference whether the variable of integration is x or a monotone function $f(x)$.

Let $D=d_1, d_2, \cdots$ be the points of discontinuity of f, and for any point d_i let (a_i, a_i') be an interval with $a_i<d_i<a_i'$ and for which a_i and a_i' are points of continuity of f. Let p_i, q_i, r_i be respectively the number of points of the set $\{x_k\}$ on $a_i<x_i<d_i$, the number of times d_i

is repeated in this set, and the number of points of the set on $d_i < x < a_i'$. Then $p_i/n \to f(d_i-0) - f(a_i)$, $q_i/n \to f(d_i+0) - f(d_i-0)$, and $r_i/n \to f(a_i') - f(d_i+0)$. Thus if $a_i' - a_i$ is sufficiently small and n is sufficiently great, the part of G_n arising from the interval (a_i, a_i') is arbitrarily near $g(d_i)\{f(d_i+0) - f(d_i-0)\} = \mathrm{CS}\int_{d_i} gdf$. Since g is bounded and f is bounded and non-decreasing, $\sum g(d_i)\{f(d_i+0) - f(d_i-0)\}$ converges. Also, there exists a set of nonoverlapping intervals $A = (a_1, a_1'), \cdots, (a_l, a_l')$ with $a_i < d_i < a_i'$ and such that for l sufficiently great and n sufficiently great we have

$$(9) \qquad \sum_{i=l+1}^{\infty} f(d_i + 0) - f(d_i - 0) < \epsilon,$$

and, on account of relations (4) and (5), the part of G_n arising from the intervals A differs from

$$\sum_{i=1}^{l} g(d_i)\{f(d_i + 0) - f(d_i - 0)\} = \sum \int_{d_i} gdf$$

by not more than ϵ. Hence the upper and lower limits of the part of G_n arising from the intervals A do not differ by more than 2ϵ.

Let e_ϵ be the set of points of (α, β) at which the saltus of g is not less than ϵ. Then e_ϵ is closed, and consequently the part e_ϵ' of e_ϵ which is not interior to A is closed. Let the common part of e_ϵ and the set of continuities of f have zero measure with respect to f. It then follows, if (9) is taken into consideration, that the closed set e_ϵ' can be put in a finite set of nonoverlapping intervals $B = (b_i, b_i')$ which do not overlap the set A, which are such that b_i, b_i' are points of continuity of f, and for which

$$\sum \{f(b_i') - f(b_i)\} < 2\epsilon.$$

It then follows that for any n the part of G_n arising from the intervals B is not greater in numerical value than $2\epsilon M$, where M is the least upper bound of $|g(x)|$.

At each point of (α, β) not interior to the intervals $A+B$ the saltus of g is less than ϵ. Hence the closed intervals complementary to the set interior to $A+B$ can be subdivided into a finite set of intervals $C = (c_i, c_i')$ such that c_i, c_i' are points of continuity of f and on each interval (c_i, c_i') the fluctuation of g is not more than ϵ. If $G_n^i = \{g(\xi_1^i) + \cdots + g(\xi_m^i)\}/n$ is the part of G_n arising from the interval (c_i, c_i'), then G_n^i lies between the values

$$\{g(c_i) - \epsilon\}p_i/n, \qquad \{g(c_i) + \epsilon\}p_i/n,$$

where p_i is the number of points of the sequence $\{x_k\}$ on $c_i<x<c_i'$. Hence as $n\to\infty$ the upper and lower limits of G_n^i do not differ by more than $2\epsilon\{f(c_i')-f(c_i)\}$. We can now conclude that the least upper bound and greatest lower bound of

$$G_n = \frac{g(\xi_1) + \cdots + g(\xi_n)}{n}$$

do not differ by more than $2\epsilon+2\epsilon M+2\epsilon\{f(\beta)-f(\alpha)\}$. Since ϵ is arbitrary, it follows that if $g(x)$ satisfies the conditions of Theorem I then the CS-integral of g with respect to the monotone function $f(x)$ exists.

If the set of discontinuities of g has measure greater than 0 with respect to f over the continuities of f, then for some $\epsilon>0$ there exists $d>0$ such that $\sum\{f(b_i')-f(b_i)\}>d$ regardless of the choice of the intervals (b_i, b_i'). Consequently for every n there exist ξ_i and ξ_i' such that the sums $\{g(\xi_1)+\cdots+g(\xi_n)\}/n$ and $\{g(\xi_1')+\cdots+g(\xi_n')\}/n$ will differ by an amount which, for the fixed ϵ in question, is not less than ϵd. From this it follows that the CS-integral of f with respect to g does not exist, and we conclude that the conditions of Theorem I are necessary.[4]

To obtain the LS-integral of g with respect to f we proceed as follows: Let (y_{i-1}, y_i) be a subdivision of the range of g. Let e_i be the part of (α, β) for which $y_{i-1}\leqq g<y_i$, and $f(e_i)$ the image of the set e_i by means of the transformation $y=f(x)$, where a point x_0 of discontinuity of f corresponds to the closed interval $\{f(x_0-0), f(x_0+0)\}$. Then $\mathrm{LS}\int_\alpha^\beta gdf=\lim\sum y_i\, mf(e_i)$ as $y_i-y_{i-1}\to 0$, provided this limit exists. This limit does exist if g is measurable with respect to f, which readily follows if the points of continuity of f at which g is discontinuous have zero measure with respect to f. If $D=d_1, d_2, \cdots$ is the set of discontinuities of f, then

$$\mathrm{LS}\int_D gdf = \sum g(d_i)\{f(d_i+0) - f(d_i-0)\} = \mathrm{CS}\int_D gdf.$$

If the intervals $A=(a_i, a_i')$ and $B=(b_i, b_i')$ are properly chosen, then $\mathrm{CS}\int_A gdf$ is arbitrarily near to $\sum g(d_i)\{f(d_i+0)-f(d_i-0)\}=\mathrm{LS}\int_D gdf$, and both $\mathrm{CS}\int_B gdf$ and $\mathrm{LS}\int_B gdf$ are less in numerical value than $2\epsilon M$.

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[4] The necessity of the conditions of Theorem I would not follow if G_n were restricted to the form $\{g(x_1)+\cdots+g(x_n)\}/n$ of Copeland's original definition. Let $f(x)=x$. The sequence $\{x_k\}$, $k=1, 2, \cdots, n$, $n=1, 2, \cdots$, is a denumerable set on (α, β). If $g(x)=0$ at each point of this denumerable set, and for the rest of (α, β) $g(x)=1$ when x is rational and $g(x)=0$ when x is irrational, then CS $\int_\alpha^\beta gdf$ exists and is equal to zero, but g does not satisfy the conditions of Theorem I.

Over the set of intervals (c_i, c_i') both the CS-integral and the LS-integral of g with respect to f lie between $\sum\{g(c_i)-\epsilon\}\{f(c_i')-f(c_i)\}$ and $\sum\{g(c_i)+\epsilon\}\{f(c_i')-f(c_i)\}$. From these considerations the truth of Theorem II readily follows.

In proceeding with the proof of Theorem III we first define the derivative of a function $F(x)$ with respect to a monotone function $f(x)$. Let

$$\begin{aligned}\psi(x, \Delta x) &= \frac{F(x+\Delta x)-F(x-0)}{f(x+\Delta x)-f(x-0)}, \\ &\qquad\qquad f(x+\Delta x)-f(x-0)\neq 0,\ \Delta x>0, \\ &= \frac{F(x+\Delta x)-F(x+0)}{f(x+\Delta x)-f(x+0)}, \\ &\qquad\qquad f(x+\Delta x)-f(x+0)\neq 0,\ \Delta x<0, \\ &= 0 \text{ otherwise.}\end{aligned}$$

If, for a fixed x, $\psi(x, \Delta x)$ tends to a limit as $\Delta x\to 0$, then this limit is the derivative of $F(x)$ with respect to $f(x)$, dF/df. Now let $F(x)=\int_a^x g\,df$. If x_0 is a point of discontinuity of f, then, for $\Delta x>0$, $F(x_0+\Delta x)-F(x_0-0)$ is the limit as $\delta\to 0$ of $F(x_0+\Delta x)-F(x_0-\delta)$, and this last is the limit of

$$\frac{g(\xi_1^-)+\cdots+g(\xi_p^-)+qg(x_0)+g(\xi_1^+)+\cdots+g(\xi_r^+)}{n} \tag{10}$$

where $\xi_1^-, \cdots, \xi_p^-, \xi_1^+, \cdots, \xi_r^+$ are points on the intervals (x_{k-1}, x_k) formed by the sequence $\{x_k\}$ on $x_0-\delta<x<x_0$, $x_0<x<x_0+\Delta x$ respectively, and q is the number of times x_0 is repeated in this sequence. Then for δ and Δx fixed $p/n\to f(x_0-0)-f(x_0-\delta+0)$, $q/n\to f(x_0+0)-f(x_0-0)$, and $r/n\to f(x_0+\Delta x-0)-f(x_0+0)$. Hence for δ and Δx sufficiently small p/n and r/n are both arbitrarily near to zero. Since g is bounded, it then follows from (10) that for δ and Δx sufficiently small $F(x_0+\Delta x)-F(x_0-\delta)$ is arbitrarily near to $g(x_0)\{f(x_0+0)-f(x_0-0)\}$, and consequently $\psi(x_0, \Delta x)$ is arbitrarily near to $g(x_0)$. Similarly it can be shown that for $\Delta x<0$, $\psi(x_0, \Delta x)\to g(x_0)$.

Next let x_0 be a point of continuity of both g and f. Then $F(x_0+\Delta x)-F(x_0-\delta)$ is the limit of a sum of the form

$$\frac{g(x_0)+t_1+\cdots+g(x_0)+t_p}{n}$$

where $|t_i|<\epsilon$ if δ and Δx are sufficiently small, and $p/n\to f(x_0+\Delta x-0)$

$-f(x_0-\delta+0)$. Hence for δ and Δx sufficiently small, and n sufficiently great

$$\psi(x_0, \Delta x) = g(x_0) + t\{f(x_0 + \Delta x - 0) - f(x_0 - \delta + 0)\},$$

where $t \to 0$ as Δx and $\delta \to 0$. We conclude therefore that, for $\Delta x > 0$, $\psi(x_0, \Delta x) \to g(x_0)$. Similarly it can be shown that, for $\Delta x < 0$, $\psi(x, \Delta x) \to g(x_0)$. Hence, at x_0, $dF/df = g(x_0)$, and Theorem III is established.

We note that the definition of dF/df given above implies the existence of $F(x-0)$ and $F(x+0)$, which can easily be shown if $F(x) = \mathrm{CS}\int_\alpha^x g\,df$. Furthermore, at points of discontinuity of $F(x)$ or of $f(x)$ the value of the derivative does not depend on the value of these functions at the point.

The next considerations are the removal of the restrictions $f(\alpha+0)=0$, $f(\beta-0)=1$, and the equivalence of the original and the modified definition. As to the first of these, it can be done, following Copeland, by setting

$$\phi(x) = \frac{f(x) - f(\alpha + 0)}{f(\beta - 0) - f(\alpha + 0)}$$

and defining

$$\mathrm{CS}\int_{\alpha<x<\beta} g\,df = \mathrm{CS}\int_{\alpha<x<\beta} g\,d\phi \cdot [f(\beta - 0) - f(\alpha + 0)]. \tag{11}$$

It is possible, however, and perhaps advisable, to forego this restriction from the start: In (3) let x_k be the greatest lower bound of numbers x for which

$$f(x - 0) \leqq f(\alpha + 0) + (k/n)\{f(\beta - 0) - f(\alpha + 0)\} \leqq f(x + 0),$$

and define

$$\mathrm{CS}\int_{\alpha<x<\beta} g\,df = \lim_{n\to\infty} \frac{g(\xi_1) + g(\xi_2) + \cdots + g(\xi_n)}{\sigma_n}, \tag{12}$$

where ξ_k is chosen as above, and $1/\sigma_n = \{f(\beta-0) - f(\alpha+0)\}/n$. If $1/\sigma_n$ replaces $1/n$ in the foregoing discussions, all the results hold without further change in the wording. When $f(\alpha+0)=0$ and $f(\beta-0)=1$, (12) reduces to (6); and when these restrictions do not hold, it can be shown that (11) and (12) are equivalent. The proof of this we leave to the reader.

As to the equivalence of the modified form and the original form, if the integral exists under the first it does under the second. For the part contributed by the discontinuities of f is the same; on the intervals (c_i, c_i') both integrals lie between the values

$$(13)\quad \sum g(c_i)\{f(c_i') - f(c_i)\} - \epsilon K, \quad \sum g(c_i)\{f(c_i') - f(c_i)\} + \epsilon K,$$

where $K=f(\beta)-f(\alpha)$. These considerations easily lead to the equality of the two. That the integral can exist under the original form and not under the modified form is shown by the example in the footnote above. The set[5] $x_1; x_2, x_3; x_4, x_5, x_6; \cdots$ on which the original definition is based is such that for any n the first n points are not arranged in increasing order of magnitude. But obviously these can be rearranged into a non-decreasing set $\{x_k\}$, and on (x_{k-1}, x_k) the ξ_k can be chosen as above. Then

$$\lim_{n\to\infty} \frac{g(\xi_1) + \cdots + g(\xi_n)}{n}$$

leads to an integral which is equivalent to that arising from the modified definition given in this paper. For the results and proof of Theorem I are the same; the part of each arising from the discontinuities of f is the same; furthermore, on the intervals (c_i, c_i') both integrals lie between the bounds given by (13).

Thus, if we get away from an integral which depends only on the values of g over a denumerable set, the sequence of sets given by (3) leads to the same result as the set $x_1; x_2, x_3; \cdots$ on which the original definition is based. The sequence given by (3) is more simply defined; the two are equivalent with respect to properties (4) and (5), but for that given by (3) these properties are more easily established.

In correspondence Copeland has stated that he had in mind statistical considerations when he developed the original definition. "These numbers," that is, the set $x_1; x_2, x_3; \cdots$, "can be interpreted as a sequence of measurements of some physical quantity. The expression $f(b+0)-f(a+0)$ is interpreted as the probability that a measurement x_k will lie in the interval $a<x\leqq b$. This probability is defined as the limit of the success ratio, that is, as the average number of points lying in the interval. If a person is to receive $g(x)$ dollars when the measurement turns out to be x, then the expected amount which he will receive is the limit of the average of his receipts. This limit is the integral $\int_\alpha^\beta g df$." Whatever the utilitarian background of the idea, a Stieltjes integral of the Riemann type applicable without modification to such a wide range of functions is interesting for its own sake.

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[5] This Bulletin, loc. cit., p. 582.

A THEOREM ON THE ROTATION GROUP OF THE TWO-SPHERE[1]

DEANE MONTGOMERY AND LEO ZIPPIN

Let R be the group of all rotations of euclidean three-dimensional space E about its origin. We shall take the domain of operation of R to be a euclidean two-sphere S with center at the origin. On the space S, R is transitive. The group R is topological and, considered as a space, is homeomorphic to projective three-space.

While studying the action of groups in certain spaces, the following theorem, which, as far as we know, is not in the literature, occurred to us.

THEOREM. *Let G be any proper subgroup of R. Then G is not transitive on S.*

The subgroup G is subject to no restrictions whatever; in particular it is not closed. The proof will be essentially topological and we begin by assuming that G is transitive on S.

The identity element e of G must be a limit point of G. For otherwise G would be finite and hence certainly not transitive on S. Let g be an element of G which is near e. The element g has a pair of fixed points on S one of which will be denoted by p. The element g is a rotation through a small angle A around the line through p and the origin. Under this rotation points near p move in a certain direction around p, say the clockwise direction.

Now let x be any point of S, and let h be any element of G such that $h(p)=x$. The element hgh^{-1} is in G, and it is a rotation through the angle A around the line through x and the origin. Points near x will be moved in a clockwise direction around x.

We have therefore shown that for every point x of the sphere, G contains a rotation through the angle A around the line through the origin and x. For each x, points near x are moved by the associated rotation in a specified sense around x. Let M be the totality of all these rotations the existence of which has just been demonstrated. The set M is homeomorphic to S and consequently is a two-sphere; furthermore M is a subset of G.

Consider the elements of G given by $g^{-1}M$. This set is a two-sphere passing through the identity element. There will be an arc L, which is in $g^{-1}M$ and therefore in G, leading from e to some element of G, call it g' distinct from e. Let U be an open three-cell subset of R which includes e. Let N be a two-sphere made up of elements of G. The existence of arbitrarily small two-spheres of this kind is proved as above by choosing the element g sufficiently near to e, and we may assume that N is in U. We may also assume that LN is in U and that $g'N$ is outside N: as must be the case if N is small enough.

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[1] Presented to the Society, October 28, 1939, under the title *Note on rotation-group of the two-sphere.*

The arc L may now be used to define a deformation of N to $g'N$. Under this deformation all points swept out by N are in G. Furthermore every point inside N is swept out by the deformation. Hence every point of R inside N is in the group G. The group G is thus seen to contain open subsets and, because of homogeneity, G is open in R. It must therefore coincide with R. The assumption that a proper subgroup G was transitive on S has now led to a contradiction, and the proof is therefore complete.

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ON ORDERED ALGEBRAS[1]

A. A. ALBERT

In his first Madison Colloquium lecture M. H. Stone connected the theory of convex bodies with linear sets over an ordered field. It was natural then to ask whether his theory could be extended by replacing these fields by ordered rings and indeed to ask whether there exist ordered rings which are not fields. I discussed this question at that time with S. MacLane and we attempted to answer it. MacLane has since found an example,[2] in the literature, of a noncommutative ordered quasi-field. It is not an algebra (of finite order) however and it is my purpose in this note to give a very brief proof in elementary language of the following decisive result.

THEOREM. *Every ordered algebra is a field.*

We first observe some known consequences of the order postulates.[3] The postulates on products imply that an ordered ring contains no divisors of zero and hence that every ordered algebra is a division algebra D. Then D has a unity quantity $1=1^2>0$, the sums

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[1] Presented to the Society, December 2, 1939.

[2] Cf. Reidemeister, *Grundlagen der Geometrie*, p. 40. It is also shown in this text that archimedian ordered quasi-fields are fields.

[3] The order postulates on page 40 of my *Modern Higher Algebra* were called postulates for an ordered field but are valid for arbitrary rings.

$1+1+\cdots+1$ are all positive, D is a normal division algebra of degree n over a non-modular field K, its centrum. If $u\neq 0$ is in D, then $uu^{-1}=1>0$, u and u^{-1} are either both positive or both negative, the transforms udu^{-1} of every positive quantity of D are all positive.

Let then $n>1$ so that D contains a quantity b not in its centrum K. The minimum function of b is $g(x)=x^n+a_1x^{n-1}+\cdots+a_n$ with $n>1$ and the a_i in K. Since K is non-modular, D contains $b-m^{-1}a_1$ and its negative. One of these is a positive quantity d of D with minimum function $h(y)=y^m+b_2y^{m-2}+\cdots+b_m$ for b_i in K. By a theorem of Wedderburn[4] $h(y)=(y-d_m)\cdots(y-d_1)$ for transforms d_i of d, the $d_i>0$, their sum is $d_1+\cdots+d_m=0$ by our choice above. This contradicts our hypothesis that D is ordered. Hence $n=1$, $D=K$ is a field.

In more technical terms our proof is simply the remark that, if it exists, a noncommutative ordered division algebra contains a positive quantity d of zero trace, whereas this trace is a sum of transforms of d and must be positive.

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[4] See the proof of Theorem 3, p. 230, of L. E. Dickson's *Algebras and their Arithmetics.*

SIMPLICIAL INTERSECTION CHAINS FOR AN ABSTRACT COMPLEX

W. W. FLEXNER

Abstract complexes have been defined by J. W. Alexander, W. Mayer, A. W. Tucker and S. Lefschetz.[1] The definition here adopted is that of Lefschetz which derives directly from that of Tucker. To conform to Lefschetz's present usage the notation here will differ from that in the article cited by having $K=\{E_i^p\}$ and $K^*=\{E_p^i\}$. It should be remarked that when K is infinite $FF=0$ implies that for given E_i^p and E_j^{p-2},

$$[E_i^p:E_k^{p-1}][E_k^{p-1}:E_j^{p-2}]\neq 0$$

for at most a finite number of E_k^{p-1}. The dual K^* here is to have the property

$$[E_i^p:E_j^{p-1}] = (-1)^{p+1}[E_{p-1}^j:E_p^i] = \mu_{ij}^p$$

where μ_{ij}^p is an abbreviated notation (compare loc. cit., p. 346 (c)). This is in order that a fourth postulate

IV. Ki $(E_i^p\cdot E_p^i)=1$

may be added to I–III (loc. cit., p. 350), where Ki means Kronecker index, the sum of the coefficients of the elements of the zero-chain in question. Here C is a p-chain of K, D a q-chain of K^*, $p>q$.

It is well known that when K is a simplicial or polyhedral manifold, for every pair E_i^p, E_q^j, $p\geqq q$, an intersection can be defined which is a chain of the simplicial regular subdivision of K. And when K is an arbitrary simplicial complex, the intersection is on K and hence simplicial. Here it is shown that *for any complex the intersections* $E_i^p\cdot E_q^j$ *can be regarded as integral chains of a simplicial complex* K^{τ}.

This complex will be defined abstractly by means of its vertices σ_i^p. The first definition is for $p=q$:

$$E_i^p\cdot E_p^j = \delta_i^j\sigma_j^p \tag{1}$$

(where $\cdot$ replaces the $\odot$ of the article cited). Then for $p>q$, if $r=p-q-1$,

$$E_i^p\cdot E_q^j = \sum_{k_1,k_2,\cdots,k_r} \mu_{ik_1}^{p}\mu_{k_1k_2}^{p-1}\cdots\mu_{k_rj}^{q+1}\sigma_i^p\sigma_{k_1}^{p-1}\cdots\sigma_{k_r}^{q+1}\sigma_j^q \tag{2}$$

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[1] S. Lefschetz, this Bulletin, vol. 43 (1937), pp. 345–359. (References to the other authors will be found on page 345.)

where $\sigma_i^p\sigma_{k_1}^{p-1}\cdots\sigma_{k_r}^{q+1}\sigma_j^q$ denotes the simplex of K^π with the given vertices; so $E_i^p\cdot E_q^j$ is an integral chain of K^π.

To verify that $E_i^p\cdot E_q^j$ is an intersection, it is sufficient, in view of (1), (2), to prove the formula III (p. 350, loc. cit.):

III. $F(E_i^p\cdot E_q^j)=(FE_i^p)\cdot E_q^j+(-1)^pE_i^p\cdot(FE_q^j)$.

When $q=p-1$, $r=0$, (2) gives $E_i^p\cdot E_{p-1}^j=\mu_{ij}^p\sigma_i^p\sigma_j^{p-1}$. Using the rule for the boundary of a simplex given by its vertices, we obtain $F(E_i^p\cdot E_{p-1}^j)=\mu_{ij}^p(\sigma_j^{p-1}-\sigma_i^p)$. By formula (2) (loc. cit.), its dual, and (1),

$$(FE_i^p)\cdot E_{p-1}^j=\sum_k\mu_{ik}^pE_k^{p-1}\cdot E_{p-1}^j=\mu_{ij}^p\sigma_j^{p-1},$$

$$E_i^p\cdot(FE_{p-1}^j)=(-1)^{p+1}\mu_{ij}^p\sigma_i^p$$

which verifies III in this case. When $q<p-1$,

$$\sum_k\mu_{ik_1}^p\cdots\mu_{k_{s-1}k_s}^{p-s+1}\mu_{k_sk_{s+1}}^{p-s}\cdots\mu_{k_rj}^{q+1}\sigma_i^p\sigma_{k_1}^{p-1}\cdots\sigma_{k_{s-1}}^{p-s+1}\sigma_{k_{s+1}}^{p-s-1}\cdots\sigma_{k_r}^{q+1}\sigma_j^q=0,\qquad 1\leqq s\leqq r,$$

since $FF=0$ implies $\sum_{k_s}\mu_{k_{s-1}k_s}^{p-s+1}\mu_{k_sk_{s+1}}^{p-s}=0$. Hence

$$\begin{aligned}F(E_i^p\cdot E_q^j)&=\sum_k\mu_{ik_1}^p\cdots\mu_{k_rj}^{q+1}(\sigma_{k_1}^{p-1}\cdots\sigma_{k_r}^{q+1}\sigma_j^q\\&\qquad+(-1)^{p-q}\sigma_i^p\sigma_{k_1}^{p-1}\cdots\sigma_{k_r}^{q+1})\\&=\sum_{k_1}\mu_{ik_1}^p(E_{k_1}^{p-1}\cdot E_q^j)+(-1)^{p-q}\sum_{k_r}\mu_{k_rj}^{q+1}(E_i^p\cdot E_{q+1}^{k_r})\end{aligned}$$

which by Lefschetz's article, formulas (2) and (1), is the right side of III.

CORNELL UNIVERSITY AND
INSTITUTE FOR ADVANCED STUDY

ON TRANSLATIONS OF FUNCTIONS AND SETS[1]

RALPH PALMER AGNEW

1. **Introduction.** It is the object of this note to prove the following theorem and two lemmas (see §3) on translations of sets which are used in the proof of the theorem.

THEOREM 1. *In order that a sequence* $x_n(t)$ *of complex-valued functions measurable over* $-\infty < t < \infty$ *may be such that, for each real sequence* λ_n,

$$\lim_{n\to\infty} x_n(t-\lambda_n) = 0 \tag{1}$$

for almost all t, *it is necessary and sufficient that for each* $\delta > 0$

$$\sum_{n=1}^{\infty} \underset{-\infty<h<\infty}{\text{l.u.b.}} \left| E_t\{h \leqq t \leqq h+1;\ |x_n(t)| \geqq \delta\}\right| < \infty. \tag{2}$$

Necessity for Theorem 1 is established by proving the following more incisive theorem.

THEOREM 2. *If a sequence* $x_n(t)$ *of complex-valued functions measurable over* $-\infty < t < \infty$ *is such that, for each real sequence* λ_n,

$$\lim_{n\to\infty} x_n(t-\lambda_n) = 0$$

for each t *in some set* D *of positive measure (where the set* D *may depend upon the sequence* λ_n), *then* (2) *holds.*

Measure is that of Lebesgue, and a property such as (1) holds for almost all t if it holds for all t in the infinite interval $-\infty < t < \infty$ with the possible exception of a null set (set of measure 0). The set

$$A \equiv A(h, t, n, \delta) = E_t\{h \leqq t \leqq h+1;\ |x_n(t)| \geqq \delta\}$$

is the set of all points t such that $h \leqq t \leqq h+1$ and $|x_n(t)| \geqq \delta$; and $|A|$ denotes the measure of A. The condition (2) implies that when n is large the function $|x_n(t)|$ is less than δ for "most" values of t in each unit interval; but (2) implies no restriction whatever on $x_n(t)$ when t lies in the "exceptional" set.

The hypothesis that (1) holds for almost all t for each real *bounded* sequence λ_n does not imply (2). For example if, for each $n = 1, 2, 3, \cdots$, $x_n(t)$ is a constant c_n over the interval $2^n < t < 2^n+1$ and is 0 otherwise, and λ_n is a bounded sequence, then (1) holds for each t; but (2) fails in case c_n fails to converge to 0 as n becomes infinite.

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[1] Presented to the Society, September 8, 1939.

2. **Proof of sufficiency for Theorem** 1. Let $x_n(t)$ be a sequence of measurable functions for which (2) holds, and let λ_n be a sequence of real numbers. It follows from (2) that, for each $\delta>0$,

$$\sum_{n=1}^{\infty} \operatorname*{l.u.b.}_{-\infty<h<\infty} \left| E_t\{h \leqq t \leqq h+1;\ |x_n(t-\lambda_n)| \geqq \delta\}\right| < \infty. \tag{3}$$

Let J denote an arbitrary finite interval. Since J can be covered by a finite set of unit intervals $h\leqq t\leqq h+1$, it follows from (3) that for each $\delta>0$

$$\sum_{n=1}^{\infty} \left| E_t\{t\,\varepsilon\, J;\ |x_n(t-\lambda_n)| \geqq \delta\}\right| < \infty. \tag{4}$$

Setting

$$A_{n,p} = E_t\{t\,\varepsilon\, J;\ |x_n(t-\lambda_n)| \geqq p^{-1}\}, \qquad n, p = 1, 2, 3, \cdots, \tag{5}$$

we see that (4) implies existence of indices $n_1<n_2<n_3<\cdots$ such that

$$\sum_{n=n_p}^{\infty} |A_{n,p}| < 2^{-p-1}, \qquad p = 1, 2, \cdots. \tag{6}$$

Setting

$$A_r = \sum_{p=r}^{\infty} \sum_{n=n_p}^{\infty} A_{n,p}, \qquad r = 1, 2, \cdots, \tag{7}$$

we find

$$|A_r| \leqq \sum_{p=r}^{\infty} \sum_{n=n_p}^{\infty} |A_{n,p}| < \sum_{p=r}^{\infty} 2^{-p-1} = 2^{-r}, \qquad r = 1, 2, \cdots. \tag{8}$$

Let

$$J_r = J - A_r, \qquad r = 1, 2, \cdots. \tag{9}$$

If $t\,\varepsilon\, J_r$ then, when $p>r$,

$$|x_n(t-\lambda_n)| < p^{-1}, \qquad n \geqq n_p, \tag{10}$$

so that $x_n(t-\lambda_n)$ converges to 0 over J_r. Hence $x_n(t-\lambda_n)$ converges to 0 over $J_1+J_2+\cdots$. But J_r is a subset of J having measure greater than $|J|-2^{-r}$; hence $J_1+J_2+\cdots$ is a subset of J having measure $|J|$. Therefore $x_n(t-\lambda_n)$ converges to 0 for almost all t in J.

Since J is an arbitrary finite interval, $x_n(t-\lambda_n)$ must converge to 0 for almost all t in $-\infty<t<\infty$ and sufficiency for Theorem 1 is proved.

3. **Lemmas on translations of sets.** In this section we prove two lemmas. The first states that if C and B are measurable subsets of unit intervals, then it is possible to translate B in such a way that the intersection of C and the translation of B will have measure at least $\frac{1}{2}|C|\,|B|$. The first lemma is used in proof of the second which specifies conditions under which a given sequence of sets can be translated so as to cover each point of the interval $-\infty<t<\infty$, with the exception of a null set, an infinite number of times. The close connection established in §4 between Lemma 2 and Theorem 2 shows that the combined proofs of Lemmas 1 and 2 furnish essentially a proof of Theorem 2.

If E is a set of points t in the interval $-\infty<t<\infty$ and λ is a real number, let $E(\lambda)$ denote the set of points t such that $t-\lambda\ \varepsilon\ E$; thus $E(\lambda)$ is the set obtained by translating the set E to the right λ units. Let U denote the unit interval $0\leqq t\leqq 1$.

LEMMA 1. *If C and B are measurable subsets of U, then*

$$\max_{-1\leqq\lambda\leqq 1}|CB(\lambda)|\geqq\tfrac{1}{2}|C|\,|B|. \tag{11}$$

Let $\phi(t)$ be the characteristic function of C, that is, $\phi(t)=1$ when $t\ \varepsilon\ C$ and $\phi(t)=0$ otherwise; and let $\psi(t)$ be the characteristic function of B. Then $\psi(t-\lambda)$ is the characteristic function of $B(\lambda)$, and $\phi(t)\psi(t-\lambda)$ is the characteristic function of the intersection $CB(\lambda)$ of C and $B(\lambda)$. Hence on denoting the measure of $CB(\lambda)$ by $\mu(\lambda)$ we have

$$\mu(\lambda)=\int_{-\infty}^{\infty}\phi(t)\psi(t-\lambda)dt. \tag{12}$$

The function $\mu(\lambda)$ is continuous since

$$\begin{aligned}|\mu(\lambda+h)-\mu(\lambda)|&\leqq\int_{-\infty}^{\infty}\phi(t)\,|\psi(t-\lambda-h)-\psi(t-\lambda)|\,dt\\&\leqq\int_{-\infty}^{\infty}|\psi(t-\lambda-h)-\psi(t-\lambda)|\,dt\\&=\int_{-\infty}^{\infty}|\psi(t-h)-\psi(t)|\,dt\end{aligned}$$

and the last integral converges to 0 with h. Hence $\mu(\lambda)$ has a maximum over the interval $-1 \leqq \lambda \leqq 1$. Since $\mu(\lambda) = 0$ when $|\lambda| > 1$, the computation

$$\int_{-1}^{1} \mu(\lambda) d\lambda = \int_{-\infty}^{\infty} d\lambda \int_{-\infty}^{\infty} \phi(t)\psi(t - \lambda) dt$$
$$= \int_{-\infty}^{\infty} \phi(t) dt \int_{-\infty}^{\infty} \psi(t - \lambda) d\lambda = |C| |B|$$

is easily justified. This equality and the inequality

$$(13) \qquad \mu(\lambda) \leqq \max_{-1 \leqq \lambda \leqq 1} \mu(\lambda), \qquad -1 \leqq \lambda \leqq 1,$$

imply that $\max |CB(\lambda)| = \max \mu(\lambda) \geqq \frac{1}{2} |C| |B|$ and Lemma 1 is established.

The fact that use of inequalities such as (13) often leads to crude results may make one suspicious that Lemma 1 holds when the factor $\frac{1}{2}$ in (11) is replaced by a greater factor. To settle this question, let $0 < \epsilon < \frac{1}{3}$, let $C = E_t\{\epsilon \leqq t \leqq 1 - \epsilon\}$, and let $B = E_t\{0 \leqq t \leqq \epsilon\} + E_t\{1 - \epsilon \leqq t \leqq 1\}$. Then $|C| = 1 - 2\epsilon$, $|B| = 2\epsilon$, and it is easy to verify that

$$(14) \qquad \max_{-1 \leqq \lambda \leqq 1} |CB(\lambda)| = \epsilon = [1/(2 - 4\epsilon)] |C| |B| > 0.$$

This shows that $\frac{1}{2}$ is the greatest factor permissible in (11).

LEMMA 2. *If* $A_1, A_2, \cdots$ *is a sequence of measurable sets and a sequence* $U_1, U_2, \cdots$ *of unit intervals exists such that*

$$(15) \qquad \sum_{n=1}^{\infty} |U_n A_n| = \infty,$$

then there exists a sequence $\lambda_1, \lambda_2, \cdots$ *such that each* t *in the interval* $-\infty < t < \infty$, *except those in some null set, lies in an infinite number of the sets* $A_n(\lambda_n)$.

Let $B_n = U_n A_n$ so that each B_n lies in some unit interval and $\sum |B_n| = \infty$. Let n be fixed. Choose λ_n such that $B_n(\lambda_n) \subset U$, where U is as before the unit interval $0 \leqq t \leqq 1$, and let $C_n = U - B_n(\lambda_n)$. Since λ'_{n+1} exists such that $B_{n+1}(\lambda'_{n+1}) \subset U$, Lemma 1 guarantees existence of λ_{n+1} such that

$$(16) \qquad |C_n B_{n+1}(\lambda_{n+1})| \geqq \tfrac{1}{2} |C_n| |B_{n+1}|.$$

Let $C_{n+1} = U - [B_n(\lambda_n) + UB_{n+1}(\lambda_{n+1})]$. Again from Lemma 1, λ_{n+2} exists such that (16) holds when n is replaced by $n+1$. In this manner, we obtain a sequence $\lambda_n, \lambda_{n+1}, \cdots$ of real numbers and a sequence

$$C_{n+p} = U - [UB_n(\lambda_n) + UB_{n+1}(\lambda_{n+1}) + \cdots + UB_{n+p}(\lambda_{n+p})] \tag{17}$$

of sets such that, for each $p = 0, 1, 2, \cdots$,

$$\sum_{k=n}^{n+p} |C_k B_{k+1}(\lambda_{k+1})| \geqq \frac{1}{2} \sum_{k=n}^{n+p} |C_k| |B_{k+1}|. \tag{18}$$

Since the sets $C_k B_{k+1}(\lambda_{k+1})$ $(k = n, n+1, \cdots, n+p)$ are subsets of U and no two have a point in common, the left member of (18) is less than or equal to unity for each $p = 0, 1, 2, \cdots$. From this it follows that $|C_{n+p}| \to 0$ as $p \to \infty$; for $|C_{n+p}|$ is monotone decreasing as $p \to \infty$ and if $|C_{n+p}|$ is bounded from 0, then the fact that $\sum |B_n| = \infty$ would imply that the right member of (18) diverges to $+\infty$ as $p \to \infty$. The conclusion that $|C_{n+p}| \to 0$ as $p \to \infty$ implies by (17) that

$$\lim_{p \to \infty} |UB_n(\lambda_n) + UB_{n+1}(\lambda_{n+1}) + \cdots + UB_{n+p}(\lambda_{n+p})| = 1. \tag{19}$$

Hence there exists a sequence $0 = n_1 < n_2 < \cdots$ of indices such that the set

$$D_k \equiv UB_{n_k+1}(\lambda_{n_k+1}) + \cdots + UB_{n_{k+1}}(\lambda_{n_{k+1}}) \tag{20}$$

has measure $|D_k| > 1 - 2^{-k-1}$ for each $k = 1, 2, \cdots$. Put $P_k = D_k D_{k+1} \cdots$ and $P = P_1 + P_2 + \cdots$. The fact that $D_k \subset U$ and $|D_k| > 1 - 2^{-k-1}$ for each $k = 1, 2, \cdots$ implies that $P_k \subset U$ and $|P_k| \geqq 1 - 2^{-k}$, and consequently $P \subset U$ and $|P| = 1$. If $t \in P$, then $t \in P_k$ for some k so that $t \in D_k$ for all sufficiently great k and $t \in B_n(\lambda_n)$ for an infinite set of n, and hence also $t \in A_n(\lambda_n)$ for an infinite set of n.

If the sequence of sets A_n is arranged in a double sequence $A_{p,q}$ $(p = 0, \pm 1, \cdots; q = 1, 2, \cdots)$ in such a way that

$$\sum_{q=1}^{\infty} |A_{p,q}| = \infty, \quad p = 0, \pm 1, \pm 2, \cdots, \tag{21}$$

it results from what we have already proved that for each fixed p there is a sequence $\lambda_{p,1}, \lambda_{p,2}, \cdots$ such that each point of a subset of $I_p \equiv E_t\{p \leqq t \leqq p+1\}$ of measure unity is contained in an infinite number of the sets $A_{p,1}(\lambda_{p,1}), A_{p,2}(\lambda_{p,2}), \cdots$. Then each point of $-\infty < t < \infty$ with the exception of a null set lies in an infinite number of sets of the double sequence $A_{p,q}(\lambda_{p,q})$ which can be arranged in the simple sequence $A_n(\lambda_n)$, and proof of Lemma 2 is complete.

The hypothesis of Lemma 2 is equivalent to the following: A_n is a sequence of measurable sets such that

$$\sum_{n=1}^{\infty} \underset{-\infty<h<\infty}{\text{l.u.b.}} \left| E_t\{h \leqq t \leqq h+1; t \in A_n\} \right| = \infty. \tag{22}$$

That the hypothesis (22) cannot be relaxed is a consequence of the following result which we give without proof. If $A_1, A_2, \cdots$ is a sequence of measurable sets, and a real sequence $\lambda_1, \lambda_2, \cdots$ and a set C of positive measure exist such that each point of C lies in an infinite number of the sets $A_n(\lambda_n)$, then (22) holds.

That the conclusion of Lemma 2 must provide for an exceptional null set becomes clear when one observes that if the sets A_n are each nondense then, however $\lambda_1, \lambda_2, \cdots$ are determined, the set $\sum A_n(\lambda_n)$ must be of the first category and hence there must be a set of the second category whose points are in *none* of the sets $A_n(\lambda_n)$.

4. **Proof of Theorem** 2. To prove Theorem 2, let $x_n(t)$ be a sequence of measurable functions for which (2) fails for some $\delta>0$. Then $\delta>0$ and a sequence $h_1, h_2, \cdots$ exist such that

$$\sum_{n=1}^{\infty} E_t\{h_n \leqq t \leqq h_n+1; \left| x_n(t) \right| \geqq \delta\} = \infty. \tag{23}$$

Let $A_n = E_t\{\left| x_n(t) \right| \geqq \delta\}$. Then by Lemma 2 there exist a sequence $\lambda_1, \lambda_2, \cdots$ and a set C whose complement is a null set such that each t in C lies in an infinite number of the sets $A_n(\lambda_n)$. Hence if $t \in C$, then $t-\lambda_n \in A_n$ for an infinite set of n so that $\left| x_n(t-\lambda_n) \right| \geqq \delta$ for an infinite set of n. This contradicts the hypothesis of Theorem 2 and completes the proof.

CORNELL UNIVERSITY

A NEW FORMULA FOR THE BERNOULLI NUMBERS[1]

H. L. GARABEDIAN

It is the purpose of this note to exhibit a unique method for deriving what appears to be a new formula for the Bernoulli numbers. We obtain the formula

$$(1)\qquad B_{k+1}=\frac{(-1)^{k+1}(k+1)}{2^{k+1}-1}\sum_{i=0}^{k}\frac{\Delta^i a_0}{2^{i+1}},$$

$$a_n=(1+n)^k,\ k=0,1,2,\cdots,$$

or

$$B_{k+1}=\frac{(-1)^{k+1}(k+1)}{2^{k+1}-1}\left[\frac{1}{2}+\frac{1}{2^2}(1-2^k)+\frac{1}{2^3}(1-2\cdot 2^k+3^k)+\cdots\right.$$
$$\left.+\frac{1}{2^{k+1}}\{1-C_{k,1}2^k+C_{k,2}3^k-+\cdots+(-1)^kC_{k,k}(k+1)^k\}\right],$$
$$k=0,1,2,\cdots.$$

Some of the Bernoulli numbers computed from (1) are $B_1=-1/2$, $B_2=1/6$, $B_3=0$, $B_4=-1/30$, $B_5=0$, $B_6=1/42$, $B_7=0$, $B_8=-1/30$, $B_9=0$, $B_{10}=5/66$. Evidently, the known formulas[2] which give the Bernoulli numbers in explicit form, in comparison with formula (1), yield only the numerical values of these numbers with even indices. While none of the formulas under discussion could possibly serve a useful purpose in computing unknown numbers with high indices, formula (1) appears to be the simplest in form and the one best adapted to computation of numbers with low indices.

Our method of obtaining (1) consists in summing the divergent series

$$(2)\qquad \sum_{n=0}^{\infty}(-1)^n(n+1)^k=1-2^k+3^k-+\cdots,\quad k=0,1,2,\cdots,$$

using two consistent methods of summation, one of which assigns a value to (2) which involves the Bernoulli numbers. Equating the two values thus obtained we get the desired formula.

First, we sum the series (2) by the method of Abel. To this end we have to make some preliminary observations. The Bernoulli numbers may be defined by the relation[3]

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[1] Presented to the Society, December 29, 1939.

[2] See, for example, Niels Nielsen, *Traité Élémentaire des Nombres de Bernoulli*, 1923.

$$(3)\qquad \frac{x}{e^x - 1} = B_0 + \frac{B_1}{1!}x + \frac{B_2}{2!}x^2 + \cdots ,$$

which is valid at least for small values of x. Now, consider the series

$$f(t) = e^{-t} - e^{-2t} + e^{-3t} - + \cdots ,$$

which converges for $e^{-t} < 1$ and hence for $t > 0$. Moreover, we can write

$$(4)\qquad \begin{aligned} f(t) &= \frac{e^{-t}}{1 + e^{-t}} = \frac{1}{e^t + 1} = \frac{e^t + 1 - 2}{e^{2t} - 1} \\ &= \frac{1}{e^t - 1} - \frac{2}{e^{2t} - 1} = \frac{1}{t} \cdot \frac{t}{e^t - 1} - \frac{1}{t} \frac{2t}{e^{2t} - 1} . \end{aligned}$$

We may now expand the last expression in (4) using (3). We obtain for small values of $t > 0$ the relation

$$\begin{aligned} f(t) = \frac{1}{t}&\left[B_0 + \frac{B_1}{1!}t + \frac{B_2}{2!}t^2 + \cdots \right] \\ &- \frac{1}{t}\left[B_0 + \frac{2B_1}{1!}t + \frac{2^2B_2}{2!}t^2 + \cdots \right], \end{aligned}$$

or

$$\sum_{n=0}^{\infty} (-1)^n e^{-(n+1)t} = -\sum_{n=0}^{\infty} \frac{2^{n+1} - 1}{(n+1)!} B_{n+1} t^n .$$

Differentiating k times with respect to t, we have

$$\begin{aligned} &\sum_{n=0}^{\infty} (-1)^n (n+1)^k e^{-(n+1)t} \\ &\qquad = (-1)^{k+1} \sum_{n=k}^{\infty} \frac{2^{n+1} - 1}{(n+1)!} B_{n+1} \cdot n(n-1) \cdots (n-k+1) t^{n-k} . \end{aligned}$$

Now, allowing t to approach zero through positive values, we obtain

$$(5)\qquad \lim_{t \to 0+} \sum_{n=0}^{\infty} (-1)^n (n+1)^k e^{-(n+1)t} = (-1)^{k+1} \frac{2^{k+1} - 1}{k+1} B_{k+1} .$$

Putting $x = e^{-t}$ in the series on the left in (5), we have a power series with radius 1. Thus, the statement (5) implies that the series (2) is summable by the method of Abel (or is summable A) to the value given in (5).[4]

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[3] Euler, *Institutiones Calculi Differentialis* . . . , vol. 2, 1755, §122.

Next, we sum the series (2) by the Cesàro method. We need the following theorem.[5]

THEOREM. *The series* $\sum_{n=0}^{\infty}(-1)^n a_n$ *is exactly summable* (C, ν), $(\nu = 1, 2, 3, \cdots)$, *to the value*

$$\sum_{n=0}^{\nu-1} \frac{\Delta^n a_0}{2^{n+1}}$$

provided that

$$\Delta^i a_0 = 0, \qquad \Delta^{\nu-1} a_0 \neq 0, \qquad i \geqq \nu \geqq 1.$$

Since the general term in the series (2) is a polynomial of degree k, its differences of order higher than k all vanish. Thus, due to our theorem, the series (2) is summable $(C, k+1)$ to the value

$$\sum_{n=0}^{k} \frac{\Delta^n a_0}{2^{n+1}}, \qquad a_n = (n+1)^k,\ k = 0, 1, 2, \cdots . \tag{6}$$

Now, it is well known that Abel summability includes Cesàro summability of positive integral orders. Then, the series (2) must be summable A to the value (6). Equating (5) and (6) we get the desired formula:

$$B_{k+1} = \frac{(-1)^{k+1}(k+1)}{2^{k+1}-1} \sum_{i=0}^{k} \frac{\Delta^i a_0}{2^{i+1}}, \qquad a_n = (n+1)^k,\ k = 0, 1, 2, \cdots .$$

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[4] K. Knopp, *Theorie und Anwendung der unendlichen Reihen*, 1922, 2d edition, 1924, p. 508.

[5] H. L. Garabedian, this Bulletin, vol. 45 (1939), pp. 592–596.

ON A PROBLEM CONCERNING PROBABILITY AND ITS CONNECTION WITH THE THEORY OF DIFFUSION[1]

M. KAC

Introduction. The present note gives a certain probabilistic approach to the problem of diffusion. The main result is that the classical solution of the differential equation of diffusion is an asymptotic formula for the statistical problem under consideration.

The method which will be used is essentially that of Steinhaus and the present author which they applied to a similar but simpler problem of P. and T. Ehrenfest.[2]

1. **The problem.** Given an infinite sequence of boxes enumerated as follows

$$\cdots, -3, -2, -1, 0, 1, 2, 3, \cdots$$

and N numbered balls which are distributed in a certain way in the boxes, one takes at random one of the numbers $-N, \cdots, -1, 1, \cdots, N$ which are suppose to be equiprobable and if the number k is drawn one moves the ball number $|k|$ from its original box to the nearest to the right or to the nearest to the left according as sign k was 1 or -1. One repeats this process n times and one asks what is the "probable value" (mathematical expectation) of the "concentration" of balls in the box number s, say. By "concentration" one simply understands the ratio of the number of balls in a certain box and N. It is, of course, understood that the successive drawings are independent in the statistical sense of this word.

2. **Reduction of the problem by means of the "function of choice."** We divide the interval $(0, 1)$ into N equal parts and we define a function on $(0, 1)$ by placing $f(x) = s$ for $l-1/N < x \leqq l/N$ if the ball number l is originally in box s. This function represents the initial state of the schema. The joint length of those intervals in which $f(x) = s$ is obviously the initial "concentration" of balls in the box number s.

Let now $\omega(x)$ (the "function of choice") be 1 for $0 < x \leqq 1/N$ and 0 for $1/N < x \leqq 1$ and let furthermore $\omega(x+1) = \omega(x)$. Then $f(x) \pm \omega(x-(p-1)/N)$ represents obviously the state of the schema after moving the ball number p from its box to the nearest to the right $(+)$ or to the nearest to the left $(-)$.

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[1] Presented to the Society, December 29, 1939.

[2] H. Steinhaus, *La Théorie et les Applications des Fonctions Indépendantes*, Actualités Scientifiques et Industrielles, Paris, 1938.

Let r be a "random variable" which takes only the values -1, $-(N-1)/N, \cdots, -1/N, 1/N, \cdots, (N-1)/N, 1$ and every one of them with the probability $1/2N$. Then $f(x)+\text{sign } r\cdot\omega(x-|r|+1/N)$ represents the "random state" after the first drawing. If $r_1, r_2, \cdots, r_n$ are independent "random variables" having the same distribution as r, then

$$\Sigma_n(x, r) = f(x) + \text{sign } r_1\omega(x - |r_1| + 1/N) + \cdots \\ + \text{sign } r_n\omega(x - |r_n| + 1/N)$$

represents the "random state" after n independent drawings. Thus the problem is reduced to evaluating the "probable value" of the joint length of those intervals in which $\Sigma_n(x, r) = s$.

3. **Evaluation by means of a "discontinuity factor."** The remaining part of the solution is now of a purely technical nature. It is clear that

$$\frac{1}{2\pi}\int_0^{2\pi} \exp\,[i\xi(\Sigma_n(x, r) - s)]d\xi$$

is equal to 1 or to 0 according as $\Sigma = s$ or $\Sigma \neq s$, and therefore the joint length of those intervals in which $\Sigma = s$ is

$$L_n(r) = \frac{1}{2\pi}\int_0^1\int_0^{2\pi} \exp\,[i\xi(\Sigma_n(x, r) - s)]d\xi dx.$$

Thus the "probable value" $P_{n,N}(s)$ of $L_n(r)$ is equal to

$$\frac{1}{2\pi}\int_0^1\int_0^{2\pi} \text{prob. val.}\,\{\exp\,[i\xi(\Sigma_n(x, r) - s)]\}d\xi dx.$$

It is easily seen that

$$\begin{aligned} &\text{prob. val.}\,\{\exp\,[i\xi(\Sigma - s)]\} \\ &\quad = \exp\,[i\xi(f(x) - s)](\text{prob. val.} \exp\,[(i\xi \text{ sign } r)\omega(x - |r| + 1/N)])^n \\ &\quad = \exp\,[i\xi(f(x) - s)]\left(1 - \frac{1}{N} + \frac{1}{N}\cos\xi\right)^n \end{aligned}$$

and finally

$$(1) \quad P_{n,N}(s) = \frac{1}{2\pi}\int_0^1\int_0^{2\pi} \cos\xi(f(x) - s)\left(1 - \frac{1}{N} + \frac{1}{N}\cos\xi\right)^n d\xi dx.$$

4. **A particular case.** The formula becomes much simpler if one assumes that $f(x) \equiv 0$, that is, all the balls were initially in the box number 0. In this case one has

$$P_{n,N}(s) = \frac{1}{2\pi}\int_0^{2\pi} \cos s\xi\left(1 - \frac{1}{N} + \frac{1}{N}\cos\xi\right)^n d\xi.$$

Suppose that $N\to\infty$ and that $n/N\to z$; then

$$P_{n,N}(s) \to \frac{e^{-z}}{2\pi}\int_0^{2\pi} \cos s\xi e^{z\cos\xi} d\xi = e^{-z}J_s(iz),$$

where $J_s(z)$ denotes the sth Bessel function.

5. **The asymptotic formula.** Suppose now that the boxes are situated on the infinite line $(-\infty, +\infty)$ and that the distance between the consecutive ones is λ. Suppose furthermore that one makes a drawing every τ seconds so that after the time t one has performed $n = [t/\tau]$ drawings. We shall be interested in evaluating the "probable value" of the "concentration" of balls on the interval (l_1, l_2) after the time t. The answer is given by the probable value of the joint length of those intervals in which $l_1 < \lambda\Sigma_n(x, r) < l_2$ and can be evaluated by means of the formula (1). However, it will be more convenient for our purpose to evaluate the "probable value" under consideration by means of the Dirichlet discontinuity factor. It is well known that

$$\frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{\sin\alpha\xi}{\xi} e^{i\gamma\xi}d\xi$$

is equal to 1, $\frac{1}{2}$ or 0 according as $|\gamma| < \alpha$, $|\gamma| = \alpha$ or $|\gamma| > \alpha$. Thus, if neither l_1 nor l_2 is a multiple of λ, the probable value one looks for is given by the following formula:

$$\begin{aligned} P_N(l_1, l_2, t) = \frac{1}{\pi}\int_{-\infty}^{+\infty} & \frac{\sin\frac{1}{2}(l_2 - l_1)\xi}{\xi} \\ & \cdot\exp\left(-i\xi\frac{l_1 + l_2}{2}\right)\left(\int_0^1 \exp(i\xi\lambda f(x))dx\right) \\ & \cdot\left(1 - \frac{1}{N} + \frac{1}{N}\cos\lambda\xi\right)^{[t/\tau]} d\xi. \end{aligned}$$

If now $\lambda^2/N\tau\to 2k$ as $N\to\infty$, and $\lambda\to 0$, one has

$$\left(1 - \frac{1}{N} + \frac{1}{N}\cos\lambda\xi\right)^{[t/\tau]} \to \exp(-kt\xi^2)$$

uniformly in every finite interval and, assuming in addition that the distribution function of $\lambda f(x)$ tends to a distribution function $\sigma(u)$, one readily deduces from the continuity theorem of Fourier-Stieltjes transforms[3] that

$$P_N(l_1, l_2, t) \to \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{\sin \frac{1}{2}(l_2 - l_1)\xi}{\xi}$$

$$\cdot \exp\left(- i\xi \frac{l_1 + l_2}{2}\right)\left(\int_{-\infty}^{+\infty} \exp (i\xi u) d\sigma(u)\right) \exp (- kt\xi^2) d\xi.$$

The integral on the right side can easily be evaluated and its value is

$$\frac{1}{2(\pi kt)^{1/2}}\int_{l_1}^{l_2}\int_{-\infty}^{+\infty} \exp \left\{- \frac{(u - l)^2}{4kt}\right\} d\sigma(u) dt.$$

In the limiting case the concentration is given by the formula

$$\frac{1}{2(\pi kt)^{1/2}}\int_{-\infty}^{+\infty} \exp \left\{- \frac{(u - l)^2}{4kt}\right\} d\sigma(u),$$

which is the classical solution of the differential equation of diffusion in the case of an infinite cylinder. In our case the classical formula turned out to be an asymptotic formula for a simple probabilistic process belonging to the category of the so-called "random walks."

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[3] Cf. E. K. Haviland, American Journal of Mathematics, vol. 56 (1934), pp. 625–658.

THE SOLUTION OF HARMONIC EQUATIONS BY MEANS OF DEFINITE INTEGRALS

H. BATEMAN

Darboux devotes a chapter of his famous *Théorie des Surfaces* to the study of partial differential equations related to some equations of mathematical physics that have sets of simple solutions of type $X(x)Y(y)$. Such equations are called harmonic equations and are of great interest on account of the various ways in which they can be reduced to the harmonic form and solved by definite integrals. The reduction problem has already been more or less solved, but it cannot be said that the definite integral problem has been fully discussed.

It is known, for instance, that with certain special forms of the functions $a(\theta)$, $b(\theta)$, $h(\theta)$, $f(\theta)$ an integral of type

$$z = \int_0^\pi F(ax^2 + 2hxy + by^2)f(\theta)d\theta \tag{1}$$

may satisfy the harmonic equation

$$r + t + up/x + vq/y = 0, \qquad u = 2k + 1, v = 2m + 1, \tag{2}$$

where u and v are constants and p, q, r, s, t denote the partial derivatives of z of the first two orders. For general values of u and v this equation is of interest in the study of solutions of the wave equation and Laplace's equation in four variables. When u and v are integers, the equation arises in the study of symmetrical solutions of Laplace's equation in N variables, in the study of the stream function of hydrodynamics and of various functions which occur in the theory of elasticity.

To find all the possible forms of $a(\theta)$, $b(\theta)$, $h(\theta)$, $f(\theta)$ when it is assumed that the integral can be differentiated in the usual way, we must discover how to satisfy the equation

$$\begin{aligned} 0 = \int_0^\pi f(\theta)d\theta\big[&4F''(w)\{(ax + by)^2 + (hx + by)^2\} \\ &+ 2F'(w)\{a + b + u(a + hy/x) + v(b + hx/y)\}\big] \end{aligned} \tag{3}$$

where $w = ax^2 + 2hxy + by^2$. As there are many different ways in which this might perhaps be done, it is not yet known whether all the useful types of integral have already been found.

If the limits were $-\pi$ and π, we could satisfy the equation by making the integrand the sum of an exact differential and an odd function of θ. When the limits are 0 and π, there is the possibility that part of the integral may be zero on account of an invariance in form but with sign changed when θ is replaced by $\pi-\theta$. There are other possibilities suggested by the theory of functional differential equations, but at present we shall consider only the case in which the equation is satisfied because the integrand is equal to $d[GF'(w)]$ where G depends only on x, y and θ in such a way that $GF'(w)$ has the same finite value at both limits. In this case the analysis depends only on the ordinary theory of exact differentials, and it is not necessary to enter into extensions of this idea and extensions of the theory of the adjoint differential equation. Such extensions are occasionally useful but are not hard to develop.

Using primes to denote derivatives of a, b, h, f, w and G with respect to θ, the equations to be satisfied are

$$\begin{aligned} Gw' &= 4f(A^2+B^2), \qquad A = ax+hy,\ B = hx+by, \\ xyG' &= 2f[(a+b)xy+uyA+vxB]. \end{aligned} \tag{4}$$

Eliminating G we obtain an equation which can be satisfied only if $uhb'=0$ and $vha'=0$. The three possibilities are (1) $u=0$, $a'=0$, (2) $v=0$, $b'=0$, (3) $h=0$.

In the first case a is constant and the equations become

$$yG' = 2f[(a+b)y+vB], \qquad Gy = 4f(A^2+B^2)/w_0 \tag{4'}$$

where $w_0=2h'x+b'y$. Eliminating yG we obtain the equation

$$\begin{aligned} &4w_0(h'yA+BB'+\tfrac{1}{2}f'A^2/f+\tfrac{1}{2}f'B^2/f) \\ &\qquad = 2w_0'(A^2+B_0^2)+w^2[(a+b)y+vB], \end{aligned} \tag{5}$$

which should hold for all values of x and y. Choosing the ratio of these quantities in such a way that $w_0=0$, we obtain the equation

$$(h''b'-b''h')[(ab'-2hh')^2+(hb'-2bh')^2] = 0. \tag{6}$$

The first factor vanishes when h'/b' is a constant. The second factor vanishes when $(a+ih)b'=2(h+ib)h'$ or

$$b = 2(a+ih)-a+c(a+ih)^2, \tag{7}$$

where c is an arbitrary constant.

Equating the coefficients of x^3 in (5), we get

$$h''/h'-f'/f = (2-v)hh'/(a^2+h^2).$$

Hence $fd\theta=dh(a^2+h^2)^{(v-1)/2}$, the constant of integration being put to unity.

An appropriate value of the constant c may be found by choosing the ratio of x/y in (5) so that $A^2+B^2=0$ but $w_0\neq 0$. The value $c=-1/a$ makes $w=A^2/a$ and the integral becomes the well known integral of Poisson[1] if we take $h=-i\cos\theta$.

If we try

$$z=\int_{-\pi}^{\pi} H(x-iy\cos\theta)f(\theta)d\theta, \tag{8}$$

the equation $r+t+vq/y=0$ will be satisfied if

$$0=[H'f\sin\theta]_{-\pi}^{\pi}-\int_{-\pi}^{\pi} H'[f'\sin\theta-(fv-1)\cos\theta]d\theta. \tag{9}$$

The integral may be made to vanish by choosing f so that $-f'\sin\theta+f(v-1)\cos\theta$ is an odd function of θ. In the case when this function is $\sin^{2n+1}\theta$, the equation for the determination of f is

$$f'(\theta)\sin\theta-(v-1)f(\theta)\cos\theta=\sin^{2n+1}\theta \tag{10}$$

and the appropriate solution is

$$f(\theta)=\sin^{v-1}\theta\int_0^\theta \sin^{2n-v+1}\theta\,d\theta. \tag{11}$$

If, for example, $v=1$ and $n=0$, we get $f(\theta)=\theta$ and the integral is zero. The device does not yield a new solution.

Passing now to the case $h=0$, the equations (4) become

$$G(a'x^2+b'y^2)=4(a^2x^2+b^2y^2)f,\ G'=[2(1+u)a+2(1+v)b]f. \tag{4''}$$

Elimination of G gives an equation

$$\begin{aligned}4(aa'x^2+bb'y^2)(a'x^2+b'y^2)-2(a^2x^2+b^2y^2)(a''x^2+b''y^2)\\+2(f'/f)(a^2x^2+b^2y^2)(a'x^2+b'y^2)\\=[(1+u)a+(1+v)b](a'x^2+b'y^2)^2,\end{aligned} \tag{12}$$

which must be satisfied for all values of x and y. Choosing the ratio of x and y so that $a'x^2+b'y^2=0$, we find that

$$(a''b'-a'b'')(a^2b'-b^2a')=0. \tag{13}$$

The first factor vanishes when b'/a' is constant, the second factor when $b'/b^2=a'/a^2$ or $b^{-1}=a^{-1}+\text{const}$. Taking the constant to be unity, we find that if $a=\tan^2\frac{1}{2}\theta$, $b=\sin^2\frac{1}{2}\theta$. Equating coefficients of x^4

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[1] Journal de l'École Polytechnique, vol. 12 (1821), pp. 215–248. See also R. Hoppe, Journal für die reine und angewandte Mathematik, vol. 58 (1861), pp. 369–373.

in (12), we find that

$$(14)\quad 4(a'/a) - 2(a''/a') + 2f'/f = (1 + u)(a'/a) + (1 + v)(ba'/a^2).$$

Now $ba'/a^2 = \cot \frac{1}{2}\theta$ and consequently $f = (\tan \frac{1}{2}\theta)^u(\sin \frac{1}{2}\theta)^{v-1}$. It may be verified that if $w = x^2 \tan^2 \frac{1}{2}\theta + y^2 \sin^2 \frac{1}{2}\theta = W^2$ and F is a suitable function, which may even be zero for $w > c^2$,

$$(15)\qquad z = \int_0^\pi F(w) \tan^u \tfrac{1}{2}\theta \sin^{v-1} \tfrac{1}{2}\theta \, d\theta$$

is indeed a solution of $r + t + up/x + vq/y = 0$ when $R(u+v) > 0$. A particular case of this result is that if $x > 0$, $R(m) > -\frac{5}{4}$, $R(k+m) > -1$,

$$(16)\quad \int_0^\pi W^{-k-m}J_{k+m}(W) \sin^{2m} \tfrac{1}{2}\theta \tan^{2k+1} \tfrac{1}{2}\theta \, d\theta = 2x^{-k}K_k(x)y^{-m}J_m(y).$$

This result may be verified by using Lommel's expansion

$$(17)\quad W^{-k-m}J_{k+m}(W) = \sum_{n=0}^{\infty} (-\tfrac{1}{2}y^2S^2)^n/n!(xT)^{-k-m-n}J_{k+m+n}(xT),$$

in which $S = \sin \frac{1}{2}\theta$, $T = \tan \frac{1}{2}\theta$. Now by a well known formula

$$(18)\quad \begin{aligned} \int_0^\pi J_{k+m+n}(xT)S^{2m+2n}T^{k+1-m-n}d\theta &= 2(\tfrac{1}{2}x)^{m+n}K_k(x)/\Gamma(m + n + 1), \\ -1 < R(k + m + n) &< R(2m + 2n + \tfrac{3}{2}). \end{aligned}$$

Hence, if $R(k+m) > -1$, $R(k-m) < \frac{3}{2}$, $R(m) > -\frac{5}{4}$, the series

$$\begin{aligned} \sum_{n=0}^{\infty} (-\tfrac{1}{2}y^2)^n/n! \int_0^\pi S^{2m+2n}T^{2k+1}(xT)^{-k-m-n}J_{k+m+n}(xT)d\theta \\ = 2x^{-k}K_k(x) \sum_{n=0}^{\infty} (-y^2)^n 2^{-m-2n}/n!\Gamma(m + n + 1) \\ = 2x^{-k}K_k(x)y^{-m}J_m(y). \end{aligned}$$

The formula (16) is obtained by changing the order of integration and summation.[2]

If in the integral (15) w is taken as the new independent variable, the integrand should satisfy the partial differential equation (2) for all values of the parameter w. If $r=(2wx^2+2x^2y^2-2y^2w+x^4+y^4+w^2)^{1/2}$, the result is that

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[2] This may be justified by forming estimates of the remainders in the two series. When $u = 0$, $v = 1$, the integral (15) is useful in potential theory and gives integral expressions for many symmetrical potentials. A list of these is being prepared. One such expression occurs in the paper of E. G. Gallop, *The distribution of electricity on the circular disc and spherical bowl*, Quarterly Journal of Mathematics, vol. 21 (1886), pp. 229–256.

$$z = r^{-1}x^{-2k}(r + w - x^2 - y^2)^{m+k}(r + w + x^2 - y^2)^{-m} \tag{19}$$

should be a solution of the partial differential equation. The solution of the partial differential equation (2) may be written in a more symmetrical form by putting

$$w = \frac{x^2}{a^2 + v} + \frac{y^2}{b^2 + v}, \qquad z = \int_C F(w)(a^2 + v)^{-k-1}(b + v)^{-m-1}dv.$$

The contour C may start at infinity and enclose either $v=-a^2$ or $v=-b^2$. It may also run from $-\infty$ to $+\infty$ along a path which avoids the points $v=-a^2$ and $v=-b^2$. The solution is now seen to be related to the integral for the product of two Bessel functions obtained by S. O. Rice, Quarterly Journal of Mathematics (Oxford series), vol. 6 (1935), pp. 52–64.

This theorem may be extended to any number of variables of type x and y. Thus if w is defined by the equation

$$w = \frac{x^2}{a^2 + v} + \frac{y^2}{b^2 + v} + \frac{z^2}{c^2 + v},$$

the integral

$$V = \int F(w)(a^2 + v)^{-\alpha-1}(b^2 + v)^{-\beta-1}(c^2 + v)^{-\gamma-1}dv$$

may be a solution of the partial differential equation

$$\begin{aligned} 0 = V_{xx} + V_{yy} + V_{zz} + (2\alpha + 1)x^{-1}V_x \\ + (2\beta + 1)y^{-1}V_y + (2\gamma + 1)z^{-1}V_z \end{aligned}$$

if the contour starts from ∞ and surrounds one of the points $-a^2$, $-b^2$, $-c^2$ and returns to ∞. The restrictions are that α, β, and γ must be such that the integral is convergent and the arbitrary func- F must be such that differentiations under the integral sign are permissible.

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SUR LE SYSTÈME DE SOUSLIN D'ENSEMBLES DANS L'ESPACE TRANSFINI

ISAIE MAXIMOFF

Soit Ω_i le premier nombre transfini que précèdent $\aleph_i$ nombres transfinis. La suite de nombres x_α, $(1 \leqq x_\alpha < \Omega_i)$:

$$(1) \qquad x = \{x_1, x_2, x_3, \cdots, x_\alpha, \cdots\}, \qquad 1 \leqq \alpha < \Omega_i,$$

sera nommée point de l'espace transfini d'ordre i. Nous le désignons par $I_x^{(i)}$.

Appelons *espace transfini à m dimensions* l'ensemble de tous les systèmes $[x^{(1)}, x^{(2)}, \cdots, x^{(m)}]$ où $x^{(s)}$, $(1 \leqq s \leqq m)$, est un point quelconque de $I_{x^{(s)}}^{(i)}$. Cet espace sera désigné par $I_{x^{(1)}x^{(2)}\ldots x^{(m)}}^{(i)}$ ou par $I_X^{(i)}$ où $X = [x^{(1)}x^{(2)} \cdots x^{(m)}]$.

Une suite d'ensembles

$$(2) \qquad E_1, E_2, E_3, \cdots, E_\alpha, \cdots, \qquad \alpha < \Omega_i,$$

de points de $I_X^{(i)}$ est appelée *convergente* (Ω_i) si pour chaque point X de cet espace il existe une valeur $\alpha = \alpha_0$, dépendant de X, telle que X appartient nécessairement à tous les E_α, $(\alpha \geqq \alpha_0)$, ou bien n'appartient à aucun des E_α, $(\alpha \geqq \alpha_0)$.

L'ensemble E de tous les points X qui appartiennent à tous les E_α, $(\alpha \geqq \alpha_0)$ (α_0 dépend de X), est appelé *limite* (Ω_i) de la suite convergente (2) et nous écrivons $E = \lim_{\alpha \to \Omega_i} E_\alpha$.

Soit

$$(3) \qquad E_1, E_2, E_3, \cdots, E_\alpha, \cdots, \qquad \alpha < \Omega_i,$$

une suite quelconque d'ensembles situés dans $I_X^{(i)}$. Nous dirons que l'ensemble $S = E_1 + E_2 + E_3 + \cdots + E_\alpha + \cdots$, $(\alpha < \Omega_i)$, est *somme* (Ω_i) *et l'ensemble* $P = E_1 \cdot E_2 \cdot E_3 \cdot \cdots \cdot E_\alpha \cdot \cdots$, $(\alpha < \Omega_i)$, est *partie commune* (Ω_i) des ensembles de la suite (3).

Il est évident que l'opération $\lim_{\alpha \to \Omega_i} E_\alpha$ se ramène encore aux sommes (Ω_i) et aux parties communes (Ω_i) par la formule suivante:

$$(4) \qquad \begin{aligned} \lim_{\alpha \to \Omega_i} E_\alpha = {} & (E_1 \cdot E_2 \cdot E_3 \cdot \cdots \cdot E_\alpha \cdot \cdots) \\ & + (E_2 \cdot E_3 \cdot E_4 \cdot \cdots \cdot E_\alpha \cdot \cdots) + \cdots \\ & + (E_\beta \cdot E_{\beta+1} \cdot E_{\beta+2} \cdot \cdots) + \cdots \end{aligned}$$

où $\alpha < \Omega_i$, $\beta < \Omega_i$.

Système initial. Prenons un système $\mathfrak{M}$ d'ensembles quelconques M de l'espace $I_X^{(i)}$. Nous dirons que $\mathfrak{M}$ est le système initial.

Tout ensemble E de $I_X^{(i)}$ qu'on obtient, à partir des ensembles du système initial au moyen des deux opérations: somme (Ω_i) et partie commune (Ω_i) répétées au plus une infinité $\aleph_i$ de fois, est dit ensemble du système B (de Borel) ou ensemble de Borel.

Soit $\nu=(n_1, n_2, n_3, \cdots, n_\kappa, \cdots)$, $\kappa<\Omega_0$, une suite quelconque de nombres n_κ tels que $1\leqq n_\kappa<\Omega_i$. Désignons par $\{\nu\}$ l'ensemble de toutes les suites ν.

Soit $F(n_1)$, $F(n_1n_2)$, $F(n_1n_2n_3)$, $\cdots$, $F(n_1n_2\cdots n_\kappa)$, $\cdots$ une suite d'ensembles quelconques de $I_X^{(i)}$. Nous introduisons la désignation

$$E(\nu) = F(n_1)\cdot F(n_1n_2)\cdot F(n_1n_2n_3)\cdot \cdots \cdot F(n_1n_2\cdots n_\kappa)\cdot \cdots .$$

Si A est la somme de tous les ensembles $E(\nu)$, ν parcourant tous les éléments de $\{\nu\}$, convenons écrire $A=\mathfrak{S}_\nu E(\nu)$ ou

$$(5)\quad A = \underset{\nu}{\mathfrak{S}} F(n_1)\cdot F(n_1n_2)\cdot F(n_1n_2n_3)\cdot \cdots \cdot F(n_1n_2\cdots n_\kappa)\cdot \cdots , \quad \kappa < \Omega_0.$$

Nous dirons que l'ensemble A appartient au système de Souslin si tous les ensembles $F(n_1)$, $F(n_1n_2)$, $F(n_1n_2n_3)$, $\cdots$, $F(n_1n_2\cdots n_\kappa)$, $\cdots$ sont contenus dans le système B.

Sans restreindre la généralité de définition on peut supposer ici que

$$(6)\qquad F(n_1)\supset F(n_1n_2)\supset F(n_1n_2n_3)\supset \cdots \supset F(n_1n_2\cdots n_\kappa)\supset \cdots .$$

Il est évident que tout ensemble du système B appartient au système de Souslin.

Par le raisonnement parfaitement analogue au raisonnement classique sur le système de Souslin dans l'espace ordinaire (voir Dr. F. Hausdorff, *Mengenlehre*, Berlin, 1927, pp. 92–93) on peut démontrer le théorème suivant.

THÉORÈME 1. *Si les ensembles* $F(n_1, n_2, \cdots, n_\kappa)$, $(1\leqq\kappa<\Omega_0)$, *dans la formule* (5) *appartiennent au système de Souslin, l'ensemble A l'est aussi.*

En effet, soient $F(n_1n_2\cdots n_\kappa)=\mathfrak{S}_{\alpha^{(\kappa)}}H_{\alpha^{(\kappa)}}$ où

$$H_{\alpha^{(\kappa)}} = M_{\alpha_1^{(\kappa)}}^{n_1n_2\cdots n_\kappa}\cdot M_{\alpha_2^{(\kappa)}}^{n_1n_2\cdots n_\kappa}\cdot M_{\alpha_3^{(\kappa)}}^{n_1n_2\cdots n_\kappa}\cdot \cdots$$

et les ensembles $M_{\alpha_r^{(\kappa)}}^{n_1n_2\cdots n_\kappa}$ appartiennent au système de Borel. Maintenant nous rangeons les éléments de la matrice

$$A = \left\|\begin{array}{llll} n_1 & n_2 & n_3 & \cdots \\ \alpha_1^1 & \alpha_2^1 & \alpha_3^1 & \cdots \\ \alpha_1^2 & \alpha_2^2 & \alpha_3^2 & \cdots \\ \cdot & \cdot & \cdot & \cdots \end{array}\right\|$$

en suite simple: $m_1, m_2, m_3, \cdots, m_s, \cdots, (s < \Omega_0)$, de telle manière que la matrice A soit égale à la matrice

$$B = \left\| \begin{matrix} m_1 & m_3 & m_5 & \cdots \\ m_2 & m_6 & m_{10} & \cdots \\ m_4 & m_{12} & m_{20} & \cdots \\ m_8 & m_{24} & m_{40} & \cdots \\ \cdot & \cdot & \cdot & \cdots \end{matrix} \right\| .$$

Après cela nous introduisons la désignation

$$\text{(a)} \qquad M[2^r(2\kappa + 1)] = M^{m_1 m_3 \cdots m_{2r+1}}_{m_{2^{r+1}} m_{2^{r+1}3} m_{2^{r+1}5} \cdots m_{2^{r+1}(2\kappa+1)}}, \quad r = 0, 1, 2, 3, \cdots, \kappa = 0, 1, 2, 3, \cdots .$$

De cette formule il suit que à tout ensemble $M[s]$ correspond un et un seul système $(m_1, m_2, \cdots, m_{2s})$ et inversement à tout système $(m_1, m_2, \cdots, m_{2s})$ correspond un et un seul ensemble $M[s]$. C'est pour cette raison on peut introduire la désignation nouvelle

$$\text{(b)} \qquad M_{m_1 m_2 \cdots m_{2s}} = M[s].$$

Maintenant nous introduisons l'ensemble $M_{m_1 m_2 \cdots m_{2s} m_{2s+1}}$ d'après

$$\text{(c)} \qquad M_{m_1 m_2 \cdots m_{2s+1}} = M_{m_1 m_2 \cdots m_{2s}}.$$

Ceci étant, on peut écrire la formule

$$A = \mathfrak{S}_{\nu} F(n_1)F(n_1 n_2) \cdots F(n_1 n_2 \cdots n_\kappa) \cdots = \mathfrak{S}_{\mu} M_{m_1} M_{m_1 m_2} M_{m_1 m_2 m_3} \cdots ,$$

où $\mu = (m_1 m_2 m_3 \cdots)$. Comme $M_{m_1 m_2 \cdots m_\kappa}$ appartient au système de Borel, l'ensemble A appartient au système de Souslin, ce qu'il fallait démontrer.

Voici les deux théorèmes qui sont conséquences immédiates du Théorème 1.

THÉORÈME 2. *Si les ensembles E_α, $(1 \leqq \alpha < \Omega_i)$, appartiennent au système de Souslin, la somme (Ω_i): $E_1 + E_2 + E_3 + \cdots + E_\alpha + \cdots$ l'est aussi.*

En effet, si nous posons $F(n_1 n_2 \cdots n_\kappa) = E_{n_1}$, nous obtenons:

$$\mathfrak{S}_{\nu} F(n_1) \cdot F(n_1 n_2) \cdot F(n_1 n_2 n_3) \cdot \cdots \cdot F(n_1 n_2 n_3 \cdots n_\kappa) \cdot \cdots = E_1 + E_2 + E_3 + \cdots + E_{n_1} + \cdots, \quad n_1 < \Omega_i,\ \kappa < \Omega_0.$$

THÉORÈME 3. *Si les ensembles E_n, $(1 \leqq n < \Omega_0)$, appartiennent au système de Souslin, l'ensemble $E_1 \cdot E_2 \cdot E_3 \cdot \cdots \cdot E_n \cdot \cdots$ l'est aussi.*

En effet, si nous posons $F(n_1 n_2 \cdots n_\kappa) = E_\kappa$, nous obtenons:

$$\underset{,}{\mathfrak{S}} F(n_1) \cdot F(n_1 n_2) \cdot \cdots \cdot F(n_1 n_2 \cdots n_\kappa) \cdot \cdots = E_1 \cdot E_2 \cdot E_3 \cdot \cdots \cdot E_\kappa \cdot \cdots , \qquad \kappa < \Omega_0.$$

Si un ensemble A appartient au système de Souslin, nous l'appelons ensemble de Souslin.

Soit A un ensemble de Souslin quelconque. Dans ce cas il y a lieu la formule (5). Nous introduisons la désignation $r = (n_1, n_2, \cdots, n_\kappa)$, $(\kappa < \Omega_0)$, et nous dirons que le système r *précède* le système $(r, n) = (n_1, n_2, \cdots, n_\kappa, n)$ ou le système (r, n) *suit* le système (r).

Soit R_0 un ensemble quelconque de systèmes $(n_1, n_2, \cdots, n_\kappa)$ où $1 \leqq n_r < \Omega_i; r = 1, 2, 3, \cdots, \kappa$; $\kappa < \Omega_0$. Si $r = (n_1 n_2 \cdots n_\kappa)$ appartient à R_0 et s'il n'y a aucun élément (r, n) qui suit l'élément r et appartient à R_0, nous dirons que r est un élément *final* de R_0.

Étant donné un ensemble quelconque R_0 d'éléments r, nous formons la suite des ensembles R_α provenant de l'ensemble donné R_0:

$$(7) \qquad R_0, R_1, R_2, \cdots, R_\alpha, \cdots, R_\gamma, \cdots$$

de manière suivante. On obtient R_γ, si γ est de première espèce, $\gamma = \gamma^* + 1$, en enlevant de l'ensemble précédent déjà défini R_{γ^*} tout élément final, et, si γ est de seconde espèce, on obtient R_γ en faisant la partie commune des ensembles précédents $R_{\gamma'}$, $(\gamma' < \gamma)$, déjà définis.

D'après ce qui précède l'inégalité $\gamma < \gamma'$ entraîne $R_\gamma \supset R_{\gamma'}$. Désignons par Δ_γ l'ensemble des éléments de $R_{\gamma+1}$ sans faire partie de R_γ. On voit que la condition nécessaire et suffisante pour que Δ_γ soit nul est que $R_{\gamma+1} = R_\gamma$. Chacun des ensembles

$$(8) \qquad \Delta_0, \Delta_1, \Delta_2, \Delta_3, \cdots, \Delta_\gamma, \cdots$$

à la puissance égale au plus à $\aleph_i$, et deux d'entre eux n'ont aucun élément commun. Il y a donc au plus une infinité $\aleph_i$ d'ensembles (8) non nuls; les indices γ de ceux des ensembles (8) qui ne sont pas nuls forment ainsi un ensemble de la puissance égale au plus à $\aleph_i$, par suite, il y a un nombre ρ, $(\rho < \Omega_{i+1})$, qui a la propriété suivante: si $\rho' \geqq \rho$, on a $\Delta_{\rho'} = 0$ d'où résulte

$$(9) \qquad R_\rho = R_{\rho+1} = R_{\rho+2} = \cdots = R_{\rho'} = \cdots , \qquad \rho' > \rho.$$

Désignons par $R_{\Omega_{i+1}}$ la partie commune à tous les ensembles (7). On voit bien que les ensembles (7) sont, pour les valeurs suffisamment grandes de l'indice, identiques à $R_{\Omega_{i+1}}$.

Il y a un nombre bien déterminé qui est le plus petit de tels qu'on ait $R_\eta = R_{\Omega_{i+1}}$. Nous le désignons par η et dirons que η est l'indice de R_0: pour $\xi < \eta$ on a $R_\xi \supset R_{\xi+1}$ et pour $\xi \geqq \eta$ on a $R_\xi = R_{\Omega_i}$.

Désignons par $K(R_0)$ l'ensemble $R = R_\eta = R_{\eta+1} = R_{\eta+2} = \cdots$ et par $R_0(X)$ l'ensemble de tous les systèmes $r = (n_1 n_2 \cdots n_\kappa)$, $\kappa < \Omega_0$, tels que X appartient à l'ensemble $F(r) = F(n_1 n_2 \cdots n_\kappa)$. Soit η le plus petit nombre tel que $KR_0(X) = R_\eta(X) = R_{\eta+1}(X) = \cdots$. Dans ce cas η est dit *l'indice* du point X, et nous écrivons l'égalité $\eta = \eta(X)$.

D'ailleurs, $R(X) = R_\eta(X)$ est aussi appelé indice du point X. Désignons par A_η (B_η) l'ensemble de tous les points X appartenant à l'ensemble A (B) où $B = CA$ et ayant l'indice η. Alors nous avons les formules

$$A = A_0 + A_1 + A_2 + \cdots + A_\gamma + \cdots,$$
$$B = B_0 + B_1 + B_2 + \cdots + B_\gamma + \cdots, \quad 0 \leqq \gamma < \Omega_{i+1}.$$

Il est évident que les ensembles

$$A_0, A_1, A_2, A_3, \cdots, A_\gamma, \cdots, \qquad B_0, B_1, B_2, B_3, \cdots, B_\gamma, \cdots,$$
$$0 \leqq \gamma < \Omega_{i+1},$$

n'ont aucun point commun deux à deux. Nous les appelons *constituantes* respectivement de A et de B.

Maintenant nous allons démontrer notre théorème fondamental suivant.

THÉORÈME 4. *Si A est un ensemble de Souslin quelconque, alors nous avons les développements*

$$A = A_0 + A_1 + A_2 + \cdots + A_\alpha + \cdots,$$
$$B = B_0 + B_1 + B_2 + \cdots + B_\alpha + \cdots, \qquad \alpha < \Omega_{i+1},$$

de l'ensemble A et de son complémentaire $B = CA$ en une infinité $\aleph_{i+1}$ d'ensembles constituantes A_α, B_α, $(0 \leqq \alpha < \Omega_{i+1})$, chacun desquels appartient au système B.

Démonstration de ce théorème consiste de quatre parties.

La première partie. Tout d'abord montrons que l'ensemble $R_\xi(X)$ défini précédemment contient tous les éléments précédents, c'est à dire,

$$(10) \qquad (r, n) \in R_\xi(X) \rightarrow (r) \in R_\xi(X),$$

où $\rightarrow$ signifie l'implication logique.

Cette proposition est vraie pour $\xi = 0$.

En effet, si (r, n) appartient à $R_0(X)$, on a $X \in F(r, n)$, par suite, $X \in F(r)$ d'où il suit $r \in R_0(X)$. Maintenant supposons que cette proposition est vraie pour $\xi=\alpha$ et démontrons la pour $\xi=\alpha+1$; d'après l'hypothèse faite $r \in R_\alpha(X)$. Donc, $R_\alpha(X)$ contient (r) et (r, n), par suite, (r) n'est pas un élément final de $R_\alpha(X)$; il en suit que $(r) \in R_{\alpha+1}(X)$.

Soit η un nombre quelconque de seconde espèce, $\eta<\Omega_{i+1}$, et soit $(r, n) \in R_\eta(X)$; alors (r, n) est contenu dans chaque $R_\xi(X)$ où $\xi<\eta$.

Si $(r, n) \in R_\xi(X)$, on a $(r) \in R_\xi(X)$ pour tout $\xi<\eta$, par suite, $r \in R_\eta(X)$.

La deuxième partie. Désignons par $F_\xi(r)$ l'ensemble des points X tels que $r \in R_\xi(X)$. Cela veut dire que la relation $r \in R_\xi(X)$ est équivalente à la relation $X \in F_\xi(r)$. Si $X \in F_\xi(r, n)$ on a $(r, n) \in R_\xi(X)$, par suite, $r \in R_\xi(X)$ d'où il suit $X \in F_\xi(r)$. Donc,

$$F_\xi(r, n) \subset F_\xi(r). \tag{11}$$

Il est évident que

$$F_0(r) = F(r), \quad F_{\xi+1}(r) = \mathfrak{S}_n F_\xi(r, n), \quad F_\eta(r) = \mathfrak{D}_{\xi<\eta} F_\xi(r), \quad n < \Omega_i, \tag{12}$$

où $\mathfrak{S}_n$ est le signe de la sommation s'étendant sur tous $n<\Omega_i$ et $\mathfrak{D}_{\xi<\eta}$ est le signe de la partie commune s'étendant sur tous $\xi<\eta$.

La première des égalités (12) est une consequence immédiate de la définition de $R_0(X)$. En effet, la relation $X=F_0(r)$ est équivalente à la relation $r \in R_0(X)$. Mais cette dernière est équivalente à la relation $X \in F(r)$. La deuxième des égalités (12) s'obtient par le raisonnement suivant: la relation $X \in F_{\xi+1}(r)$ est équivalente à la relation $r \in R_{\xi+1}(X)$. Mais dans ce cas r n'est pas élément final de $R_\xi(X)$, par suite, $(r, n) \in R_\xi(X)$ d'où $X \in F_\xi(r, n)$.

La troisième des égalités s'obtient par le raisonnement analogue: la relation $X \in F_\eta(r)$ est équivalente à la relation $r \in R_\eta(X) = \mathfrak{D}_{\xi<\eta} R_\xi(X)$. Mais la dernière est équivalente à la suite des relations:

$$r \in R_0(X),\ r \in R_1(X), \cdots, r \in R_\xi(X), \cdots, \qquad \xi < \eta,$$

équivalentes respectivement aux relations

$$X \in F_0(r),\ X \in F_1(r), \cdots, X \in F_\xi(r), \cdots, \qquad \xi < \eta.$$

Mais le dernier système des relations est équivalent à la relation $X \in \mathfrak{D}_{\xi<\eta} F_\xi(r)$ d'où il suit $F_\eta(r) = \mathfrak{D}_{\xi<\eta} F_\xi(r)$.

La troisième partie. Maintenant posons

$$S_\xi = \mathfrak{S}_r F_\xi(r), \tag{13}$$

$$T_\xi = \mathfrak{S}_r [F_\xi(r) - F_{\xi+1}(r)]. \tag{14}$$

$F_0(r)$ est ensemble de Borel, puisque $F_0(r)=F(r)$. En partant des

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ensembles $F_0(r)$ et en tenant compte de formules (12) on peut démontrer que tous les ensembles $F_\xi(r)$ appartiennent au système B, par suite, S_ξ et T_ξ appartiennent aussi à ce système.

Il est à remarquer que S_ξ est l'ensemble des points X pour lesquels $R_\xi(X)\neq 0$, et T_ξ est l'ensemble des points X pour lesquels

$$R_\xi(X) - R_{\xi+1}(X) \neq 0.$$

La quatrième partie. Maintenant nous allons démontrer que

$$S_\xi = A + \sum_{\eta>\xi} B_\eta, \qquad T_\xi = \sum_{\eta>\xi} A_\eta + \sum_{\eta>\xi} B_\eta. \tag{15}$$

Pour cela nous faisons les remarques suivantes: En premier lieu, si $X \varepsilon A$ on a $KR_0(X)\neq 0$ et inversement si $KR_0(X)\neq 0$ on a $X \varepsilon A$. En second lieu, si $X \varepsilon B$ on a $KR_0(X)=0$ et inversement si $KR_0(X)=0$ on a $X \varepsilon B$. En troisième lieu, $R_\xi(X)\supset R_{\xi+1}(X)$ pour $\xi<\eta(X)$ et $R_\xi(X)=R_{\xi+1}(X)$ pour $\xi\geqq\eta(X)$ où $\eta(X)$ est l'indice du point X.

Soit X un point quelconque de $I_X^{(i)}$. Nous considérons les cas possibles suivants:

Le premier cas: $X \varepsilon S_\xi$. Dans ce cas $R_\xi(X)\neq 0$. Inversement, si $R_\xi(X)\neq 0$ on a $X \varepsilon S_\xi$. Si $X \varepsilon A$ on a $KR_0(X)\neq 0$, par suite, $R_\xi(X)=0$ d'où $X \varepsilon S_\xi$, c'est à dire, $A\subset S_\xi$. Si $X \varepsilon B$ on a $KR_0(X)=0$, par suite, $R_\eta(X)=0$, mais $X \varepsilon S_\xi$, par suite, $X \varepsilon R_\xi(r)$ d'où $\xi<\eta$, c'est à dire, l'indice de X est supérieure à ξ. Donc $S_\xi=A+\sum_{\eta>\xi}B_\eta$.

Le second cas: $X \varepsilon T_\xi$. Dans ce cas $R_\xi(X)-R_{\xi+1}(X)\neq 0$, alors $\eta>\xi$, par suite, $T_\xi=\sum_{\eta>\xi}A_\eta+\sum_{\eta>\xi}B_\eta$. De formules (15) on tire

$$\begin{aligned} S_\xi - T_\xi &= A_0 + A_1 + A_2 + \cdots + A_\xi,\\ I_x^{(i)} - S_\xi &= B_0 + B_1 + B_2 + \cdots + \beta_\xi. \end{aligned} \tag{16}$$

Au moyen de ces formules par le procédé d'induction on peut démontrer que les ensembles A_ξ et B_ξ sont contenus dans le système B, ce qu'il fallait démontrer.

Maintenant posons les définitions suivantes:

(a) Si pour un point $x=\{x_1, x_2, x_3, \cdots, x_\alpha, \cdots\}$, $(\alpha<\Omega_i)$, de l'espace $I_x^{(i)}$ tous les nombres x_n, $(1\leqq n<\Omega_0)$, sont finis, $x_n<\Omega_0$, alors nous dirons que x possède un *noyau* $\bar{x}=\{x_1, x_2, x_3, \cdots, x_n, \cdots\}$, $(1\leqq n<\Omega_0)$, qui est un point de l'espace de Baire $I^{(0)}$.

(b) Désignons par $n(E)$ l'ensemble formé des noyaux de tout les points de E; $n(E)$ est dit le noyau de l'ensemble E.

(c) Si E_0 est le noyau d'un ensemble E de Borel, nous dirons que E_0 est ensemble de Borel $(\aleph_i)$ dans $I_x^{(0)}$.

(d) Si E_0 est le noyau d'un ensemble E de Souslin, nous dirons que E_0 est ensemble de Souslin $(\aleph_i)$ dans $I_x^{(0)}$.

(e) Si E contient tout ensemble $\mathcal{E}$ tel que $\boldsymbol{n}(E)=\boldsymbol{n}(\mathcal{E})$, nous dirons que E est ***normal.***

Ceci étant, prenons un ensemble de Souslin quelconque A de points de $I_x^{(i)}$. D'après le Théorème 4 on a

$$\begin{aligned} A &= A_0 + A_1 + A_2 + \cdots + A_\alpha + \cdots , \\ CA &= B_0 + B_1 + B_2 + \cdots + B_\alpha + \cdots , \qquad 0 \leqq \alpha < \Omega_{i+1}. \end{aligned}$$

Il est évident que si E est normal on a

$$\boldsymbol{n}(CE) = C[\boldsymbol{n}(E)] \tag{17}$$

d'où il suit pour A normal

$$\begin{aligned} \boldsymbol{n}(A) &= \boldsymbol{n}(A_0) + \boldsymbol{n}(A_1) + \boldsymbol{n}(A_2) + \cdots + \boldsymbol{n}(A_\alpha) + \cdots , \\ C[\boldsymbol{n}(A)] &= \boldsymbol{n}(B_0) + \boldsymbol{n}(B_1) + \boldsymbol{n}(B_2) + \cdots + \boldsymbol{n}(B_\alpha) + \cdots , \\ & \qquad \alpha < \Omega_{i+1}. \end{aligned} \tag{18}$$

Donc, nous avons obtenu le théorème suivant.

THÉORÈME 5. *Pour tout ensemble a de Souslin* $(\aleph_i)$ *dans* $I_x^{(0)}$ *on peut trouver les ensembles de Borel* $(\aleph_i)$:

$$a_0, a_1, a_2, \cdots , a_\alpha, \cdots , \qquad b_0, b_1, b_2, \cdots , b_\alpha, \cdots , \qquad \alpha < \Omega_{i+1},$$

de cet espace tels que

$$\begin{aligned} a &= a_0 + a_1 + a_2 + \cdots + a_\alpha + \cdots , \\ b &= b_0 + b_1 + b_2 + \cdots + b_\alpha + \cdots , \qquad \alpha < \Omega_{i+1}. \end{aligned}$$

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ON LINEAR POLYGON TRANSFORMATIONS[1]

JESSE DOUGLAS

1. Introduction, definitions. In a recent paper[2] the author has developed a general theory of linear transformations of polygons, regarded as lying in the complex plane. By a *polygon* we understand a system of points, or complex numbers: $(z_1, z_2, \cdots, z_n)$, called *vertices*, which are taken in a definite *cyclic* order. This is to say that two n-gons, $(z_1, z_2, \cdots, z_n)$ and $(w_1, w_2, \cdots, w_n)$, are the same when and only when $z_i = w_{i+k}$ for $i = 1, 2, \cdots, n$, where k has any fixed one of the values $0, 1, 2, \cdots, n-1$ and all indices are taken modulo n. To be distinguished from a polygon is a *multipoint*, where the definition of equivalence is the identity of corresponding points in the order given: $z_i = w_i$ for $i = 1, 2, \cdots, n$.

A linear multipoint transformation is simply the general linear transformation of n complex variables:

$$z_i' = \sum_{j=1}^{n} a_{ij} z_j, \qquad i = 1, 2, \cdots, n, \tag{1}$$

where the coefficients a_{ij} may be any complex numbers. For a linear polygon transformation, on the other hand, a certain cyclicity is required: a cyclic permutation of the z's must produce the same cyclic permutation of the z''s. Thus we may write down arbitrarily the first line of an L.P.T.:[3]

$$z_1' = \alpha_0 z_1 + \alpha_1 z_2 + \cdots + \alpha_{r-1} z_r,$$

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[1] Presented to the Society, October 29, 1938, under the title *Geometry of polygons in the complex plane.*

[2] *Geometry of polygons in the complex plane*, Journal of Mathematics and Physics, vol. 19 (1940), pp. 93–130, and this Bulletin, abstract 44-9-390. See, as a preliminary to the present theory, papers by E. Kasner and his students in Scripta Mathematica, vol. 2 (1934), pp. 131–138, and vol. 4 (1936), pp. 37–49. Kasner considers the polygon derived from a given one by taking the midpoint of each side, and, more generally, by taking the centroid of r consecutive vertices. These are special linear polygon transformations (2), where $\alpha_0 = \alpha_1 = \cdots = \alpha_{r-1} = 1/r$. Kasner uses real cartesian coordinates, and his polygons may lie in euclidean space of any number of dimensions.

The basic ideas of the present paper are (i) to regard the polygons as lying in the *complex plane*, (ii) to consider general linear polygon transformations (2) with any *complex* coefficients. As pointed out in §1, this is equivalent to taking a "center of gravity" with complex "weights."

The author delivered a series of lectures on the present topic at Columbia University in July, 1939.

[3] L.P.T. denotes "linear polygon transformation" throughout this paper.

but then the whole transformation is determined by cyclic permutation, that is:

$$z_k' = \alpha_0 z_k + \alpha_1 z_{k+1} + \cdots + \alpha_{r-1} z_{k+r-1}, \qquad k = 1, 2, \cdots, n, \tag{2}$$

where the indices of z are to be taken modulo n. Here r may have any value from 1 to n inclusive, and we may suppose $\alpha_0 \neq 0$,[4] $\alpha_{r-1} \neq 0$, the transformation being then called *r-ary*. If $r < n$, we may fill out formula (2) to the general form (1) by using zero coefficients in each line of (2) for those z's which do not effectively appear.

Formula (2) represents the general *linear polygon transformation* L.[5] Its characteristic property is to be transformed into itself by a cyclic permutation $C = (z_1, z_2, \cdots, z_n)$ of the z's: $C^{-1}LC = L$, or to be commutative with such a permutation: $CL = LC$. Since $C^{-1}LC = L$ and $C^{-1}L'C = L'$ imply by multiplication $C^{-1}LL'C = LL'$, it follows that *the product of two L.P.T.'s is again an L.P.T.* Thus, all L.P.T.'s for a given value of n form a group,[6] a subgroup G_n of n complex parameters in the total linear group G_{n^2} of n^2 complex parameters represented by (1). In §2 another proof of the same fact will appear (Theorem II).

Particularly interesting are the special L.P.T.'s, to be denoted by M, for which

$$\alpha_0 + \alpha_1 + \cdots + \alpha_{r-1} = 1. \tag{3}$$

As is immediately verifiable, (3) is the necessary and sufficient condition that a given L.P.T. (2) be permutable with an arbitrary similitude transformation $S: z' = Az + B$ (A, B complex); that is, $MS = SM$, or $S^{-1}MS = M$, for every S and every M. It follows, as in the preceding paragraph, that *the transformations M form a group*, a subgroup G_{n-1} of the group G_n of all L.P.T.'s.

If the polygon P' is the image of the polygon P by any transformation M, the remarks just made state that the relation between P and P' is invariant under an arbitrary similitude transformation; we may say that P' is a "similitude concomitant" of P. Accordingly, we shall term any L.P.T., M, which obeys the condition (3) a *similitude*

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[4] Evidently we can always bring about $\alpha_0 \neq 0$ by cyclic renumbering of the vertices z.

[5] We dispense with the consideration of nonhomogeneous L.P.T.'s, since these have the same constant term in each line and therefore differ from a homogeneous L.P.T. only by a translation.

[6] It should be emphasized that here, and throughout this paper, we use the term "group" only in the sense of possession of the specific group property: closure with respect to composition of elements. Since the determinant of an L.P.T. may be zero, an inverse transformation does not always exist.

construction on the polygon $P=(z_1, z_2, \cdots, z_n)$ and a *binary, ternary,* $\cdots$, *r-ary* S.C.[7] according as two, three, $\cdots$, r consecutive vertices of P are effectively involved in the formula (2) of the transformation ("effectively" meaning $\alpha_0 \neq 0$, $\alpha_{r-1} \neq 0$).

If furthermore the α's are *real*, then, as is easily verified, we have for every affine transformation a, that is, $z'=Az+B\bar{z}+C$ (A, B, C complex, $\bar{z}$ the conjugate of z), the relation $a^{-1}Ma=M$ or $Ma=aM$. The polygon P' is then an "affine concomitant" of the polygon P; accordingly, for *real* values of the α's obeying (3), we term M an *affine construction.*

Evidently, under the condition (3), we can consider that (2) defines z_k' as a kind of "center of gravity" of $z_k, z_{k+1}, \cdots, z_{k+r-1}$ with *complex* "weights" proportional to $\alpha_0, \alpha_1, \cdots, \alpha_{r-1}$. In the case of an affine construction, where the weights α are real, we have an actual center of gravity in the ordinary sense.

We may observe here, as in our cited paper,[8] that the centroid of any polygon is invariant under any S.C., M. For by addition of all the equations (2),

$$\begin{aligned} (3a)\qquad & z_1' + z_2' + \cdots + z_n' \\ & \qquad = (\alpha_0 + \alpha_1 + \cdots + \alpha_{r-1})(z_1 + z_2 + \cdots + z_n); \end{aligned}$$

then take account of (3) and divide by n.

Of special importance is the *binary* S.C., M_2:

$$(4)\qquad z_k' = \alpha_0 z_k + \alpha_1 z_{k+1}, \qquad k = 1, 2, \cdots, n,$$

$$(4')\qquad \alpha_0 + \alpha_1 = 1.$$

The point z_k' is here determined by the simple geometric condition that *the triangle* (z_k, z_{k+1}, z_k') *is directly*[9] *similar to the fixed triangle* $(-\alpha_1, \alpha_0, 0)$. This is seen by considering the equations in A, B

$$(5)\qquad z_k = -\alpha_1 A + B, \quad z_{k+1} = \alpha_0 A + B, \quad z_k' = B,$$

whose solvability is the condition for the existence of a similitude transformation $z'=Az+B$ converting the triangle $(-\alpha_1, \alpha_0, 0)$ into (z_k, z_{k+1}, z_k'). By use of (4'), the solution of the first two of these equations is

$$A = z_{k+1} - z_k, \; B = \alpha_0 z_k + \alpha_1 z_{k+1};$$

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[7] S.C. is "similitude construction" throughout this paper.

[8] In this way we shall always refer to the paper whose title is given in the first footnote.

[9] That is, with preservation of sense.

and then equation (4) expresses exactly that the third equation (5) is satisfied.

If α_0, α_1 are real, the point z_k' lies upon the side $z_k z_{k+1}$ and divides it in the fixed ratio $z_k z_k' : z_k' z_{k+1} = \alpha_1 : \alpha_0$.

2. L.P.T.'s **and cyclic matrices.** The matrix of the general L.P.T. is a *cyclic matrix*

$$(6) \qquad L = \left\| \begin{matrix} \alpha_0 & \alpha_1 & \cdots & \alpha_{n-2} & \alpha_{n-1} \\ \alpha_{n-1} & \alpha_0 & \alpha_1 \cdots & \cdot & \alpha_{n-2} \\ \alpha_{n-2} & \alpha_{n-1} & \alpha_0 \cdots & \cdot & \alpha_{n-3} \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ \alpha_1 & \alpha_2 & \cdots & \alpha_{n-1} & \alpha_0 \end{matrix} \right\|,$$

where the circular derivation of each row from the preceding is the characteristic feature. Any number of the α's may be zero, so that the L.P.T. can be r-ary with $r=1, 2, \cdots, n$. We shall speak interchangeably of an L.P.T., L, and its corresponding cyclic matrix L.

The simplest L.P.T. is a cyclic permutation of vertices, C:

$$z_1' = z_2, \quad z_2' = z_3, \quad \cdots, \quad z_{n-1}' = z_n, \quad z_n' = z_1.$$

As a matrix, this is

$$C = \left\| \begin{matrix} 0 & 1 & 0 \cdots 0 \\ 0 & 0 & 1 \cdots 0 \\ \cdot & \cdot & \cdots\cdot \\ 0 & 0 & \cdots 1 \\ 1 & 0 & \cdots 0 \end{matrix} \right\|.$$

We may say that C has 1's in its second "cyclic diagonal" and 0's everywhere else. In general, as is seen at once, C^k has 1's in its $(k+1)$th cyclic diagonal and 0's elsewhere. Obviously, $C^n = I$, the identity matrix, which has 1's in its first (cyclic) diagonal and 0's elsewhere.

From these remarks we have directly by comparison with (6):

$$(7) \qquad L = \alpha_0 I + \alpha_1 C + \alpha_2 C^2 + \cdots + \alpha_{n-1} C^{n-1}.$$

Conversely, every polynomial in C of this form represents a cyclic matrix, namely the one whose first, or generating, row consists of the coefficients of the polynomial in order. Hence we have the following theorem.

THEOREM I. *The necessary and sufficient condition that an n-rowed square matrix L be cyclic is that it be expressible as a polynomial of degree $n-1$ at most in the matrix C of a cyclic permutation. This mode of representation of L is unique.*

If in (7) we replace C by a (complex) numerical variable x, and I by 1, we obtain a polynomial $\phi(x)=\alpha_0+\alpha_1x+\cdots+\alpha_{n-1}x^{n-1}$, that we shall term the *auxiliary polynomial*[10] of L.

If any two polynomials of the form (7) are multiplied together, the product can be reduced to the degree $n-1$ at most by means of the relation $C^n=I$. From this remark and Theorem I the following three theorems result immediately.

THEOREM II. *The product of two cyclic matrices is again a cyclic matrix.*

THEOREM III. *The multiplication of cyclic matrices is isomorphic with the multiplication of their auxiliary polynomials, modulo x^n-1.*

THEOREM IV. *The multiplication of cyclic matrices is commutative.*

Thus cyclic matrices form a commutative subalgebra of order n in the non-commutative algebra of order n^2 formed by general n-rowed square matrices.

The following alternative form of proof of these theorems, directly in terms of the explicit representation (6) of a cyclic matrix, may be of interest. Let α_m, $(m=0, 1, \cdots, n-1)$, denote any n complex numbers. Then a cyclic matrix of order n, $\|a_{ij}\|$, is one where $a_{ij}=\alpha_{j-i}$, *the index $j-i$ being taken modulo n*; that is, in case $j-i$ is negative, $j-i+n$ is to be used instead. If $\|b_{ij}\|$ is another cyclic matrix: $b_{ij}=\beta_{j-i}$, $j-i$ taken modulo n; then $\|a_{ij}\|\times\|b_{ij}\|=\|c_{ij}\|$, where

$$c_{ij}=\sum_{k=1}^{n}a_{ik}b_{kj}=\sum_{k=1}^{n}\alpha_{k-i}\beta_{j-k}=\sum\alpha_s\beta_t$$

(in the last summation s, t evidently take once and only once every combination of values such that $s+t\equiv j-i$, modulo n). In other words, defining

$$\gamma_m=\sum\alpha_s\beta_t,\qquad s+t\equiv m \pmod{n}, \tag{8}$$

we have $c_{ij}=\gamma_{j-i}$, $j-i$ taken modulo n.

This proves Theorem II. Theorem III follows because (8) represents the law of multiplication of the auxiliary polynomials modulo x^n-1:

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[10] This is only of degree $r-1$ if L is r-ary (see (9)). In our cited paper we defined the auxiliary polynomial as $\alpha_0x^{r-1}+\alpha_1x^{r-2}+\cdots+\alpha_{r-1}$. With this difference of notation in mind, no ambiguity should arise in comparing the two papers.

$$\sum_{m=0}^{n-1} \gamma_m x^m \equiv \sum_{s=0}^{n-1} \alpha_s x^s \cdot \sum_{t=0}^{n-1} \beta_t x^t \pmod{(x^n-1)}.$$

Theorem IV expresses the symmetry of (8) in the indices s, t of α, β.

As has been pointed out, all these theorems can be stated in terms of L.P.T.'s[11] instead of cyclic matrices. The auxiliary polynomial of an r-ary L.P.T. is of degree $r-1$:

$$\phi(x) = \alpha_0 + \alpha_1 x + \cdots + \alpha_{r-1} x^{r-1}. \tag{9}$$

We term $\phi(x)=0$ the *auxiliary equation* of the L.P.T.

By factorization of its auxiliary polynomial, or solution of its auxiliary equation, *every r-ary L.P.T. can be factored into $r-1$ binary L.P.T.'s*. This results from Theorem III; in fact, if

$$\phi(x) = (\alpha_0^{(1)} + \alpha_1^{(1)} x)(\alpha_0^{(2)} + \alpha_1^{(2)} x) \cdots (\alpha_0^{(r-1)} + \alpha_1^{(r-1)} x), \tag{10}$$

then L is equal to the product of the $r-1$ binary L.P.T.'s

$$L_2^{(j)} = \alpha_0^{(j)} I + \alpha_1^{(j)} C, \qquad j = 1, 2, \cdots, r-1.$$

The binary components $L_2^{(j)}$ are evidently not unique; rather, as is evident, each is subject to multiplication by a (complex) scalar $\lambda^{(j)}$ provided that the product of all the $\lambda^{(j)}$'s is unity.

If L is an S.C., M, so that (3) is verified, we have $\phi(1)=1$, or by (10),

$$1 = (\alpha_0^{(1)} + \alpha_1^{(1)})(\alpha_0^{(2)} + \alpha_1^{(2)}) \cdots (\alpha_0^{(r-1)} + \alpha_1^{(r-1)}). \tag{11}$$

Dividing (10) by (11), we obtain

$$\phi(x) = (\beta_0^{(1)} + \beta_1^{(1)} x)(\beta_0^{(2)} + \beta_1^{(2)} x) \cdots (\beta_0^{(r-1)} + \beta_1^{(r-1)} x)$$

where $\beta_0^{(j)} = \alpha_0^{(j)}/(\alpha_0^{(j)}+\alpha_1^{(j)})$, $\beta_1^{(j)} = \alpha_1^{(j)}/(\alpha_0^{(j)}+\alpha_1^{(j)})$; hence

$$\beta_0^{(j)} + \beta_1^{(j)} = 1. \tag{12}$$

Accordingly $M_2^{(j)} = \beta_0^{(j)} I + \beta_1^{(j)} C$ is a binary S.C. for $j=1, 2, \cdots, r-1$, and we have the following theorem.

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[11] The commutativity of L.P.T.'s seems the more noteworthy because it is certainly not evident geometrically that two binary S.C.'s are commutative. See the geometric interpretation of a binary S.C. at the end of §1.

THEOREM V. *Every r-ary S.C., M_r, is representable uniquely as the product of $r-1$ binary S.C.'s*: $M_r = \prod_{j=1}^{r-1} M_2^{(j)}$.

The uniqueness is a consequence of the fact that the components $M_2^{(j)}$ are not subject to multiplication by a scalar, because that would disturb condition (12).

An interesting illustration of Theorem V is furnished by the ternary centroidal construction

$$z_k' = \tfrac{1}{3}z_k + \tfrac{1}{3}z_{k+1} + \tfrac{1}{3}z_{k+2}.$$

Here the auxiliary polynomial factors as follows:

$$\frac{1}{3}(1 + x + x^2) = \frac{\omega - x}{\omega - 1}\cdot\frac{\omega^2 - x}{\omega^2 - 1},$$

where $\omega = e^{2\pi i/3}$ is a complex cube root of unity. By the remarks associated with (4), the component binary S.C.'s are those which employ triangles similar respectively to $(1, \omega, 0)$, $(1, \omega^2, 0)$. Hence the centroid of any three points z_1, z_2, z_3 will be arrived at as follows. On z_1z_2, z_2z_3 construct 120° isosceles triangles to the left; let z_1', z_2' be their respective vertices. On $z_1'z_2'$ construct a 120° isosceles triangle to the right; its vertex z_1'' is the centroid of z_1, z_2, z_3.

We conclude this section by observing the following factorization of a cyclic determinant:

$$\det L = \phi(1)\cdot\phi(\omega)\cdot\phi(\omega^2)\cdot\;\cdots\;\cdot\phi(\omega^{n-1}), \tag{13}$$

where $\omega = e^{2\pi i/n}$ is a primitive nth root of unity. The proof is easily given as follows. Det L is a homogeneous polynomial of degree n in the α's. By addition of all the columns of L (refer to (6)), $\phi(1)$ is seen to be a factor of det L. By using $1, \omega, \omega^2, \cdots, \omega^{n-1}$ as multipliers of the successive columns and adding, $\phi(\omega)$ is seen to be a factor of det L. Similarly $\phi(\omega^2), \cdots, \phi(\omega^{n-1})$ are factors. Each of these n factors is a linear polynomial in the α's. Hence (13) holds to within a numerical factor, which is verified to be 1 by comparison of the terms in α_0^n.

3. **Geometrical applications.** In our cited paper, we consider the relations

$$R_p \equiv \sum_{k=1}^{n} \omega^{(k-1)p} z_k = 0, \quad p = 1, 2, \cdots, n-1, \tag{14}$$

as applied to any polygon. Each relation $R_p = 0$ expresses an intrinsic geometric property of the polygon, being invariant under any similitude transformation, $z_k' = Az_k + B$; this is easily verified if account is taken of the well known equation[12] $\sum_{k=0}^{n-1} \omega^{kp} = 0$, $(p = 1, 2, \cdots, n-1)$.

A polygon is called *regular of ω^q-type*[13] if it is similar to the polygon $(1, \omega^q, \omega^{2q}, \cdots, \omega^{(n-1)q})$. A regular polygon of ω-type is simply a (convex) regular polygon in the ordinary sense. A regular polygon of ω^q-type may be star-shaped (for example, $n=5$, $q=2$: a star-shaped regular pentagon), or it may resolve into an s-gon described t times, where $st = n$ (for example, $n=6$, $q=2$: an equilateral triangle described twice, considered as a form of regular hexagon).

It was shown (loc. cit., §5) that *the criterion for a polygon to be regular of ω^q-type is that it obey the following $n-2$ conditions*:.

$$(15) \qquad R_p = 0 \qquad \text{for } p = 1, 2, \cdots, n-1, \text{ except } n - q.$$

Each condition $R_p = 0$ may therefore be regarded as expressing a "degree of regularity," and any number $k < n-2$ of these conditions as expressing "partial regularity" of degree k.

By multiplying equation (2) by $\omega^{(k-1)p}$ and summing for $k = 1, 2, \cdots, n$, we find

$$(16) \qquad R_p' = \phi(\omega^{-p})R_p = \phi(\omega^{n-p})R_p,$$

or, interchanging p and $n-p$,

$$(16') \qquad R_{n-p}' = \phi(\omega^p)R_{n-p}.$$

We infer from (16) or (16′) the following two facts:

(i) *If a polygon obeys the relation $R_p = 0$, this relation persists after any L.P.T.* (That is, R_p is a *relative invariant* of the group G_n of all L.P.T.'s.)

(ii) *If ω^p is a root of the auxiliary equation $\phi(x) = 0$ of an L.P.T., then this L.P.T. converts every polygon P into a polygon P' obeying the relation $R_{n-p} = 0$.*

A binary S.C., $M_2 = \alpha_0 I + \alpha_1 C$, $(\alpha_0 + \alpha_1 = 1)$, has the auxiliary linear polynomial $\phi(x) = \alpha_0 + \alpha_1 x$, and hence obeys the hypothesis of (ii): $\phi(\omega^p) = 0$, when and only when $-\alpha_1 : \alpha_0 = 1 : \omega^p$. By the statement following (4′), this means that the triangle (z_k, z_{k+1}, z_k') is similar to $(1, \omega^p, 0)$. Then z_k' is at the vertex of an isosceles triangle with vertex angle $2p\pi/n$ based on $z_k z_{k+1}$ and to the left[14] of this base. We shall denote this construction whereby the polygon (z_k') is derived from the polygon (z_k) by $\Delta(2p\pi/n)$.

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[12] ω^p is a root, not 1, of $x^n - 1 = (x-1)(1 + x + x^2 + \cdots + x^{n-1}) = 0$.

[13] As defined, loc. cit., §2, Definition (iii).

[14] With this interpretation: an isosceles triangle to the left with vertex angle $\pi + \theta$ is an isosceles triangle to the right with vertex angle $\pi - \theta$.

We may now put (i), (ii), and the criterion (15) together so as to give the next theorem:

THEOREM VI. *If, on an* arbitrary *polygon* P_0, *the* $n-2$ *constructions* $\Delta(2p\pi/n)$ *for* $p=1, 2, \cdots, n-1$ *except* q, *are performed successively in any order, giving the series of polygons* $P_1, P_2, \cdots, P_{n-2}$, *then* P_{n-2} *is* regular *of* ω^q*-type. Further,* P_{n-2} *is independent of the order of these constructions. Also, the centroid of* P_{n-2} *and of all the intermediate polygons* $P_1, P_2, \cdots$ *is the same as that of* P_0.[15]

(The last statement refers to the remark associated with (3a).)

PROOF. Each operation $\Delta(2p\pi/n)$ confers the property $R_{n-p}=0$ on the new polygon, according to (ii); and the subsequent operations allow the polygon to keep this property, according to (i); therefore the final polygon P_{n-2} has all the properties (15) requisite for regularity of ω^q-type.

The case $n=3$ of Theorem VI is the following well known theorem of elementary geometry: If on each side of an *arbitrary* triangle as base, a 120° isosceles triangle is constructed, always outward or always inward, then the vertices of these isosceles triangles form an *equilateral* triangle.

In conclusion, we consider the effect on a polygon of any r-ary S.C., M. Since for an S.C., $\phi(1)=1$ by (3), the determinant of M is by (13)

$$\det M = \phi(\omega)\cdot\phi(\omega^2)\cdot \cdots \cdot\phi(\omega^{n-1}).$$

Hence, if the auxiliary polynomial $\phi(x)$ is prime to x^n-1, then $\det M\neq 0$, and the transformation M is (uniquely) reversible. This means that for every polygon P' there is a (unique) polygon P such that $P'=MP$; accordingly, P' can have no special properties if P is general.

On the other hand, if the greatest common divisor of $\phi(x)$ and x^n-1 is

$$\psi(x) = (x-\omega^{p_1})(x-\omega^{p_2})\cdots(x-\omega^{p_k}),$$

so that

$$\begin{aligned} &\phi(\omega^{p_1}) = 0, \quad \phi(\omega^{p_2}) = 0, \cdots, \phi(\omega^{p_k}) = 0, \\ &\phi(\omega^{p}) \neq 0 \qquad \text{for } p \neq p_1 \text{ or } p_2 \cdots \text{ or } p_k, \end{aligned} \tag{17}$$

then by (16′) the transformed polygon P' has the special properties (partial regularity of degree k)

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[15] Theorem VI is the main one of our cited paper (Theorem A or A′, §7).

$$(18)\qquad R'_{n-p_1} = 0, \qquad R'_{n-p_2} = 0, \qquad \cdots, \qquad R'_{n-p_k} = 0$$

for an arbitrary original polygon P. Further, no other relations $R'_{n-p}=0$ ($p\neq p_1$ or $p_2 \cdots$ or p_k) are satisfied by P' if P remains general (P' has no higher than the kth degree of regularity). This is also seen from (16′), where $\phi(\omega^p)\neq 0$, $R_{n-p}\neq 0$ (since P is general); therefore $R'_{n-p}\neq 0$.

In fact, no relations of any kind besides (18) are satisfied by $P'=MP$ if P remains general. This is because, by the general theory of systems of linear equations, it can be readily shown that if the conditions (17) are satisfied by the coefficients α in (2), then the conditions (18) are sufficient as well as necessary in order that (2) be solvable for the z's in terms of the z''s. This is to say that for *any* polygon P' obeying (18) a polygon P can be found such that $P'=MP$; indeed, the class of such polygons P depends linearly on k complex parameters.

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AXIOMS FOR MOORE SPACES AND METRIC SPACES[1]

C. W. VICKERY

We shall consider a set of five axioms in terms of the undefined notions of *point* and *region*. It will be shown that these axioms are independent and that they constitute a set of conditions necessary and sufficient for a space to be a complete metric space. It will also be shown that certain subsets of this set of axioms constitute necessary and sufficient conditions for a space to be (1) a metric space, (2) a Moore space, (3) a complete Moore space. Axiom 2 and a more general form of Axiom 1 have been stated by the author in an earlier paper [1]. Following terminology of F. B. Jones [2], a space is said to be a *Moore space* provided conditions (1), (2), and (3) of Axiom 1 (that is, Axiom 1_0) of R. L. Moore's *Foundations of Point Set Theory* [3] are satisfied. A space is said to be a *complete Moore space* provided it satisfies all the conditions of that axiom. Wherever the notion of region is employed, whether as a defined or an undefined notion, it is understood that a necessary and sufficient condition that a point P be a limit point of a point set M is that every region containing P contain a point of M distinct from P. The letter S is used to denote the set of all points.

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[1] Presented to the Society, April 20, 1935, under the title *Sets of independent axioms for complete Moore space and complete metric space.*

AXIOM 0. *Every region is a point set.*

AXIOM 1. *There exists a countable family F such that* (1) *every element of F is a collection of regions covering S,* (2) *if R is a region and A and B are points of R, there exists a collection G of F such that if g is a region of G that contains A, $\bar{g}$ is a subset of $(R-B)+A$.*

AXIOM 2. *If P is a point and H and K are regions containing P, there exists a region R containing P which is a subset both of the region H and of K.*

AXIOM 3. *If α is a monotonic descending sequence of closed point sets $A_1, A_2, \cdots$ such that for each n there exists a monotonic descending sequence ρ_n of distinct regions $R_1, R_2, \cdots, R_n$ containing A_n, then there exists a point common to all the elements of α.*

AXIOM 4. *If G is a collection of regions covering S, there exists a collection H of regions covering S such that if h_1 and h_2 are intersecting regions of H, then (h_1+h_2) is a subset of a region of G.*

THEOREM 1. *In order that a space be a Moore space, it is necessary and sufficient that it satisfy Axioms* 0, 1, *and* 2.

The necessity of these conditions is evident. We shall undertake to show their sufficiency without changing the notion of region. Let $H_1, H_2, \cdots$ be a type ω sequence of all the elements of family F postulated by Axiom 1. Let G_1 denote the collection of all regions R such that R is a subset of a region of H_1. Let G_2 denote the collection of all regions R such that R is a subset of a region of H_1 and a region of H_2. For each positive integer n let G_n denote the collection of all regions R such that R is a subset of a region of H_i for each $i \leqq n$. For each n, G_n covers S, by Axiom 2. Furthermore, for each n, G_n contains all the regions of G_{n+1}. The sequence $G_1, G_2, \cdots$ satisfies all the conditions of Axiom 1_0 of R. L. Moore.

As a means to proving the next theorem, we shall prove the following lemma on the basis of Moore's Axioms 0 and 1_0:

LEMMA 1. *If M is a set of points and G is a collection of domains covering M, there exists a collection H of domains covering M such that no domain of H is a subset of another domain of H and such that every domain of H is a subset of some domain of G.*

Suppose that M is a set of points and G a collection of domains covering M. For each positive integer n let T_n denote the set of all points P of M such that some domain of G contains every region of G_n that contains P. Then $M=\sum_{n=1}^{\infty}T_n$. For each positive integer n

let θ_n denote a well-ordered sequence of the points of T_n. Let θ denote the sequence obtained by taking first the elements of θ_1, then the elements of θ_2, and so on. Let $t_{i,\mu}$ denote the first element of θ. where i is the smallest integer such that $t_{i,\mu}$ is an element of θ_i and where μ is an ordinal number denoting the order of $t_{i,\mu}$ in θ_i. (Some sets T_n may be vacuous.) We shall now define a sequence Δ of domains $D_1, D_2, \cdots$. Let D_1 denote the sum of all the regions of G_i that contain $t_{i,\mu}$. Let $t_{j,\nu}$ denote the first point of θ not contained in D_1. Let D_2 denote the sum of all the regions of G_j that contain $t_{j,\nu}$. In general, suppose that Δ_α denotes any *abschnitt* of Δ; then let $t_{k,\xi}$ denote the first point of θ not contained in any domain of Δ_α and let D_α denote the sum of all the regions of G_k that contain $t_{k,\xi}$. Let H denote the collection of all the domains of Δ. Then H has the required properties.

THEOREM 2. *In order that a space be a complete Moore space, it is necessary and sufficient that it satisfy Axioms* 0, 1, 2, *and* 3.

We shall first show the sufficiency of these conditions. Let $H_1, H_2, \cdots$ denote a type ω sequence of the elements of family F of Axiom 1. For each positive integer n let G_n denote the collection of all regions R such that R is a point or a proper subset of a region of H_n and of a region of G_{n-1}. It follows, with the help of Axiom 2, that sequence $G_1, G_2, \cdots$ satisfies conditions (1), (2), and (3) of Moore's Axiom 1. It remains to be shown that it satisfies condition (4). Suppose that $M_1, M_2, \cdots$ is a type ω sequence of nondegenerate closed point sets such that for each n, M_n contains M_{n+1} and is a subset of some region of G_n. Let R_n denote a region of G_n that contains M_n. Then R_n is a proper subset of a region R_{n-1} of G_{n-1}. Similarly R_{n-1} is a proper subset of a region R_{n-2} of G_{n-2}. Thus the conditions of Axioms 3 are satisfied and hence there exists a point common to all the sets $M_1, M_2, \cdots$.

We shall now show the necessity of these conditions by redefining region. Let $G_1, G_2, \cdots$ be a sequence of collections of regions postulated by Moore's Axiom 1. For each n let H_n denote a collection of domains covering S such that no domain of H_n is a subset of another domain of H_n and such that every domain of H_n is a subset of a region of G_n. For each n, H_n exists, by Lemma 1. Let F denote the family of all collections H_n. Let $H=\sum_{n=1}^{\omega} H_n$. If the domains of H are called regions and if nothing else is called a region, then Axioms 0, 1, 2, and 3 are satisfied. (1) Clearly Axioms 0 and 1 are satisfied. (2) We shall show that Axiom 2 is satisfied. Let h and k denote two domains of H having a point P in common. There exists an integer n such that every region of G_n that contains P is a subset of $h\cdot k$. Let R denote a domain of H_n. Then R is a subset of some region of G_n and hence of $h\cdot k$. (3) We shall now show that Axiom 3 is satisfied. Let α denote a type ω sequence of closed point sets $A_1, A_2, \cdots$, and for each n let ρ_n denote a type n sequence of domains of H, $R_1, R_2, \cdots, R_n$ satisfying the conditions of Axiom 3. Since for each n no domain of H_n is a subset of another domain of H_n, it follows that there exists an $i \geqq n$ such that some domain of ρ_n belongs to H_i and hence is a subset of a region of G_i. It follows that for each n, A_n is a subset of a region of G_n and hence there exists a point common to all the elements of α.

THEOREM 3. *In order that a metric space be complete it is necessary and sufficient that it satisfy Axiom* 3.

This follows immediately with the aid of Theorem 2 and a result of J. H. Roberts [4] to the effect that every metric space that satisfies Axiom 1 of R. L. Moore is complete. In a metric space every interpretation of region that preserves the notion of limit point satisfies Axioms 0, 1, 2, and 4.

THEOREM 4. *In order that a space be metric it is necessary and sufficient that it satisfy Axioms* 0, 1, 2, *and* 4.

We shall first show the sufficiency of these conditions. We have shown that Moore's Axiom 1_0 follows from Axioms 0, 1, and 2. If Axiom 4 be added, it can be shown that the following stronger analogue (due to R. L. Moore) of Moore's Axiom 1_0 follows: "There exists a sequence $G_1, G_2, \cdots$ such that (1) for each n, G_n is a collection of regions covering S, (2) for each n, G_n contains G_{n+1}, (3) if R is a region and A and B are points of R, there exists an integer n such that if h and k are two regions of G_n having a point in common and such that h contains A, then $h+k$ is a subset of $(R-B)+A$." Moore has shown that this proposition is a necessary and sufficient condition for a space to be metric.

We shall show the necessity of these conditions. Suppose that S denotes a space $(\mathfrak{D})$. Let all spheroids be called regions. Let collection H_n of family F be the set of all spheroids of radius less than $1/n$. Clearly Axioms 0, 1, and 2 are satisfied. We shall show that Axiom 4 is satisfied. Let G denote a collection of spheroids covering S. For each positive integer n let T_n denote the set of all points P such that there exists a spheroid of G containing the sum of every two-linked chain of spheroids of radius less than $1/n$ that contains P. Then $S=\sum_{n=1}^{\infty}T_n$. Let Q_n denote the collection of all spheroids of radius less than $1/n$ containing a point of T_n. Let $Q=\sum_{n=1}^{\infty}Q_n$. Then Q is the required collection, for the sum of every two-linked chain of regions of Q is a subset of some region of G.

THEOREM 5. *In order that a space be a complete metric space, it is necessary and sufficient that it satisfy Axioms* 0, 1, 2, 3, *and* 4.

This is an immediate consequence of Theorems 3 and 4.

INDEPENDENCE EXAMPLES

For Axiom 1. Let S be the set of all real numbers between 0 and 1. Let p and q denote two real numbers such that $0<p<q<1$. Let the collection of all regions be the collection of all segments ab such that either (1) $0<a<p$ and $q<b<1$, or (2) $0<a<p$ and $0<b<p$, or (3) $q<a<1$ and $q<b<1$.

For Axiom 2. Let S be the set of all points on the x axis between $(-1, 0)$ and $(+1, 0)$. Let every segment of S not containing O $(0, 0)$ or having O as an end point be taken as a region. Furthermore, let every point set consisting of O together with a segment of S having O as an end point be taken as a region. If n is an odd positive integer, let collection H_n of family F of Axiom 1 be the collection of all regions not containing O and of length less than $1/n$, together with all left-hand regions containing O and of length less than $1/n$. If n is even, we have the same statement except that we substitute right-hand regions containing O for left-hand regions.

For Axiom 3. Let S be the set of all rational points on the x axis. Let the sets of all rational points of all segments be called regions.

For Axiom 4. Let S be the set of all points on or above the x axis. Let regions be the interiors of all circles lying wholly above the x axis, together with all point sets Q such that Q is the interior of a circle tangent to the x axis plus the point of tangency. (Example due to R. L. Moore.)

REFERENCES

1. C. W. Vickery, *Spaces in which there exist uncountable convergent sequences of points*, Tôhoku Mathematical Journal, vol. 40 (1934), pp. 1–26.

2. F. B. Jones, *Concerning normal and completely normal spaces*, this Bulletin, vol. 43 (1937), pp. 671–677.

3. R. L. Moore, *Foundations of Point Set Theory*, American Mathematical Society Colloquium Publications, vol. 13, New York, 1932.

4. J. H. Roberts, *A property related to completeness*, this Bulletin, vol. 38 (1932), pp. 835–838.

AUSTIN, TEXAS

THE AUTOMORPHISMS OF THE SYMMETRIC GROUP

IRVING E. SEGAL

The purpose of this note is to give a proof of the following well known theorem: *The group of automorphisms of the symmetric group* S_n *on* n *letters is isomorphic with* S_n, *except when* $n=6$. The proofs of this in the literature are complicated[1] and involve the use of lemmas whose relevance is not plain.

Let A be an automorphism of S_n. Then it is clear that A takes a class of similar elements into a class of similar elements, and that it takes an element of order m into an element with the same order. Hence suppose $A(1r)=t_1(r)\cdot t_2(r)\cdot \cdots \cdot t_k(r)$ $(k\geqq 1)$, where the $t_i(r)$ are disjoint transpositions. A simple calculation shows that there are $n(n-1)/2$ elements similar to $(1r)$, and that there are $n!/2^k k!(n-2k)!$ elements similar to $t_1(r)\cdot t_2(r)\cdot \cdots \cdot t_k(r)$. Hence

$$\frac{n(n-1)}{2}=\frac{n!}{2^k k!(n-2k)!}.$$

If $n\neq 6$ this equation is satisfied for no k $(k\geqq 1)$ except $k=1$.

Suppose now that $n\neq 6$. Then $A(1r)=(a_r b_r)$ say. If $r\neq 2$, $(12)(1r)=(12r)$ (multiplying from right to left), and evidently, $A(12r)=(a_2b_2)(a_rb_r)$. Since $(12r)$ has the order 3, so has $(a_2b_2)(a_rb_r)$ and the transpositions (a_2b_2) and (a_rb_r) must have a letter in common. Then it is no loss to assume $a_2=a_r$ or $b_2=b_r$. However, if $a_2=a_r$ and $b_2=b_s$ $(r\neq 2,\ s\neq 2)$, then $r\neq s$ and $A(12r)=A(12)\cdot A(1r)=(a_2b_2)(a_2b_r)=(b_ra_2b_2)$. Similarly $A(12s)=(a_sb_2a_2)$. Hence $A((12r)\cdot(12s))=A(12r)\cdot A(12s)=(b_ra_2b_2)(a_sb_2a_2)=(b_ra_sb_2)$ which is of order 3, while $(12r)\cdot(12s)=(1s)(2r)$, which is of order 2. Hence one must have $a_2=a_r$ for all r or $b_2=b_r$ for all r; of course one can let $a_2=a_r$ $(r=2,3,\cdots,n)$. Then $A(1r)=(a_2b_r)$. Hence A is precisely the automorphism A defined by $Ax=t^{-1}xt$, where

$$t=\begin{pmatrix}1 & 2 & \cdots & r & \cdots & n\\ a_2 & b_2 & \cdots & b_r & \cdots & b_n\end{pmatrix}.$$

For $Ax=t^{-1}xt$ when $x=(1r)$, and the elements $\{(1r)\}$ $(r=2,3,\cdots,n)$ generate S_n.

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[1] The first proof is by O. Hölder, Mathematische Annalen, vol. 46 (1895), especially pp. 340–345.

A CORRECTION TO "A NOTE ON LINEAR FUNCTIONALS"[1]

R. P. BOAS, JR., AND J. W. TUKEY

R. S. Phillips has called our attention to an error in our paper *A note on linear functionals*. On page 526, we have misquoted a theorem of Lebesgue's: the statement in the last display on that page is incorrect. It is, in fact, contradicted by the Riemann-Lebesgue theorem whenever the functions $x_n(t)$ are the elements of a uniformly bounded orthonormal set. Fortunately, however, the error does not affect the validity of any of our results. The correct consequence of Lebesgue's theorem is that

$$\sup_{0 \leqq n < \infty} \ \sup^0_{0 \leqq t \leqq 1} | x_n(t) | < \infty; \tag{1}$$

that is, that $\sup_{0 \leqq n < \infty} \|x_n\|_B < \infty$. From this it still follows that any linear functional on B is a linear functional on R; and we used our incorrect statement only to deduce this. This consequence is true in virtue of the following simple lemma.

LEMMA. *If a set* $\{x\}$ *forms a normed vector space under two norms,* $\|x\|$ *and* $\|x\|_B$, *and if* $\lim_{n\to\infty}\|x_n\| = 0$ *implies that* $\sup_{0 \leqq n < \infty}\|x_n\|_B < \infty$, *then any distributive functional continuous with respect to the second norm is also continuous with respect to the first norm.*

PROOF. Let f be a distributive functional, continuous with respect to the norm $\| \cdots \|_B$, so that for some number H,

$$| f(x) | \leqq H \|x\|_B \tag{2}$$

for every x. Suppose that f is not continuous with respect to the norm $\| \cdots \|$; then, as is well known (cf. S. Banach, *Théorie des Opérations Linéaires*, 1932, p. 55) there exist elements y_n such that $\|y_n\| = 1$, $|f(y_n)| > n$. The elements $z_n = n^{-1/2} y_n$ have the properties

$$\|z_n\| \to 0, \tag{3}$$

$$| f(z_n) | > n^{1/2}. \tag{4}$$

By hypothesis, (3) implies that $\|z_n\|_B < K$, $n = 0, 1, 2, \cdots$, for some finite K. Then, by (2), $|f(z_n)| \leqq HK$, contradicting (4) for large n.

DUKE UNIVERSITY AND
PRINCETON UNIVERSITY

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[1] This Bulletin, vol. 44 (1938), pp. 523–528.

THE APRIL MEETING IN BERKELEY

The three hundred sixty-seventh meeting of the American Mathematical Society was held at the University of California, Berkeley, California, on Saturday, April 6, 1940. The attendance was about eighty including the following forty-one members of the Society:

H. M. Bacon, G. A. Baker, Clifford Bell, B. A. Bernstein, H. F. Blichfeldt, J. L. Brenner, D. G. Chapman, C. L. Clark, P. H. Daus, L. E. Dickson, F. W. Dresch, G. C. Evans, A. L. Foster, E. C. Goldsworthy, E. R. Hedrick, A. E. Ingham, C. G. Jaeger, S. H. Levy, Hans Lewy, A. E. Marston, C. B. Morrey, F. R. Morris, Jerzy Neyman, A. R. Noble, C. A. Noble, C. D. Olds, W. G. Palm, W. T. Puckett, T. M. Putnam, S. E. Rauch, R. M. Robinson, E. B. Roessler, A. C. Schaeffer, W. H. Simons, Gabor Szegö, A. E. Taylor, J. V. Uspensky, F. A. Valentine, W. F. Whitmore, A. R. Williams, O. V. Wood.

General sessions were held both Saturday morning and afternoon. Professors Hedrick, Szegö and Blichfeldt presided over the meeting at various times. The following resolution was passed during the morning session:

"Those in attendance at this meeting of the American Mathematical Society wish to go on record as favoring that a program of mathematics be provided in the secondary schools, beginning normally with algebra in the ninth year, to be available for those who wish to elect it or who otherwise need it in preparation for college work.

"It is felt that a capable student should be able to secure solid geometry and trigonometry in the secondary school."

The titles and cross references to abstracts of papers read at the meeting are given below. Papers whose abstract numbers are followed by the letter *t* were read by title. Professor A. G. Clark and Mr. G. B. Dantzig were introduced by Professor Jerzy Neyman, Mr. J. R. Woolson, Mr. S. P. Avann, Mr. J. P. LaSalle, and Mr. A. B. Mewborn by Professor A. D. Michal, and Mr. R. W. Shephard and Mr. Lewis Nelson by Professor G. C. Evans.

1. J. L. Brenner: *Well-orderable abelian groups whose elements have prime-power order.* (Abstract 46-5-240.)

2. P. H. Daus: *Bisecting circles.* (Abstract 46-5-256.)

3. B. A. Bernstein: *Groups in terms of addition and negation.* (Abstract 46-5-236.)

4. C. D. Olds: *On the number of representations of the square of an integer as the sum of an odd number of squares.* (Abstract 46-5-309.)

5. G. B. Dantzig: *On the non-existence of tests of "Student's" hypothesis, the power function of which would be independent of σ.* (Abstract 46-5-255.)

6. F. W. Dresch: *A mathematical model of a dynamic economic system.* (Abstract 46-5-263.)

7. J. R. Woolson: *The mean of the iteration of linear operators in reflexive Banach spaces.* Preliminary report. (Abstract 46-5-351.)

8. A. C. Schaeffer and Gabor Szegö: *Inequalities for harmonic polynomials in two and three dimensions.* (Abstract 46-5-321.)

9. Lewis Nelson: *An Abel integral equation with constant limits of integration.* (Abstract 46-5-305.)

10. A. R. Williams: *On a certain Cremona transformation between two $(n-1)$-spaces in S_n.* (Abstract 46-5-350.)

11. W. T. Puckett: *On arc-preserving transformations.* (Abstract 46-5-316.)

12. R. W. Shephard: *The length of production and related dynamic aspects of a simplified economic system.* Preliminary report. (Abstract 46-5-323.)

13. S. P. Avann: *Lattice automorphisms.* (Abstract 46-5-227-*t*.)

14. J. P. LaSalle: *Applications of the pseudo-norm to the study of linear topological spaces.* (Abstract 46-5-290-*t*.)

15. A. D. Michal and Max Wyman: *Characterization of complex couple spaces.* (Abstract 46-5-299-*t*.)

16. G. B. Dantzig: *Existence of the Neyman-Pearson unbiased test of Type* A. (Abstract 46-5-254-*t*.)

17. A. B. Mewborn: *Abstract local geometry of paths.* II. (Abstract 46-5-297-*t*.)

18. E. T. Bell: *Note on a certain type of diophantine system.* (Abstract 46-5-233-*t*.)

19. E. T. Bell: *Postulational basis for the umbral calculus.* (Abstract 46-5-234-*t*.)

20. A. G. Clark: *Two sample problem concerning Poisson and binomial distributions.* Preliminary report. (Abstract 46-5-249-*t*.)

T. M. PUTNAM,
Associate Secretary

THE APRIL MEETING IN CHICAGO

The three hundred sixty-eighth meeting of the American Mathematical Society was held at the University of Chicago on Friday and Saturday, April 12–13, 1940. Two hundred twelve persons registered including the following one hundred eighty-one members of the Society:

A. A. Albert, G. E. Albert, P. H. Anderson, T. W. Anderson, Max Astrachan, J. V. Atanasoff, W. L. Ayres, Reinhold Baer, R. H. Bardell, R. W. Barnard, R. C. F. Bartels, R. A. Beaumont, J. H. Bell, H. R. Beveridge, S. F. Bibb, G. A. Bliss, Morris Bloom, Henry Blumberg, L. M. Blumenthal, F. L. Brooks, R. S. Burington, H. E. Burns, L. E. Bush, W. H. Bussey, J. W. Calkin, C. C. Camp, H. H. Campaigne, T. E. Caywood, E. W. Chittenden, R. V. Churchill, M. D. Clement, R. H. Cole, B. H. Colvin, Max Coral, J. J. Corliss, E. L. Crow, D. R. Curtiss, J. E. Davis, W. M. Davis, A. H. Diamond, J. M. Dobbie, J. L. Doob, D. M. Dribin, R. J. Duffin, Ben Dushnik, J. J. Eachus, Samuel Eilenberg, W. S. Erickson, E. B. Escott, H. P. Evans, H. S. Everett, L. R. Ford, Bernard Friedman, D. G. Fulton, H. L. Garabedian, E. R. Garbe, R. E. Gaskell, B. H. Gere, F. J. Gerst, B. C. Getchell, H. A. Giddings, E. L. Godfrey, H. H. Goldstine, G. D. Gore, Cornelius Gouwens, L. M. Graves, J. W. Green, L. W. Griffiths, V. G. Grove, O. H. Hamilton, P. C. Hammer, M. L. Hartung, G. E. Hay, K. E. Hazard, O. C. Hazlett, A. E. Heins, E. D. Hellinger, M. R. Hestenes, J. F. Heyda, T. H. Hildebrandt, J. J. L. Hinrichsen, D. L. Holl, A. S. Householder, D. H. Hyers, M. H. Ingraham, S. B. Jackson, Fritz John, Chosaburo Kato, J. F. Kenney, W. C. Krathwohl, A. C. Ladner, R. E. Langer, Lincoln LaPaz, D. H. Leavens, R. A. Leibler, J. H. Levin, P. E. Lewis, A. O. Lindstrum, M. I. Logsdon, A. T. Lonseth, W. C. McDaniel, C. C. MacDuffee, H. M. MacNeille, Morris Marden, J. R. Mayor, Karl Menger, H. L. Meyer, A. N. Milgram, D. D. Miller, W. E. Milne, W. L. Mitchell, M. G. Moore, R. L. Moore, C. W. Moran, E. J. Moulton, A. L. Nelson, H. E. Newell, K. L. Nielsen, I. M. Niven, E. A. Nordhaus, M. A. Norval, E. N. Oberg, J. W. Odle, Rufus Oldenburger, E. W. Paxson, J. F. Paydon, E. R. Peck, G. H. Peebles, I. E. Perlin, Sam Perlis, G. B. Price, Tibor Radó, E. D. Rainville, R. B. Rasmusen, W. T. Reid, C. E. Rickart, R. F. Rinehart, W. E. Roth, E. H. Rothe, M. A. Sadowsky, R. G. Sanger, W. T. Scott, C. E. Sealander, M. E. Shanks, C. G. Shover, H. A. Simmons, M. M. Slotnick, F. C. Smith, G. G. Speeker, D. W. Starr, N. E. Steenrod, B. M. Stewart, H. P. Thielman, R. M. Thrall, Gerhard Tintner, L. W. Tordella, Leonard Tornheim, H. C. Trimble, W. J. Trjitzinsky, P. L. Trump, J. W. Tukey, V. J. Varino, J. I. Vass, H. E. Vaughan, H. S. Wall, Henry Wallman, T. O. Walton, L. E. Ward, K. W. Wegner, M. J. Weiss, C. P. Wells, M. E. Wescott, G. W. Whaples, G. W. Whitehead, L. R. Wilcox, K. P. Williams, L. A. Wolf, M. C. Wolf, Y. K. Wong, F. E. Wood, J. W. T. Youngs.

On Friday morning the meeting was held in three sections, Analysis, Professor R. E. Langer presiding; Geometry and Algebra, Professor Reinhold Baer presiding; and Topology and the Theory of Sets, Professor N. E. Steenrod presiding. On Friday afternoon Professor W. J. Trjitzinsky gave the Symposium Lecture entitled *General the-*

ory of functions of a complex variable. Professor E. W. Chittenden presided at this lecture. On Saturday morning there were two sections, Analysis with Professor W. T. Reid presiding and Algebra with Professor C. C. MacDuffee presiding.

All sessions were held in Eckhart Hall and the adjoining Ryerson Laboratory.

On Friday afternoon a tea was given by the ladies of the Department of Mathematics of the University of Chicago in the Commons Room of Eckhart Hall.

On Friday evening a dinner was held at Burton Court for members and guests with an attendance of one hundred sixty-eight. Professor H. S. Everett of the University of Chicago acted as toastmaster, and brief talks were given by Professors W. H. Bussey, Marie J. Weiss, G. B. Price, L. R. Ford, and R. L. Moore.

Titles and cross references to the abstracts of the papers read at this meeting follow below. Papers numbered 1 to 9 were read before the Analysis section Friday morning, papers 10 to 16 before the section for Geometry and Algebra, papers 17 to 24 before the section for Topology and the Theory of Sets, papers 25 to 32 before the Analysis section Saturday morning, and papers 33 to 39 before the Algebra section Saturday morning. Papers 40 to 61, whose abstract numbers are followed by the letter *t*, were read by title. Mr. F. P. Jenks and Mr. J. C. Abbott were introduced by Professor Karl Menger, Dr. W. S. Snyder by Professor Tibor Radó, Mr. H. J. Miser by Professor L. R. Ford, and Mr. Roy Dubisch by Professor A. A. Albert. Paper number 2 was read by Dr. Duffin, 3 by Dr. Bartels, 4 by Dr. Leibler, 18 by Professor Dushnik, 29 by Dr. Reichelderfer, and 37 by Mr. Porges.

1. R. E. Gaskell: *A problem in heat conduction and an expansion theorem.* (Abstract 46-5-271.)

2. R. J. Duffin and J. J. Eachus: *The converse of a closure theorem of Paley and Wiener.* (Abstract 46-5-265.)

3. R. C. F. Bartels and R. V. Churchill: *An extension of Duhamel's theorem.* (Abstract 46-5-230.)

4. J. L. Doob and R. A. Leibler: *On the spectral analysis of a certain transformation.* Preliminary report. (Abstract 46-5-261.)

5. Fritz John: *The Dirichlet problem for a hyperbolic equation.* (Abstract 46-5-284.)

6. R. V. Churchill: *Termwise integration of Sturm-Liouville expansions.* Preliminary report. (Abstract 46-5-247.)

7. A. E. Heins: *On the solution of partial difference equations: symmetric boundary conditions.* (Abstract 46-5-277.)

8. K. L. Nielsen: *Concerning general boundary value problems for linear differential equations.* Preliminary report. (Abstract 46-5-307.)

9. Gerhard Tintner: *Fourier integrals as Bessel differential operators.* Preliminary report. (Abstract 46-5-333.)

10. F. P. Jenks: *Order and parallelism in the non-euclidean geometry of joining and intersecting.* (Abstract 46-5-283.)

11. J. C. Abbott: *Projective theory of congruence in non-euclidean geometry.* Preliminary report. (Abstract 46-5-225.)

12. L. M. Blumenthal: *A new concept in distance geometry, with applications to spherical subsets.* (Abstract 46-7-354.)

13. Sam Perlis: *Scalar extensions of algebras with exponent equal to index.* (Abstract 46-5-315.)

14. D. M. Dribin: *The norm residue symbol for solvable algebraic number fields.* (Abstract 46-5-264.)

15. W. L. Mitchell: *Topological rings and infinite matrices.* (Abstract 46-5-302.)

16. V. J. Varino: *A note on matrices over a principal ideal ring.* (Abstract 46-5-335.)

17. E. W. Paxson: *Strictly convex metric spaces.* (Abstract 46-5-312.)

18. Ben Dushnik and E. W. Miller: *On partially ordered sets.* (Abstract 46-5-266.)

19. A. N. Milgram: *Partially ordered sets and the covering theorems of topology.* (Abstract 46-5-301.)

20. Samuel Eilenberg: *Ordered topological spaces.* (Abstract 46-5-268.)

21. G. E. Albert: *On contiguous point spaces and their applications.* (Abstract 46-3-138.)

22. D. D. Miller: *Hereditary properties of continuous transformations.* Preliminary report. (Abstract 46-5-211.)

23. J. W. T. Youngs: *On parametric representations of surfaces.* Preliminary report. (Abstract 46-5-352.)

24. R. L. Moore: *Concerning separability.* (Abstract 46-5-303.)

25. H. P. Thielman: *On the convex solution of a certain functional equation.* (Abstract 46-5-331.)

26. H. H. Goldstine: *Linear functionals and integrals in abstract spaces.* (Abstract 46-5-273.)

27. Tibor Radó: *On a lemma of McShane.* (Abstract 46-3-150.)

28. Karl Menger: *On Green's formula and the integral of derivatives.* (Abstract 46-5-210.)

29. Tibor Radó and P. V. Reichelderfer: *Note on an inequality of Steiner.* (Abstract 46-3-184.)

30. W. S. Snyder: *Functions of simple figures.* Preliminary report. (Abstract 46-5-325.)

31. J. W. Calkin: *A quotient ring over the ring of bounded operators in Hilbert space.* I. (Abstract 46-5-203.)

32. H. J. Miser: *Regions and their "patterns" in conformal mapping.* (Abstract 46-7-365.)

33. R. A. Beaumont: *Projections of non-abelian groups upon abelian groups.* Preliminary report. (Abstract 46-5-232.)

34. R. M. Thrall: *A note on a theorem by Witt.* (Abstract 46-5-332.)

35. P. E. Lewis: *Characters of abelian groups.* (Abstract 46-5-291.)

36. H. H. Campaigne: *Multiplication systems.* Preliminary report. (Abstract 46-5-242.)

37. Rufus Oldenburger and A. E. Porges: *The minimal numbers of binary forms.* (Abstract 46-5-212.)

38. Leonard Tornheim: *Linear forms in function fields.* (Abstract 46-3-196.)

39. Roy Dubisch: *Non-cyclic algebras of degree four and exponent two with pure maximal subfields.* (Abstract 46-5-206.)

40. H. L. Garabedian and H. S. Wall: *Hausdorff matrices and continued fractions.* (Abstract 46-3-171-*t*.)

41. G. E. Albert: *On quasi-metric spaces.* Preliminary report. (Abstract 46-3-154-*t*.)

42. Y. K. Wong: *On non-modular matrices.* (Abstract 46-5-224-*t*.)

43. Y. K. Wong: *Interchange of J-processes in bilinear forms involving non-modular matrices.* (Abstract 46-5-223-*t*.)

44. Gordon Pall: *On the arithmetic of quaternions.* (Abstract 46-5-215-*t*.)

45. J. W. Calkin: *A generalization of a theorem of Weyl.* (Abstract 46-5-204-*t*.)

46. J. W. Calkin: *Functions of several variables and absolute continuity.* I. (Abstract 46-5-205-*t*.)

47. Rufus Oldenburger: *On a class of non-negative matrices.* (Abstract 46-5-214-*t*.)

48. Rufus Oldenburger: *Polynomials in several variables.* (Abstract 46-5-213-*t*.)

49. M. S. Webster: *Maximum of certain fundamental Lagrange interpolation polynomials.* (Abstract 46-5-342-*t*.)

50. C. W. Vickery: *On spaces* ($\mathfrak{E}$) *and Moore spaces.* (Abstract 46-5-337-*t*.)

51. Reinhold Baer: *Sylow theorems for infinite groups.* (Abstract 46-5-229-*t*.)

52. Reinhold Baer: *Nets and groups.* II. (Abstract 46-5-228-*t*.)

53. Karl Menger: *On shortest polygonal approximations to a curve.* (Abstract 46-5-296-*t*.)

54. A. N. Milgram: *On the length of continuous curves.* (Abstract 46-5-300-*t*.)

55. Samuel Eilenberg: *On spherical cycles.* (Abstract 46-5-267-*t*.)

56. W. H. Ingram: *A generalization of Erhard Schmidt's solution of the nonhomogeneous integral equation.* Preliminary report. (Abstract 46-5-282-*t*.)

57. I. E. Perlin: *Sufficient conditions that polynomials in several variables be positive.* (Abstract 46-5-314-*t*.)

58. H. H. Campaigne: *A lower limit on the number of hypergroups of a given order*. (Abstract 46-5-241-*t*.)

59. J. M. Feld: *Whirl-similitudes, euclidean kinematics and non-euclidean geometry*. Preliminary report. (Abstract 46-5-270-*t*.)

60. R. V. Churchill: *A problem in the conduction of heat.* (Abstract 46-5-246-*t*.)

61. R. L. Wilder: *Characterization of the lower dimensional generalized manifolds by positional properties in S_n.* (Abstract 46-5-349-*t*.)

W. L. AYRES,
Associate Secretary

THE APRIL MEETING IN WASHINGTON

The three hundred sixty-ninth meeting of the American Mathematical Society was held at The George Washington University, Washington, D. C., on Friday and Saturday, April 26–27, 1940. The attendance included the following one hundred sixty members of the Society:

C. R. Adams, C. B. Allendoerfer, R. C. Archibald, J. V. Atanasoff, Harry Bateman, E. E. Betz, Archie Blake, R. P. Boas, A. T. Brauer, Richard Brauer, R. S. Burington, Herbert Busemann, W. E. Byrne, S. S. Cairns, J. W. Calkin, W. B. Campbell, C. E. Carey, Randolph Church, Paul Civin, J. M. Clarkson, G. R. Clements, Abraham Cohen, I. S. Cohen, Nancy Cole, M. J. Cox, H. S. M. Coxeter, P. D. Crout, H. B. Curry, Tobias Dantzig, C. H. Dowker, Arnold Dresden, J. E. Eaton, Samuel Eilenberg, L. P. Eisenhart, M. L. Elveback, Paul Erdös, W. K. Feller, E. J. Finan, D. A. Flanders, W. W. Flexner, R. M. Foster, Hilda Geiringer, Abe Gelbart, P. W. Gilbert, Wallace Givens, Michael Goldberg, Michael Golomb, B. L. Hagen, D. W. Hall, O. G. Harrold, Philip Hartman, G. A. Hedlund, Edward Helly, Olaf Helmer, J. G. Herriot, Einar Hille, M. P. Hollcroft, T. R. Hollcroft, Witold Hurewicz, Nathan Jacobson, S. A. Jennings, Evan Johnson, R. F. Johnson, F. E. Johnston, H. A. Jordan, E. R. van Kampen, Wilfred Kaplan, J. L. Kelley, S. C. Kleene, J. R. Kline, E. R. Kolchin, H. L. Krall, W. D. Lambert, O. E. Lancaster, A. E. Landry, Solomon Lefschetz, D. T. McClay, N. H. McCoy, Brockway McMillan, E. J. McShane, H. M. MacNeille, Dorothy Manning, A. V. Martin, M. H. Martin, F. M. Mears, A. D. Michal, Don Mittleman, T. W. Moore, R. C. Morrow, Marston Morse, D. C. Murdoch, F. D. Murnaghan, F. G. Myers, John von Neumann, C. O. Oakley, Oystein Ore, F. W. Owens, H. B. Owens, J. C. Oxtoby, N. G. Parke, G. W. Patterson, P. M. Pepper, R. S. Phillips, H. A. Rademacher, O. J. Ramler, H. J. Riblet, J. N. Rice, R. G. D. Richardson, J. F. Ritt, J. H. Roberts, H. P. Robertson, G. de B. Robinson, L. B. Robinson, R. E. Root, J. E. Rosenthal, Barkley Rosser, O. F. G. Schilling, Abraham Schwartz, G. E. Schweigert, C. H. W. Sedgewick, I. E. Segal, Wladimir Seidel, Abraham Seidenberg, L. W. Sheridan, R. G. Simond, Abraham Sinkov, H. L. Smith, J. P. Smith, P. A. Smith, Virgil Snyder, F. W. Sohon, V. E. Spencer, J. M. Stetson, E. B. Stouffer, J. L. Synge, J. H. Taylor, J. M. Thomas, E. W. Titt, C. B. Tompkins, A. W. Tucker, L. B. Tuckerman, J. W. Tukey, Oswald Veblen, G. L. Walker, A. D. Wallace, J. V. Wehausen, F. M. Weida, Albert Wertheimer, F. J. Weyl, Hermann Weyl, A. P. Wheeler, C. H. Wheeler, P. A. White, A. L. Whiteman, G. T. Whyburn, S. S. Wilks, John Williamson, H. A. Wood, J. W. Wrench, J. W. T. Youngs.

The meeting opened Friday morning in two sections, Analysis, Professor J. L. Synge presiding, and Topology, Professor A. D. Michal presiding.

At the general session Friday afternoon, Vice President F. D. Murnaghan presiding, Professor A. W. Tucker of Princeton University gave an address entitled *Complexes and tensors*. An informal meeting of those especially interested in the application of tensor analysis to electrical engineering was held immediately following this address.

The general session was followed by two sections: Algebra, Professor Hermann Weyl presiding; Applied Mathematics, Geometry and Logic, Professor S. C. Kleene presiding.

Two general sessions were held Saturday morning. At the first, Professor Arnold Dresden presiding, Professor Wladimir Seidel of the University of Rochester gave an address entitled *Univalent and multivalent functions*. The second, Vice President C. R. Adams presiding, was for short papers on various subjects.

All sessions were held in the Hall of Government, The George Washington University.

On Friday afternoon, tea was served in Columbian House, Mrs. J. H. Taylor, hostess.

The dinner Friday evening at the Cosmos Club was attended by one hundred thirty-eight mathematicians and guests. The toastmaster, Professor Marston Morse, introduced President C. H. Marvin of The George Washington University who welcomed the mathematicians and generously invited the Society to hold its meetings at The George Washington University whenever it chose to do so. President Marvin was followed by three speakers, Professors Oswald Veblen, Oystein Ore and E. J. McShane, who spoke briefly on *The Mathematical Reviews*, *Art and mathematics*, and *Mathematics in South America*, respectively.

The Council met at 9:00 P.M., Friday, April 26, in the Holmes Room of the Cosmos Club.

At the general session Saturday morning Professor Barkley Rosser presented a resolution expressing the appreciation and thanks of the Society to the President of The George Washington University, the members of the local committee, the Mathematics Department and all others concerned for their competent arrangements and cordial hospitality. This resolution was unanimously adopted.

The Secretary announced the election of the following seventy-three persons to membership in the Society:

Mr. Lucius Anthony, Seattle, Wash.;
Professor José Barral-Souto, University of Buenos Aires;
Dr. Alfred Basch, Holy Cross College, Worcester, Mass.;
Mr. Jacob Bearman, University of Minnesota;
Professor Richard A. Beth, Michigan State College;
Dr. Willard E. Bleick, Gibbs and Cox, Incorporated, New York, N. Y.;
Mr. Augusto Bobonis, University of Puerto Rico;
Professor Alexander William Boldyreff, University of Arizona;
Dr. Achille Capecelatro, New York, N. Y.;

Mr. George P. Cowan, St. John's University, Brooklyn, N. Y.;
Miss Mary Elizabeth Agnes Dever, Manhattanville College of the Sacred Heart;
Dean Alfonso Elder, North Carolina College for Negroes, Durham, N. C.;
Miss Frances E. Falvey, Ward-Belmont School, Nashville, Tenn.;
Professor Joseph Homer Forrest, Langston University, Langston, Okla.;
Mr. Franklin H. Fowler, Jr., New York, N. Y.;
Mr. Arthur Tapley French, State Teachers College, Winona, Minn.;
Mr. John Raymond Fritz, Wyomissing Polytechnic Institute, Wyomissing, Pa.;
Mr. Robert Elston Fullerton, Syracuse University;
Professor Andre Gleyzal, St. Michael's College, Winooski Park, Vt.;
Dr. Hans-Karl Hammer, Texas Company Research Laboratories, Beacon, N. Y.;
Mr. Wesley Turnell Hanson, Technical High School, Atlanta, Ga.;
Dr. George Edward Hay, Armour Institute of Technology;
Dr. Edward Helly, New York, N. Y.;
Dr. Morris Hendrickson, Ohio State University;
Mrs. Frances Louise Campbell Hinds, George Pepperdine College, Los Angeles, Calif.;
Sister Mary Charlotte Holland, St. Xavier College for Women, Chicago, Ill.;
Professor Elvin Albert Hoy, University of Hawaii;
Mr. Robert Sigurd Jacobson, Luther College;
Mr. E. A. Johnson, Phoenix, Ariz.;
Mr. Andrew Richard Kirby, College of the City of New York and Fordham University;
Mr. Calvin John Kirchen, Itasca Junior College, Coleraine, Minn.;
Mr. Stephen Anthony Kiss, Standard Oil Development Company, New York, N. Y.;
Mr. Richard Frederick Koch, Bard College, Columbia University;
Mr. Paul Herbert Kratz, Temple University;
Mr. Joseph S. Leech, College of Mines and Metallurgy, El Paso, Texas;
Mr. John N. B. Livingood, Gettysburg College;
Mr. Jack Lorell, Brown University;
Mr. Charles Edward McCauley, Woodstock College, Woodstock, Md.;
Mr. Charles B. McMullen, State Teachers College, Hyannis, Mass.;
Dr. Joanna Mayer, Xavier University, New Orleans, La.;
Mr. Herman Meyer, University of Chicago;
Professor José M. Mijares, Mapua Institute of Technology, Manila, P. I.;
Mr. Fred A. Miller, Wentworth Military Academy, Lexington, Mo.;
Dr. Richard von Mises, Harvard University;
Mr. Nathan Morrison, Division of Placement of Unemployment Insurance, New York State Department of Labor, New York, N. Y.;
Professor Mario John Nardelli, Fordham University;
Professor Edwin Burman Ogden, Union College, Lincoln, Neb.;
Mr. William Gray Palm, Jr., University of Nevada;
Mr. George William Patterson, National Bureau of Standards, Washington, D. C.;
Professor William Anderson Patterson, Fenn College, Cleveland, Ohio;
Professor Francisco Perez, University of the Philippines;
Miss Edris Pauline Rahn, Hayward Union High School, Hayward, Calif.;
Reverend John H. Raymond, St. Martin's College, Lacey, Wash.;
Dr. Paul V. Reichelderfer, Ohio State University;
Dr. Irving F. Ritter, New York University;
Mr. Walter C. Roberts, Glendale Junior College, Glendale, Calif.;
Professor Alexander Russ, Mississippi Woman's College, Hattiesburg, Miss.;

Professor Thomas Justin Saunders, University of San Francisco;
Reverend Maurice Andrew Scheier, St. Bonaventure College, St. Bonaventure, N. Y.;
Dr. Herman M. Schwartz, Duquesne University;
Dr. Andrew Sobczyk, Oregon State College;
Miss Ethel Sutherland, Teachers College, Columbia University;
Mr. Francis Xavier Sutton, Princeton University;
Professor Hubert M. Thaxton, Agricultural and Technical College of North Carolina, Greensboro, N. C.;
Dr. George Raymond Thurman, University of Missouri;
Professor Gerhard Tintner, Iowa State College;
Mr. Louis Bryant Tuckerman, III, Princeton University;
Professor Enrique T. Virata, University of the Philippines;
Dr. Abraham Wald, Columbia University;
Professor Marjorie Watson, Huntingdon College, Montgomery, Ala.;
Mr. William Wernick, New York, N. Y.;
Mr. Hugh S. Westbrook, U. S. Army Air Corps, Chicago, Ill.;
Mr. John Davis Williams, Princeton University.

It was reported that the following had been elected as nominees on the Institutional Memberships of the institutions indicated:

California Institute of Technology: Messrs. J. P. LaSalle, A. B. Mewborn, and J.R. Woolson.
Duke University: Messrs. Paul Civin and Milton Preston Jarnagin, Jr.
Pennsylvania State College: Messrs. Paul M. Kendig (Schuylkill Undergraduate Center, Pottsville, Pa.) and H. H. Reichard (Hazleton Undergraduate Center, Hazleton, Pa.).
Wayne University: Mr. Herbert Kapfel Brown.
Yale University: Messrs. Bengt Edvin Carlson, Eli Sherman Grable, and Wendell Gayton Swope.

It was announced that Dr. Joseph Gillis of Queen's University, Belfast, Northern Ireland, and Professor John Greenlees Semple of King's College, University of London, have been admitted to membership in the Society in accordance with the reciprocity agreement with the London Mathematical Society.

The Secretary announced that the meeting in Seattle, arranged to be held with the American Association for the Advancement of Science, is set for June 20, 1940. It was voted to hold an autumn meeting at Wayne University on November 22–23, 1940, and another at the University of California at Los Angeles on November 23; also a spring meeting at the University of Chicago on April 11–12, 1941. An invitation to hold the Annual Meeting of 1941 at the University of Florida was received.

The following appointments by President G. C. Evans to the four Committees on Invited Addresses were reported: for the Annual and Summer Meetings, Professor L. M. Graves (two years); for Eastern Meetings, Professors Einar Hille (two years) and J. L. Walsh (one year to replace Dean L. P. Eisenhart, resigned); for Western Meetings, Professor E. W. Chittenden (two years); for Far Western Meetings, Professor Gabor Szegö (two years).

On recommendation of the Editorial Committee on Colloquium, a book by Professor Norman Levinson entitled *Tauberian Theorems and Gap Theorems* was accepted for publication in the Colloquium Series.

A special and unusual program is being arranged for the Summer Meeting of 1941 to be held at the University of Chicago in connection with the fiftieth anniversary of the founding of that institution. The following Committee on Arrangements was appointed: Professors L. M. Graves (chairman), A. A. Albert, W. L. Ayres, G. A. Bliss, W. D. Cairns, L. R. Ford, E. J. Moulton, and Dr. R. G. Sanger. A special Committee on Program was also appointed, consisting of Professors A. A. Albert (chairman), W. L. Ayres, G. A. Bliss, and T. H. Hildebrandt.

Titles and cross references to the abstracts of the papers read follow below. Papers whose abstract numbers are followed by the letter *t* were read by title. The papers numbered 1 to 8 were read before the section for Analysis; those numbered 9 to 18 before the section for Topology; those numbered 19 to 24 before the section for Algebra; those numbered 25 to 30 before the section for Applied Mathematics, Geometry and Logic; those numbered 31 to 36 before the general session on Saturday morning; and papers numbered 37 to 92 were read by title. Paper 11 was read by Dr. Eilenberg, paper 29 by Professor Crout, and paper 33 by Dr. Schilling. Dr. Bergman was introduced by Professor W. T. Martin, Dr. Hua by Professor Hermann Weyl, Mr. Hurwitz by Professor G. D. Birkhoff, and Mr. Western by Dr. D. W. Hall.

1. Paul Civin: *Inequalities for trigonometric integrals.* Preliminary report. (Abstract 46-5-248.)

2. F. J. Weyl: *On the defect relation for meromorphic curves.* (Abstract 46-5-346.)

3. Stefan Bergman: *On the surface integrals of functions of two complex variables.* (Abstract 46-5-235.)

4. H. L. Krall: *Orthogonal polynomial solutions of a certain fourth order differential equation.* (Abstract 46-5-289.)

5. E. W. Titt: *On the characteristic theory of a first order partial differential equation.* (Abstract 46-7-367.)

6. R. P. Boas: *Univalent derivatives of entire functions.* (Abstract 46-5-238.)

7. Wilfred Kaplan: *Singularities of a curve-family on a surface, with applications to differential equations.* (Abstract 46-5-285.)

8. Michael Golomb: *On the boundary-value problems of the equation* $\Delta u = g(x, y)$ *in the infinite half-strip.* (Abstract 46-5-274.)

9. J. L. Kelley: *On the hyperspaces of a continuum.* (Abstract 46-5-287.)

10. P. M. Pepper: *Concerning pseudo-planar-quintuples.* (Abstract 46-5-313.)

11. Samuel Eilenberg and R. L. Wilder: *Uniform local contractibility.* (Abstract 46-5-269.)

12. A. D. Wallace: *On separation spaces.* (Abstract 46-7-368.)

13. G. E. Schweigert: *Equivalence of pointwise periodic and interior transformations on dendrites.* (Abstract 46-5-322.)

14. S. S. Cairns: *Triangulated manifolds which are not Brouwer manifolds.* (Abstract 46-7-355.)

15. O. G. Harrold (National Research Fellow): *Minimal coverings of Peano spaces by maps of a circle.* (Abstract 46-5-276.)

16. I. E. Segal: *Certain topological rings.* (Abstract 46-7-366.)

17. P. A. White: *On certain relatively non-alternating transformations.* (Abstract 46-5-347.)

18. D. W. Hall: *On arc and tree preserving transformations.* (Abstract 46-7-360.)

19. H. J. Riblet: *Factorization of differential ideals in an algebraic differential field.* (Abstract 46-5-318.)

20. H. S. M. Coxeter: *The binary polyhedral groups, and other generalizations of the quaternion group.* (Abstract 46-5-252.)

21. Olaf Helmer: *A new type of transcendence proof.* (Abstract 46-5-278.)

22. Richard Brauer: *On the Cartan invariants of a group of finite order.* (Abstract 46-5-239.)

23. J. E. Eaton: *On the extension of groups.* I. *Normal extensions.* II. *Non-normal extensions.* (Abstract 46-7-359.)

24. D. C. Murdoch: *A characterization of abelian quasi-groups.* (Abstract 46-5-304.)

25. M. H. Martin: *Quasi-Lagrangian systems.* Preliminary report. (Abstract 46-7-364.)

26. J. M. Clarkson: *An involutorial line transformation associated with a quadric.* (Abstract 46-5-250.)

27. Harry Bateman: *Hulthen's integral equation.* Preliminary report. (Abstract 46-5-231.)

28. H. B. Curry: *A formulation of recursive arithmetic.* (Abstract 46-5-253.)

29. F. B. Hildebrand and P. D. Crout: *A least square procedure for*

improving a method for solving integral equations by polynomial approximation. (Abstract 46-3-175.)

30. D. T. McClay: *On certain manifolds of somas.* (Abstract 46-5-292.)

31. J. H. Roberts: *A theorem on dimension.* (Abstract 46-5-319.)

32. J. L. Synge: *On the electromagnetic two-body problem.* (Abstract 46-5-329.)

33. Saunders MacLane and O. F. G. Schilling: *Normal algebraic number fields.* (Abstract 46-5-294.)

34. E. J. McShane: *On the second variation in certain anormal problems of the calculus of variations.* (Abstract 46-5-209.)

35. Oystein Ore: *The extension problem for groups.* (Abstract 46-5-311.)

36. A. D. Michal: *Higher order differentials of functions with arguments and values in topological abelian groups.* (Abstract 46-5-298.)

37. R. P. Agnew: *Some remarks on a paper entitled "General Tauberian theorems."* (Abstract 46-5-226-*t*.)

38. Stefan Bergman: *On a generalized Green's function and certain of its applications.* (Abstract 46-7-353-*t*.)

39. R. P. Boas: *Expansions of analytic functions.* (Abstract 46-5-237-*t*.)

40. Leonard Carlitz: *A set of polynomials.* (Abstract 46-5-243-*t*.)

41. Leonard Carlitz: *On certain exponential sums.* (Abstract 46-5-244-*t*.)

42. Randolph Church: *Numerical analysis of certain free distributive structures.* (Abstract 46-5-245-*t*.)

43. Nancy Cole: *The index theorem for a calculus of variations problem in which the integrand is discontinuous.* (Abstract 46-5-251-*t*.)

44. J. L. Coolidge: *Analytic systems of central conics in space.* (Abstract 48-7-356-*t*.)

45. Richard Courant: *On a generalized form of Plateau's problem.* (Abstract 46-7-357-*t*.)

46. M. M. Day: *A norm ergodic theorem.* (Abstract 46-5-257-*t*.)

47. M. M. Day: *Linear methods of summability.* II. (Abstract 46-5-258-*t*.)

48. A. González Domínguez: *Some theorems on the Hermite singular kernel.* (Abstract 46-7-358-*t*.)

49. J. J. DeCicco: *The affinelong near-Laguerre transformations.* (Abstract 46-5-259-*t*.)

50. E. L. Dodd: *The substitutive mean and certain subclasses of this general mean.* (Abstract 46-5-260-*t*.)

51. Jesse Douglas: *A new special form of the linear element of a surface.* (Abstract 46-5-262-*t*.)

52. Abe Gelbart: *On the growth of a function of two complex variables satisfying certain partial differential equations.* (Abstract 46-5-272-*t*.)

53. O. G. Harrold (National Research Fellow): *Characterizations of a class of continua by means of continuous functions.* (Abstract 46-5-275-*t*.)

54. M. H. Heins: *A note on a theorem of Radó.* (Abstract 46-5-207-*t*.)

55. L. K. Hua: *On Waring's problem for cubic polynomial summands.* (Abstract 46-5-280-*t*.)

56. L. K. Hua: *On the number of partitions of a number into unequal parts.* (Abstract 46-5-279-*t*.)

57. Henry Hurwitz: *On certain wave equations analogous to the Dirac equations.* (Abstract 46-5-281-*t*.)

58. Edward Kasner and J. J. DeCicco: *Families of curves conformally equivalent to circles.* (Abstract 46-5-286-*t*.)

59. B. O. Koopman: *Intuitive probability and sequences.* (Abstract 46-5-288-*t*.)

60. A. N. Lowan and Hyman Serbin: *On upper bounds for the remainder in the evaluation of differences.* (Abstract 46-7-363-*t*.)

61. A. N. Lowan and Jack Laderman: *On the distribution of errors in the computation of tables by differences.* (Abstract 46-7-362-*t*.)

62. E. J. McShane: *On the theory of relative extrema.* (Abstract 46-5-293-*t*.)

63. Margaret P. Martin: *A sequence of tests for the convergence and divergence of infinite series.* (Abstract 46-5-295-*t*.)

64. Philip Newman: *The geometry of the* (2, 2) *planar connex.* Preliminary report. (Abstract 46-5-306-*t*.)

65. Rufus Oldenburger: *Binary forms.* (Abstract 46-5-308-*t*.)

66. Oystein Ore: *Remarks on structures and group relations.* (Abstract 46-5-310-*t*.)

67. H. A. Rademacher and A. L. Whiteman: *On Dedekind sums.* (Abstract 46-5-317-*t*.)

68. Barkley Rosser: *An additional criterion for the first case of Fermat's last theorem.* (Abstract 46-5-320-*t*.)

69. Max Shiffman: *The Morse relations in the Plateau problem for several boundaries.* (Abstract 46-5-324-*t*.)

70. A. R. Schweitzer: *Concerning general abstract relational spaces.* (Abstract 46-3-190-*t*.)

71. D. C. Spencer: *On finitely mean valent functions.* (Abstract 46-5-326-*t*.)

72. D. C. Spencer: *On finitely mean valent functions.* II. (Abstract 46-5-327-*t*.)

73. Alvin Sugar: *On a result of Hua for cubic polynomials.* (Abstract 46-5-328-*t*.)

74. Olga Taussky and John Todd: *On determinants of quaternions.* (Abstract 46-5-330-*t*.)

75. W. J. Trjitzinsky: *Developments in the analytic theory of algebraic differential equations.* (Abstract 46-5-334-*t*.)

76. C. W. Vickery: *On cyclically invariant graduation.* (Abstract 46-5-336-*t*.)

77. T. L. Wade: *Subgeometries of projective geometries as theories of tensors.* (Abstract 46-5-338-*t*.)

78. H. S. Wall: *A class of power series bounded in the unit circle.* (Abstract 46-5-219-*t*.)

79. A. D. Wallace: *An analysis of non-alternating transformations.* (Abstract 46-5-339-*t*.)

80. A. D. Wallace: *On certain classes of subcontinua.* (Abstract 46-5-340-*t*.)

81. J. L. Walsh: *Note on overconvergent power series.* (Abstract 46-5-220-*t*.)

82. J. L. Walsh and W. E. Sewell: *On the degree of convergence of harmonic polynomials to harmonic functions.* (Abstract 46-5-341-*t*.)

83. J. V. Wehausen: *A space isomorphic to Hilbert space.* (Abstract 46-7-370-*t*.)

84. Louis Weisner: *Moduli of the roots of polynomials and power series.* (Abstract 46-5-343-*t*.)

85. Louis Weisner: *Power series, the roots of whose partial sums lie in a sector.* (Abstract 46-5-344-*t*.)

86. D. W. Western: *A method for analytic continuation.* (Abstract 46-5-345-*t*.)

87. Hassler Whitney: *On regular families of curves.* (Abstract 46-5-348-*t*.)

88. G. T. Whyburn: *Mapping theorems.* (Abstract 46-5-221-*t*.)

89. G. T. Whyburn: *On non-alternating transformations.* (Abstract 46-5-222-*t*.)

90. G. T. Whyburn: *On multicoherence.* (Abstract 46-7-371-*t*.)

91. G. T. Whyburn: *On singularities and inversion under interior transformations.* (Abstract 46-7-372-*t*.)

92. G. T. Whyburn: *Transformations on boundary curves.* (Abstract 46-7-373-*t*.)

T. R. HOLLCROFT,
Associate Secretary

necessary analytical tools: representation of nonnegative polynomials, bounds for the derivative of a polynomial or trigonometric sum, trigonometric and polynomial approximation of continuous functions, the concept of closure, linear functional operators, the Gamma-function, Bessel's functions. We also find here interesting "comparison theorems" of Sturm's type for linear differential equations. Chapter II treats the existence of OP and their representation in determinantal and multiple-integral form. The starting point here is the orthogonality property. The general principles are illustrated on the classical OP—of Jacobi, Hermite and Laguerre—also on OP corresponding to some special distribution-functions absolutely continuous and discrete. Chapter III deals with the general properties of OP: some extremal properties, closure, integral approximation in the sense of the least squares and p-powers, the all-important recurrence-relation, Darboux' formula. We also find here a discussion of the elementary properties of the zeros of OP and the corresponding mechanical quadrature formula. For the coefficients of the latter formula the author derives, in three different ways, the celebrated Tchebyscheff inequalities ("separation theorem"). The chapter closes with a brief discussion of algebraic continued fractions, in their relation to OP. Chapters IV, V, VIII, IX are devoted to the classical OP. In Chapters IV–V we find their differential equation, also representation by means of trigonometric and Bessel functions, hypergeometrical series, contour-integrals. In Chapter VIII the author makes use of the powerful tools of modern analysis in order to obtain asymptotic expressions for the classical OP in various parts of the interval of orthogonality (with an estimate of the remainder), also in the complex domain. This yields asymptotic estimates of their zeros. The interesting feature is the use of Bessel's functions which offers the advantage of extending the above asymptotic formulas to the end-points of the orthogonality interval—of great importance in the study of expansion problems. Such expansions in series of the classical OP are taken up in Chapter IX: ordinary convergence and (C, k) summability for various values of k. These two chapters form, we believe, one of the most interesting parts of the book. Many results are new; many known results are presented in a decidedly more precise form. The study is based upon the corresponding Lebesgue constants. A masterly use of these constants enables the author to derive conditions for convergence and summability which, from the point of view of generality and precision, leave practically nothing to be desired. This is in sharp contrast to many older investigations, where the said conditions are frequently of an artificially restricted character. The author is the more justified in devoting so much attention to the classical OP, for they form the backbone of the whole theory and serve as a yardstick in dealing with the general OP.

We turn back to Chapter VI. Here a study is made of the zeros of OP: analytic distribution, variation with a parameter. The general results are considerably elaborated for the classical OP (discriminant, extremal properties), where use is made of the corresponding differential equation (Sturm's methods). The results are extended to Fejér's generalization of Legendre polynomials, also to the generalized (non-orthogonal) Jacobi polynomials. In Chapter VII the author derives various inequalities for some special classes of OP (classical OP, monotonic weight-function) and related functions. These inequalities are of importance in mechanical quadratures, interpolation, also in evaluating certain integrals involving OP. We find here also a refinement, for polynomials, of the mean-value theorem of the integral calculus. Chapters X–XIII and XVI, based in the main upon the modern theory of functions of a complex variable, contains results due almost entirely to Szegö. Chapter X is devoted to the analytical representation of positive functions. This is made use of in the next chapter for a study of OP on the unit-circle (these were introduced in Analysis by Szegö). This discussion is continued and generalized in Chapters XII and XIII. It is extended in Chapter XVI to a rectifiable Jordan curve. The simple transformation $x=\frac{1}{2}(z+z^{-1})$ enables the author to return to the real domain and to derive important results concerning OP on a finite segment of the real axis. We note the asymptotic expression for a certain class of OP on $(-1, 1)$ corresponding to the so-called trigonometric weight-function (S. Bernstein). Szegö's method of deriving the said asymptotic expression is much simpler than that of Bernstein. Several general equiconvergence theorems for the real and complex domains constitute another important contribution. Chapters XIV and XV deal with interpolation and closely related mechanical quadrature (they are less extensive than other parts of the book). Here the author makes extensive use of the results developed in the preceding chapters. In Chapter XV a study is made of Lagrange and Hermite interpolation formulae based upon the zeros of OP: convergence, mean-convergence in the sense of pth powers. For the classical OP Szegö derives results more precise than those obtained by other writers. In Chapter XVI the author studies, mainly for the classical OP, the convergence of mechanical quadrature formulae based on Lagrange interpolation. Two types of mechanical quadrature formulae are considered: with and without weight-function. Concerning the latter type, the author takes up the question of positiveness of the coefficients ("Cotes' numbers") which is of cardinal importance in this theory.

In closing the author gives a stimulating set of problems and exercises, also a considerable bibliography.

It goes without saying that the writer of a scientific book is the supreme judge in choosing its content. It is only natural for him to favor his own brain-children and, from amongst the works of others, those closely related to his own.

In the Introduction the author disclaims completeness of treatment. This disarming remark is necessary but not sufficient. In a book of this kind one would like to find as complete a treatment as possible, preparing the reader for future progress in various branches of the subject.

The omission of the problem of moments is, we believe, very regrettable. The problem of moments is important historically and is growing in importance both in pure and applied analysis. Its inclusion would have been much to the point. Moreover, it would have enabled the author (on the basis of the important results of M. Riesz) to give a complete treatment of closure for an infinite interval (the importance of closure in the theory of OP cannot be underestimated). As it is, the author discusses closure, in case of an infinite interval, for Laguerre and Hermite polynomials only. Furthermore, this inclusion of the moment-problem would have made possible a discussion of the limiting values of certain important definite integrals. Another regrettable feature is, we believe, the far too short space (3½ pages) devoted to continued fractions. The author remarks that continued fractions have been gradually abandoned as a starting point for the theory of OP. Be that as it may, one should not underrate continued fractions as a powerful tool in the theory and application of OP. We may mention a few other omissions: OP in several variables; other methods of summability of series of OP in addition to (C, k) summability, for example, the interesting results of Hille concerning Abel summability of expansions in series of Laguerre polynomials; application of OP to algebra (zeros of polynomials).

The reviewer did not pursue the thankless task of hunting misprints, minor errors or minor faults in exposition. While no book is free of these, the known thoroughness of the author offers sufficient assurance in this respect.

With all these omissions the beautifully printed book of Szegö is an excellent addition to the Colloquium Publications of the Society. It is a remarkable source of powerful methods and far-reaching results in an important field, which will inspire and stimulate further research.

J. SHOHAT

Structure of Algebras. By A. Adrian Albert. (American Mathematical Society Colloquium Publications, vol. 24.) New York, American Mathematical Society, 1939. 11+210 pp.

The study of algebras has been one of the most significant features of the present period in the history of mathematics; and in the theory of algebras a monument has been erected recording some of the characteristic traits of contemporary mathematical thought.

One may say that algebras have been pushed into the centre of attention by the publication of Dickson's *Algebras and their Arithmetics*; and from that moment on they have kept the interest of the mathematical public. In the meantime a great number of the problems has been solved, methods have been streamlined so that a moment propitious for the survey of the results has arrived. One of the principal actors in the movement has given an account of its results. The mathematical public certainly will be grateful for his effort, as he has been able to capture the inherent beauty of the theory of algebras and to communicate it to the reader.

The book may be divided roughly into two parts, the first being concerned with the general theory, the second containing applications to related problems. It should be mentioned at once that the theory of representations has been "put in its place"; that is, it appears as an application of the general theory and has not been used for the derivation of the results of the general theory.

The general theory of algebras may be defined as that part of the theory in which no special restrictions are imposed upon the field of reference. There are two main topics of discussion. The first is the reduction to simple algebras and the second the discussion of the simple algebras themselves.

The reduction theory proceeds in two steps. One shows first the existence of a radical and the semisimplicity of the algebra modulo its radical. This semisimple difference algebra may be represented by some subalgebra of the original algebra, provided the difference algebra stays semisimple under every scalar extension (this result belongs to a later phase of the theory, since changes of the field of reference have to be considered). But since both the investigation of nilpotent algebras and of the possible extensions of a nilpotent algebra by a semisimple algebra have started only recently, the state of the theory allows one only to speak of a theory of semisimple algebras and not yet of a theory of algebras. The second step in the reduction theory is the proof of the theorems that every semisimple algebra possesses an identity element and is the direct sum of uniquely determined (not only in the sense of equivalence) simple algebras; and thus it is clear how to construct the most general semisimple algebra, as soon as the simple algebras are known. It should be noted that the theorems concerning semisimple algebras are gotten as special cases of corresponding theorems concerning algebras which may or may not possess a radical. However the reduction contained in these more general theorems does not lead to a class of algebras as tractable as the simple ones.

Every simple algebra is a direct product of a total matric algebra and a division algebra; and any two representations of this kind are conjugate, that is, may be transformed into each other by means of inner automorphisms. The inner automorphisms of simple algebras are exactly those automorphisms which leave the elements of the centrum invariant. The centrum of a simple algebra being a field, it may be taken as the field of reference, in which case we say that the the algebra is normal simple. Then the above theorem states that all the automorphisms of a normal simple algebra are inner. This theorem is supplemented by the fact that any equivalence of two simple subalgebras of a normal simple algebra may be induced by an automorphism of the containing algebra, provided the three algebras have the same identity element.

It follows from the above that the structure of a normal simple algebra is completely determined by two invariants: the one numerical, namely, the degree of the total matric algebra in question, the other a structural invariant, namely, the structure of the division algebra mentioned above. Thus one may be led to the following construction. One considers all the normal simple algebras over a fixed field F. Those normal simple algebras over F which are only distinguished by the nature of the total matric algebra involved and lead to the same division algebra are put into one class and these classes form an abelian group under direct multiplication (of its elements).

If the field N is a finite, normal, and separable extension of the field F, then N determines a subgroup of the group of classes of normal simple algebras over F as follows. There exist some normal simple algebras over F which contain N as a maximal subfield. The product of such an algebra by N is a total matric algebra over N, that is, is split by N; and each such algebra is a crossed product of N by the

Galois group of N over F. The classes of normal simple algebras over F which are split by N form a subgroup of the group of classes of algebras over F; and this subgroup is practically the same as the group of classes of associated factor sets of N over F. As each normal simple algebra over F is split by some such N, it suffices to consider these subgroups of the group of classes; and thus it is possible to use the theory of factor sets for proving theorems like the following one: every element of the group of classes is of finite order in this group; this order divides the degree of the characteristic division algebra and is divisible by every prime divisor of this degree.

As to the methods of proof employed in the derivation of this theory two tools ought to be mentioned which the author uses with much dexterity and success. There is first the commutator of a subalgebra in a containing algebra, that is, the set of all those elements in the containing algebra which permute with every element in the given subalgebra—the adoption of this term commutator may be regretted, as this concept is called centralizer everywhere outside this theory. The importance of the commutator may be seen from the following theorems. If some algebra is the direct product of two algebras B and C such that B is normal simple and such that B and C have the same identity, then C is the commutator of B in their direct product, that is, one factor of the direct product determines the other one in a unique fashion. More generally: the simple subalgebras of a normal simple algebra occur in pairs such that each member of such a pair is the commutator of the other one; and forming the commutator is an involutorial operation as far as simple subalgebras of a normal simple algebra are concerned.

Observing that multiplication of the elements of an algebra by a fixed element is a linear transformation of the algebra, that the linear transformations form a total matric algebra, that the transformations gotten by left-multiplication with an element form exactly the commutator of those gotten by right-multiplication (and conversely), and that the second of these sets of transformations is isomorphic, the first anti-isomorphic to the given algebra, one proves the theorem that the direct product of a normal simple algebra by its reciprocal (anti-isomorphic) is a total matric algebra. This theorem is so powerful a weapon in the theory because it permits a reduction of the proofs of theorems concerning normal simple algebras to the proofs of the corresponding theorems concerning total matric algebras.

The following partial converse of the last theorem ought to be mentioned. The direct product of two division algebras is a total matric algebra if, and only if, these two division algebras are reciprocal.

As an illustration of the author's methods in handling these tools we indicate a proof of the theorem that every automorphism of a normal simple algebra is an inner automorphism (our reason for choosing just this example is that the author does not use his methods for a proof of this theorem and that the proof given in the text is inconclusive). If A is a normal simple algebra, A' its reciprocal, then their direct product is a total matric algebra $T=A\times A'$. If f is an automorphism of A, then there exists one and only one automorphism of T which induces f in A and leaves every element in A' invariant As every automorphism of a total matric algebra is an inner automorphism, there exists some element t in T such that transformation by t induces in T just this automorphism. This element belongs clearly to the commutator of A' in T. Since the commutator of A' in T is just A, it follows that t belongs to A so that f is the inner automorphism of A, induced by this element t.

As a first application of the theory of algebras, the theory of representations has been treated. This theory is no longer concerned with the study of the structure of algebras but with the study of the structure of homomorphisms between algebras. If both algebras are considered over the same field, then there exists however a connection between the two theories which is very near to an equivalence. The following theorems may be mentioned as typical examples of this situation. A representation of the algebra A over F in the subalgebra A^* of a total matric algebra M over F is irreducible if, and only if, A^* is simple and the commutator of A^* in M is a division algebra; it is fully decomposable if, and only if, A^* is semisimple; it stays irreducible under every scalar extension if, and only if, A^* itself is a total matric algebra. On the other hand it must be admitted that by its very definition the theory of representations is much more complicated than the theory of algebras proper; and thus both theories are greatly improved by doing the simpler thing first and using it then for the derivation of the more involved theory.

The most important application of the general theory and perhaps the most interesting part of the whole work is the enumeration of the normal simple algebras over finite algebraic number fields or—what amounts to the same thing—of the simple algebras over the field of rational numbers. The problem of how to construct all these algebras is settled by the famous theorem that every simple algebra over the field of rational numbers is not only similar to, but may actually be represented as, the crossed product of a cyclic extension of an algebraic number field by its cyclic Galois group. In order to classify these algebras the author proceeds as follows. If the field F is a finite extension of the field of rational numbers, V a valuation of F and A a normal simple algebra over F, then denote by F_V the essentially uniquely determined derived field of F with respect to the valuation V and by A_V the direct product of A and F_V (that is, the scalar extension of A by F_V). A reduction of the problem "in the large" to a problem "in the small" is contained in the theorem that the normal simple algebras A and B over F are equivalent if, and only if, A_V and B_V are equivalent for every valuation V. This theorem is a comparatively simple consequence of the following lemma (which the author states without giving a proof): an element in F is the norm of an element in the given finite and normal extension N of prime degree over F if, and only if, it is a norm of an element in N_V for every valuation V of N. Thus it suffices to determine these algebras A_V. There exists at most a finite number of valuations V of F such that A_V is not a total matric algebra. If A_V is not a total matric algebra, and if V is an archimedian valuation, then A_V is a real quaternion algebra. If A_V is not a total matric algebra, and if V is a non-archimedian valuation, then A_V is completely determined by two numerical invariants one of which is just the degree of the division algebra characteristic for the class of A_V whereas the other one is derived from a certain normal form of representing A_V as a crossed product.

There is a great number of further equally interesting applications of the theory which find treatment in this book. But it would lead too far to discuss them here at great length. Suffice it to mention such topics as the cyclic systems, the modern theory of Riemann matrices and of involutions, and more special problems like the enumeration of all normal division algebras of degrees three and four.

The standard source of reference for the theory of algebras has been in recent years Deuring's report on this topic. As to the material covered none of these works has been a subset of the other and even in their treatment of the common parts they differ widely. In the reviewer's opinion Albert's text is the more easily accessible one of the two, so much so that he feels that the book will serve admirably as an introduction into the theory to all those who know the rudiments of present-day algebra. The expert on the other hand will find this exposition interesting, stimulating and useful both because of the new material included and because of the more "streamlined" treatment of it.

An extensive bibliography of recent publications on related topics has been added which will be welcome and helpful to every student of the field.

REINHOLD BAER

The Classical Groups. By Hermann Weyl. Princeton, University Press, 1939. 11+302 pp.

It is a curious fact that while almost all the textbooks on higher algebra written prior to 1930 devote considerable space to the subject of invariants, the recent ones written from the axiomatic point of view disregard it completely. Because of this neglect the phrase "invariant theory" is apt to suggest a subject that was once of great interest but one that has little bearing on modern algebraic developments. For this reason it is an important and original accomplishment that Professor Weyl has made here in connecting the theory of invariants with the main stream of algebra and in indicating that the subject has a future as well as a distinguished past.

In his treatment the theory of invariants becomes a part of the theory of representations. The most natural way to begin the study of invariants of a particular group is therefore to determine its representations. A large part of the book is concerned with this problem as it applies to the "classical" groups $GL(n)$, the full linear group; $O(n)$, the orthogonal, and $S(n)$, the symplectic (complex or abelian) group.

In discussing the representations of an abstract group $\mathfrak{g}$ one finds it convenient to adjoin to the set of representing matrices their linear combinations. The resulting set is an algebra, the enveloping algebra of the original set, and defines a representation of a certain abstract algebra, the group algebra, that is completely determined by $\mathfrak{g}$ and the field of the coefficients. In this way the theory of algebra is applicable. The author has gone somewhat beyond his immediate needs in discussing this domain. Consequently his book may serve also as an excellent introduction to this theory.

The book begins with a brief review of the basic concepts of field, abstract group, vector space, linear transformation (matrix) and representation of a group. These are used to give the following definition of an invariant: Let $x, y, \cdots$ be variable vectors varying in different representation spaces of an abstract group $\mathfrak{g}$ and let $A(s), B(s), \cdots$ be the corresponding matrices. A polynomial $f(x, y, \cdots)$ in the coordinates of $x, y, \cdots$ is an invariant of $\mathfrak{g}$ if $f(A(s)x, B(s)y, \cdots) = f(x, y, \cdots)$ for all s in $\mathfrak{g}$. Relative invariants and covariants are defined along similar lines. The central problems to which this definition gives rise are: (1) Given a class of invariants, to determine, if possible, a basis, that is, a finite number $I_1, I_2, \cdots, I_r$ of these invariants such that every invariant in the class is expressible as a polynomial in the I's. (2) To determine a fundamental set of relations obtaining between the basic invariants. The solution of these problems for vector invariants[1] of the unimodular group $SL(n)$ and of the orthogonal group is the subject of Chapter II. There are two essential steps in the proofs, the formal part by which the theorems are reduced by certain identities (Capelli's) to the cases where the number of vector variables of each kind does not exceed $(n-1)$ and the numerical argument which disposes of the reduced cases. The proofs are less direct for $O(n)$ than for $SL(n)$. For here it is necessary to suppose first that the underlying field is real and then to extend the results to arbitrary fields of characteristic 0 by using Cayley's parametrization of the "general" orthogonal matrix of determinant 1 as $(E-S)(E+S)^{-1}$ where E is the identity and S is an arbitrary skew-symmetric matrix.

The third chapter introduces the theory of algebras. The treatment given is based on the concept of a completely reducible matrix algebra. It is shown that any representation of an algebra $\mathfrak{A}$ of this type is completely reducible and there is a reciprocity between $\mathfrak{A}$ and the algebra $\mathfrak{B}$ consisting of the matrices commutative with those of $\mathfrak{A}$: $\mathfrak{B}$ is completely reducible and $\mathfrak{A}$ in turn may be characterized as the set of matrices commutative with those of $\mathfrak{B}$. Together with Schur's lemma this result yields Wedderburn's fundamental theorems. The methods used here are well suited for handling problems of the type discussed in this book, where algebras occur in concrete form as sets of matrices. More abstract methods would be required if one wished to obtain these theorems in their most general form. In the second half of the chapter an independent treatment of group algebras is given. Their complete reducibility and reciprocity with the commutator algebras is derived in a very elementary way. As a consequence the results, particularly the latter, are given a much more explicit form than in the general situation.

After this machinery has been set up, it is possible to determine the (integral) representations of $GL(n)$. One must establish first the complete reducibility of the tensor representation $A \rightarrow A \times A \times \cdots \times A$ (Kronecker product of f factors) and to split this representation into its irreducible parts. This is accomplished by showing that the enveloping algebra $\mathfrak{A}_f$ is the commutator of a certain representation of σ_f, the group algebra of the symmetric group on f letters. Explicit formulas for the idempotent elements of σ_f (Young symmetrizers) yield the irreducible subspaces of tensor space. These results may be extended to the representation obtained by stringing along the diagonal the tensor representations of ranks $1, 2, \cdots, f$. The enveloping algebra is here denoted as $\mathfrak{A}^{(f)}$. Now, if $A \rightarrow T(A)$ is any representation of $GL(n)$ whose coordinates are polynomials of degree not greater than f in the coordinates of A, then the enveloping algebra of $T(A)$ represents $\mathfrak{A}^{(f)}$. The general theory shows that $T(A)$ is completely reducible into parts similar to some of those contained in the tensor representations.

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[1] That is, functions of an arbitrary number of covariant and contravariant vectors where by a covariant vector we mean one belonging to the original representation $x \rightarrow Ax$ and contravariant refers to vectors in $\xi \rightarrow (A')^{-1}\xi$, A' the transposed matrix. There is no distinction between covariant and contravariant vectors if A is orthogonal.

To obtain these results for $O(n)$ the author follows a procedure due to Brauer (used first by Weyl for $S(n)$). The complete reducibility of the tensor representation is established for the case of real fields. Its commutator algebra is determined by using the first main theorem for orthogonal invariants. Finally one obtains the enveloping algebra of the tensor representation by the reciprocity theorem. The tool by which the reality restriction is removed is again Cayley's parametrization. The real significance of the parametrization is brought out in the following beautiful theorem: The ideal generated by the relations defining an orthogonal matrix of determinant 1 is prime and has the generic zero $(E-S)(E+S)^{-1}$. To obtain the irreducible parts of tensor space relative to $O(n)$ a process of contraction is used. By this means the problem is reduced to one of decomposing certain spaces relative to the symmetric group. These results are then extended to the proper orthogonal group. The discussion of the symplectic group is reserved for a separate chapter. It parallels that of $O(n)$ and enables the reader to get a bird's-eye view of the entire structure as it appears up to this point.

In cases such as those under discussion here, in which the representations decompose into absolutely irreducible parts, a knowledge of these parts, though not their explicit construction, may be obtained by expressing the character of the given representation as a sum of primitive (irreducible) characters. This is done in Chapter VII for $GL(n)$, $O(n)$ and $S(n)$. The method is that of Weyl's "unitarian trick." One replaces these groups by certain compact Lie groups (thus, in place of $GL(n)$ one uses the group of unitary matrices). The characters of the latter may be obtained by using an invariant integration defined on the group manifolds. Since any algebraic relation with rational coefficients in the elements of the compact group $\mathfrak{u}$ holds also for the original group $\mathfrak{g}$, the formulas obtained for $\mathfrak{u}$ are valid for $\mathfrak{g}$ after being expressed as polynomials in the coefficients of the matrices. It is shown incidentally that the representations, determined by the earlier constructions form a complete set of continuous irreducible representations for the compact groups. As an illustration of the use of characters, the author determines the irreducible components of the Kronecker product of irreducible representations.

In the first part of Chapter VIII the general definition of an invariant is specialized to the classical case. Here the representation space is determined by the coefficients of a homogeneous polynomial. The symbolic method as well as certain irrational methods are discussed as tools to obtain the first fundamental theorem. Hilbert's basis theorem for polynomial ideals settles the question of the finiteness of relations among the basic invariants. The symbolic method is then used to extend these results to general orthogonal and symplectic invariants. The ideas in the last half of the chapter cluster around the concept of a Lie group and its Lie algebra. The local properties of these groups are all expressible in terms of the algebras and if the group is simply connected one may readily pass to its properties in the large. On the other hand Lie algebras may be studied independently of the group theory and questions analogous to those regarding representations, invariants, and so on, may be raised. For compact Lie groups the powerful integration method finds another application here in a very simple derivation of the first fundamental theorem for general invariants. The last chapter resumes the discussion of algebras. A number of interesting results concerning automorphisms and direct products are obtained by matrix methods.

We have confined our description to the broad outlines of the subject. Space does not permit a detailed account of the interesting bypaths indicated by the author. We may mention, for example, a new formulation of Klein's Erlanger program, the close connection between representation theory and the theory of almost periodic functions and Fourier series, the topology of the classical groups. There are ample indications in the notes and bibliography to enable the reader to pursue these questions further.

In short, the book is heartily recommended to the present generation of algebraists as an introduction to a rich and rather neglected field and to those educated in the classical tradition for an insight into important recent algebraic ideas and their applicability to familiar problems.

N. JACOBSON

Aspects of the Calculus of Variations. Notes by J. W. Green after lectures by Hans Lewy. Berkeley, University of California Press, 1939. 6+96 pp.

As indicated in the introduction, the goal of the lecturer was to acquaint the reader with a skeleton of methods such as he is apt to encounter in the calculus of variations, rather than to formulate results as generally as possible. The first chapter is devoted to a discussion of various problems, and the derivation of the first order necessary conditions for an extremum. Chapter II treats quadratic problems; in particular, the theory of integral equations with real symmetric kernel and boundary value problems associated with a second order linear differential equation. Chapter III, which deals with sufficient conditions, is extremely brief; nowhere in the lectures is the Weierstrass $\mathcal{E}$-function mentioned. In Chapter IV the work of Lewy (Mathematische Annalen, vol. 98 (1928), pp. 107–124) on the absolute minimum for nonparametric problems in the plane is presented. Chapter V is concerned with harmonic functions and associated boundary value problems; this chapter is preliminary to the discussion of the problem of Plateau and conformal mapping given in Chapter VI. In this last chapter the problem of Plateau is solved for a single contour in three dimensions; the method of proof is that of Courant (Annals of Mathematics, (2), vol. 38 (1937), pp. 676–724).

W. T. Reid

Die Zylinderfunktionen und ihre Anwendungen. By R. Weyrich. Leipzig, Teubner, 1937. 5+137 pp.

A cylinder function may be defined as any solution of Bessel's differential equation. They include functions of the first kind, $J_p(z)$, also called Bessel functions, which are regular at $z=0$, and functions of the second kind which are not regular at $z=0$. There has been considerable confusion in the notation and canonical form of the cylinder functions of the second kind. They have been denoted by Y, G, K, and so on, by various authors and often the same notation is used with different meanings. Following the tables of Jahnke and Emde, our author uses the notation $N_p(z)$ for functions of the second kind and calls them Neumann functions. They are the same as those denoted by $Y_p(z)$ by Nielsen and also by Watson. (See Watson, *Theory of Bessel functions*, p. 57, for a further discussion.) In addition there are the Hankel cylinder functions, which are really functions of the second kind, defined by

$$H_p^{(m)}(z) = J_p(z)+i(-1)^{m+1}N_p(z), \qquad m=1,\ 2,\ i=(-1)^{1/2}.$$

The importance of the Hankel functions arises from the fact that alone among the cylinder functions $H_p^{(m)}(re^{i\theta})\to 0$ as $r\to\infty$, provided that $m=1$, $0\leqq\theta\leqq\pi$, and that $m=2$, $\pi\leqq\theta\leqq 2\pi$.

One way to introduce cylinder functions is to define them, as above, as solutions of Bessel's differential equation. Our author, however, follows a different path. His starting point is the wave equation, from which he derives first plane waves, and then, by superposition of these, spherical and cylindrical waves. This leads immediately to the Sommerfeld integral representation of the Hankel functions, which then serve as the basis for defining the J and N functions and for deriving the important properties of the cylinder functions.

The author discusses the following aspects of cylinder functions: power series, asymptotic expansions of Hankel and Debye, various integral representations, recurrence relations, zeros, definite and indefinite integrals, boundary value problems and applications. It will be noted that the author has succeeded in covering the most important topics in a remarkably small number of pages. But the value of the book must not be judged by its brevity. It contains a carefully planned exposition of the theory and will serve as a valuable study and reference book.

C. A. SHOOK

Differentialgeometrie der Kurven und Flächen und Tensorrechnung. By Václav Hlavatý. Groningen, Noordhoff, 1939. 11+569 pp.

This treatise presents a large portion of the classical differential geometry of one and two dimensional subspaces of ordinary euclidean space. Just enough vector and tensor analysis is given to enable the reader to manage profitably the abbreviated symbolism. Definitions and results are stated in such a way as to generalize readily to higher dimensions.

The first chapter is devoted to curves. After the theory is developed in terms of a general parameter, an account is given of the various specializations arising from the use of the arc length as parameter. In particular, the construction of a curve from its curvature and torsion is treated carefully.

The second chapter concerns those properties of a surface which depend only on its metric tensor. The absolute differential is used systematically. There is an unusually full discussion of the applicability of surfaces, including explicit equations for developing a surface on a plane or on a surface of revolution.

The normal to a surface leads, on differentiation, to a tensor associated with the behavior of the surface toward the ambient space. In the third chapter, those properties are discussed which depend on this (second fundamental) tensor. A feature of this chapter is the discussion of the explicit construction of surfaces having prescribed first or first and second fundamental tensors.

The fourth chapter contains applications to a variety of special surfaces, notably ruled surfaces and minimal surfaces. There is no treatment of quadrics nor of congruences of lines.

The book is carefully written, and a high level of explicitness is maintained in the details of the argument and in the treatment of special cases. The reviewer feels that two exceptions to this statement may confuse readers unfamiliar with tensor analysis. "Tensor" is so defined that there is no distinction between a tensor and the set of its components in any particular coordinate system. This distinction—analogous to that between the number two and a couple of apples—is the key to the geometric significance of tensors, and this point seems to have been obscured by the form given to the definition.

In the second place, "differential invariant" is defined in the algebraic sense—as a function which is transformed by substituting for the old variables the appropriate functions of the new variables and multiplying by a certain power of the Jacobian. It is then stated that "The problem of metric differential geometry . . . consists in the study of the invariants . . . which can be obtained from the equations of the surface (or curve)." As examples, two invariants for each dimension are constructed, the whole treatment covering about five pages. No further mention is made of the concept. This procedure appears open to several objections. First, no explicit connection is established between the remainder of the text and "the problem of differential geometry." Second, it seems pedagogically unsound to introduce such a basic concept without giving the reader more definite occasion to become familiar with its applications and to grasp at least one precise significance of a word having a confusing variety of meanings. Finally, the term has been so used in other books that tensors are "invariants." If a practice opposed to this usage is adopted, a word of caution to readers seems desirable.

The book has an ingenious index which greatly enhances its usefulness as a source for reference. There is no bibliography, nor exact reference to sources, nor exercises. In dealing with surfaces, roman numerals I and II are used as indices for quantities pertaining to the surface; the gain in emphasis seems offset by the visual hesitation in distinguishing I II from II I.

Chiefly for its thoroughgoing use of tensor methods, the book is a valuable complement to the available treatises. It is distinguished from them by numerous interesting details in the presentation.

F. A. FICKEN

NOTES

Lattice Theory, by Assistant Professor Garrett Birkhoff of Harvard University, has just been published as volume 25 of the series of American Mathematical Society Colloquium Publications. The volume contains 155 pages. The list price is $2.50, with a discount of twenty-five per cent to members. Orders should be sent to the Society office at 531 West 116th Street, New York City.

In March of this year the Mathematical Society of Hamburg celebrated the two hundred fiftieth anniversary of its founding.

The National Council of Teachers of Mathematics will hold its 1940 summer meeting July 1-3 in Milwaukee, Wisconsin. The theme of the meetings is *How mathematics serves our community.*

The fourth annual mathematical symposium at the University of Notre Dame was held April 10–11, 1940, on the foundations of topology. The following papers were presented: *On contiguous points*, Professor R. L. Moore; *The topology of lumps*, Professor Karl Menger; *Classification of topological functions*, Professor E. W. Chittenden; *The equal generality of "convergence," "closure," and "neighborhoods,"* Dr. J. W. Tukey; *Partially ordered sets and topology*, Dr. A. N. Milgram; *The foundations of algebraic topology*, Professor Solomon Lefschetz; *Lattices and connectivity*, Dr. Henry Wallman.

At the meeting of the National Academy of Sciences in Washington, D. C., April 22-23, papers were presented by Professors Harry Bateman, G. D. Birkhoff, Gregory Breit, Edward Kasner, andMarston Morse.

Professor W. D. Cairns, Secretary-Treasurer of the Mathematical Association of America, has announced that the Department of Mathematics of the University of Toronto has won the first prize of $500 in the third annual William Lowell Putnam Mathematical Competition, held March 2, 1940. The members of the team were W. J. R. Crosby, J. C. Maynard, G. H. K. Strathy. The second prize of $300 is awarded to the Department of Mathematics of Yale University, members of whose team were J. E. Brewster, A. M. Gleason, G. R. MacLane. For the third prize of $200 there is a tie between the Department of Mathematics of Columbia University and the Department of Mathematics of Cooper Union Institute of Technology. The members of the Columbia University team were Laurence Annenberg, Julius Ashkin, Paul Marcus; the members of the Cooper

Union team were Murray Klamkin, Benjamin Lax, Samuel Manson. In addition to these prizes to the departments of mathematics with winning teams, a prize of $50 each is awarded to the following five persons whose scores ranked highest in the six-hour examination (names arranged alphabetically): W. J. R. Crosby, University of Toronto; A. N. Gleason, Yale University; E. L. Kaplan, Carnegie Institute of Technology; J. C. Maynard, University of Toronto; R. M. Snow, George Washington University. One of these five will be chosen to receive a $1000 year scholarship at Harvard University, this award to be announced later. The members of the four winning teams will receive individual cash awards according to the ranks of their teams, and all individuals receiving awards will also receive medals. Honorable mention has been awarded this year to three teams and to five individuals. The teams are from the Department of Mathematics, University of California at Berkeley, members of the team being Julia H. Bowman, W. M. Kincaid, C. W. Lippman; the Department of Mathematics, University of California at Los Angeles, members being Richard Arens, Robert James, Harold Shniad; and the Department of Mathematics, Carnegie Institute of Technology, members being S. N. Foner, E. L. Kaplan, W. E. Stuermann. The five individuals receiving honorable mention are: G. R. MacLane, Yale University; Samuel Manson, Cooper Union Institute of Technology; Paul Marcus, Columbia University; G. H. K. Strathy, University of Toronto; J. E. Wilkins, Jr., University of Chicago.

Among those recently elected to fellowships in the Royal Society of Edinburgh are Dr. A. F. Buchan of James Gillespie's School, Edinburgh, Dr. H. Davenport of the University of Manchester, and Professor H. S. W. Massey of University College, London.

The following persons have been awarded fellowships by the Guggenheim Foundation: Dr. Jesse Douglas, Brooklyn, N. Y., Professor R. L. Wilder of the University of Michigan, Associate Professor A. F. Wintner of the Johns Hopkins University, and Assistant Professor Gordon Pall of McGill University.

Dr. A. S. Gale of the University of Rochester has been made dean emeritus of the physical sciences.

Professor Hermann Weyl of the Institute for Advanced Study has been elected a member of the National Academy of Sciences.

Professor G. Vranceanu of the University of Cernauti has been appointed to a professorship at the University of Bucharest.

Assistant Professor J. E. Eaton of Hofstra College is now head of the department of mathematics.

Dr. K. O. Friedrichs of New York University has been promoted to an associate professorship.

Associate Professor Rufus Oldenburger of Armour Institute of Technology will deliver a series of lectures on modern algebra and dynamics at the University of Mexico during July and August.

Assistant Professor Mina S. Rees of Hunter College has been promoted to an associate professorship.

Dr. M. S. Robertson of Rutgers University has been promoted to an assistant professorship.

Associate Professor D. J. Struik of Massachusetts Institute of Technology has been promoted to a professorship.

Professor L. C. Karpinski of the University of Michigan will be on leave for the first semester of 1940–1941.

Professor A. Zygmund, formerly of the University of Wilno, has accepted a position at Mt. Holyoke College for 1940–1941.

Sir Thomas Heath died on March 16, 1940, at the age of seventy-eight years.

The Polish Government Information Department has reported the death of Professor A. Hoborski.

Professor J. H. Michell of the University of Melbourne died on February 3, 1940, at the age of seventy-six years.

The death of Professor Ernesto Pascal of the University of Naples on January 25, 1940, has been reported.

Professor Otto Toeplitz, formerly of the University at Bonn, died on March 15, 1940.

Colonel C. P. Echols, retired professor of mathematics of the United States Military Academy at West Point, died May 21, 1940. He had been a member of the Society for many years.

Associate Professor Beulah Russell of the College of William and Mary died February 23, 1940.

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ABSTRACTS OF PAPERS

SUBMITTED FOR PRESENTATION TO THE SOCIETY

The following papers have been submitted to the Secretary and the Associate Secretaries of the Society for presentation at meetings of the Society. They are numbered serially throughout this volume. Cross references to them in the reports of the meetings will give the number of this volume, the number of this issue, and the serial number of the abstract.

353. Stefan Bergman: *On a generalized Green's function and certain of its applications.*

A theorem of Blaschke states: $\sum_{\nu=1}^{\infty} \lg |a_\nu| > -\infty$, $|a_\nu| < 1$, is the necessary and sufficient condition for the existence of a nonnegative harmonic function $H(z)$, $H(0) < \infty$, $z \in \mathfrak{E}^2 - \mathfrak{S}_\nu \{a_\nu\}$, $\mathfrak{E}^2 = \mathrm{E}[|z| < 1]$ which possesses the property that $H(z) + \log|z - a_\nu|$, $\nu = 1, 2, \cdots$, is regular at $z = a_\nu$. Let $\mathfrak{M}^4 = \mathrm{E}[|z_2| < 1, z_1 \in \mathfrak{B}^2(z_2)]$, $\mathfrak{B}^2(z_2) = \mathrm{E}[z_1 = sh(z_2, \lambda), 0 \leqq s < 1, 0 \leqq \lambda \leqq 2\pi]$, $h(z_2, \lambda)$, $h(z_2, 0) = h(z_2, 2\pi)$, being for every λ an analytic function of z_2, $z_2 \in \overline{\mathfrak{E}}^2$. In the paper (Compositio Mathematica, vol. 6 (1939), pp. 307–335) there was introduced for every domain $\mathfrak{M}^4$ an "extended class of functions" possessing properties analogous to those of harmonic functions. If $g(z_1, z_2)$ is in $\mathfrak{M}^4$ an analytic function, it is then possible to construct the "Green's function of the extended class" $\Gamma(z_1, z_2; g; \mathfrak{M}^4)$ vanishing on the boundary surface $\mathrm{E}[|z_2| = 1, z_1 = h(z_2, \lambda), 0 \leqq \lambda \leqq 2\pi]$ such that $\Gamma(z_1, z_2; g; \mathfrak{M}^4) + \lg|g(z_1, z_2)|$ is a function of extended class. Let $g_\nu(z_1, z_2)$ designate a set of functions analytic in $\mathfrak{M}^4$ satisfying a certain additional restriction concerning the distribution of zero surfaces. The condition $\sum_{\nu=1}^{\infty} \Gamma(0, 0; g_\nu; \mathfrak{M}^4) < \infty$ is necessary and sufficient for the existence of a nonnegative function $H(z_1, z_2)$, $H(z_1, z_2) < \infty$ for $(z_1, z_2) \in \mathfrak{M}^4 - \mathfrak{S}_\nu \mathfrak{G}_\nu^2$, $\mathfrak{G}_\nu^2 = \mathrm{E}[g_\nu(z_1, z_2) = 0, (z_1, z_2) \in \mathfrak{M}^4]$ such that for every n, $H(z_1, z_2) + \sum_{\nu=1}^{n} \lg |g_\nu(z_1, z_2)|$, belongs to the extended class of functions of $\mathfrak{M}^4_{n_0}$, $n_0 \leqq n$, where $\lim_{n \to \infty} \mathfrak{M}^4_n = \mathfrak{M}^4$. (Received March 28, 1940.)

354. L. M. Blumenthal: *A new concept in distance geometry, with applications to spherical subsets.*

If $\sigma \subset S$, a semimetric space, then S has σ-relative congruence indices $\{n, k\}$ with respect to a given class (Σ) of semimetric spaces provided any space Σ of (Σ) with more than $n+k$ pairwise distinct points is congruently contained in S whenever each n of its points is congruently contained in σ. This concept gives rise to new problems in distance geometry. For $\sigma = S$ the indices $\{n, k\}$ are called merely congruence indices of S with respect to (Σ). The notions of congruence and quasi congruence orders introduced by Menger correspond to congruence indices $\{n, 0\}$, $\{n, 1\}$, respectively, while $\{n, 2\}$ signifies a property weaker than quasi congruence order n but stronger than quasi congruence order $n+1$. Let $\kappa_{2,\rho}$ denote a cap with spherical radius ρ of the sphere $S_{2,r}$. It is shown that (1) if $\sigma = \kappa_{2,\pi r/2}$, with base circle removed, then $S_{2,r}$ has σ-relative congruence indices $\{4, 2\}$ with respect to all semimetric spaces (Σ); (2) the closed cap $\kappa_{2,\rho}$, $\rho < \pi r/2$, has congruence indices $\{4, 2\}$ with respect to (Σ); (3) $\kappa_{2,\rho}$, $\rho < r \cos^{-1} 1/3$, has indices $\{4, 1\}$. Analogous theorems for caps of $S_{n,r}$ are obtained. (Received April 12, 1940.)

355. S. S. Cairns: *Triangulated manifolds which are not Brouwer manifolds.*

This paper relates to problems involving Brouwer manifolds, differentiable manifolds, and polyhedral representations of complexes. A *Brouwer m-manifold* (L. E. J. Brouwer, Mathematische Annalen, vol. 71 (1912), pp. 97–115) is a triangulated topological m-manifold such that each star has a piecewise linear homeomorphic image in an euclidean m-space E^m. (I) A triangulated topological m-manifold is not necessarily a Brouwer manifold if $m>3$. This is one consequence of a constructed triangulation of an m-simplex ($m=3, 4, 5, \cdots$) not homeomorphic to any rectilinear triangulation of it. Other consequences follow. (II) Triangulated m-spheres exist ($m=3, 4, 5, \cdots$) which cannot be represented as convex polyhedral spheres in an E^{m+1}. (III) The lowest-dimensional euclidean space in which a complex K has a nonsingular polyhedral representation is not, for every K, invariant under subdivision. (IV) For every $m>3$, there exist stars of simplexes which are m-cells but which do not admit transversal $(n-m)$-planes, no matter how they are polyhedrally represented in any E^n. A plane is *transversal* to a point set if it makes angles bounded away from zero with secants of the set. (V) Brouwer's definition of an m-manifold (loc. cit.) is not invariant ($m>3$) under subdivision. For the connection with questions of differentiability, see abstracts 45-3-110 and 45-9-306, the results of which are to appear in the Annals of Mathematics. (Received March 29, 1940.)

356. J. L. Coolidge: *Analytic systems of central conics in space.*

Systems of central conics in three-space have been little studied. Spottiswoode and Johnson have studied algebraic systems and there is a paper by Blutel dealing with surfaces generated by conics of a particular type. In the present paper there is a study of general one- and two-parameter analytic systems. The technique is that of the moving tetrahedron first developed by Darboux and employed in more recent times by Cartan. (Received April 9, 1940.)

357. Richard Courant: *On a generalized form of Plateau's problem.*

From a geometrical viewpoint it is natural to replace the Plateau problem, in which a continuous and monotonic description of the prescribed boundary is required, by another problem where no condition of this type is stipulated. For the solution of this generalized problem, methods quite different from those usually used for Plateau's problem are needed. The main objective of the paper is to show that the solution of the ordinary Plateau problem also solves the new problem, while the converse is not necessarily true. (Received April 11, 1940.)

358. A. González Domínguez: *Some theorems on the Hermite singular kernel.*

Let $g(x)$ be a function of bounded variation in $(-\infty, \infty)$ normalized in the usual way, and let $K(x, y, r)$ be the singular kernel which appears in the Abel summation of Hermite series. Then the following equality holds: $\lim_{r\to 1} \int_0^a dy \int_{-\infty}^{\infty} K(x, y, r)dg(x) = g(a)-g(0)$. This result, and others of a similar character, are used to obtain a uniqueness theorem for Hermite-Stieltjes series, a new limit-theorem for distributions functions, and also to solve the problem of the "classes" for Hermite series. (Received April 4, 1940.)

359. J. E. Eaton: *On the extension of groups. I. Normal extensions. II. Non-normal extensions.*

I. The problem of finding all groups $\mathfrak{G}$ such that $\mathfrak{G}/\mathfrak{N}\cong\mathfrak{H}$ for given groups $\mathfrak{H}$ and $\mathfrak{N}$ has been considered by Schreier. In this paper Schreier's n^3 associativity conditions on n^2 elements of $\mathfrak{N}$ are replaced by kn more complicated conditions on rn elements, wherein n is the number of elements of $\mathfrak{H}$, k the number of generating relations defining $\mathfrak{H}$, and r the number of generators. The problem of obtaining all non-redundant solutions of these conditions is then considered. II. By using the results of a previous paper (abstract 45-9-310) one may formulate the problem of non-normal extensions of groups: *to determine all groups $\mathfrak{G}$ such that $\mathfrak{G}/\mathfrak{H}\cong\mathfrak{C}$ where $\mathfrak{H}$ is a given group and $\mathfrak{C}$ a given "cogroup."* If one calls those groups $\mathfrak{G}$ for which no proper subgroup of $\mathfrak{H}$ is normal in $\mathfrak{G}$ a ***purely non-normal extension***, the problem of non-normal extensions may be reduced to purely non-normal ones followed by normal extensions. Conditions for purely non-normal extensions, analogous to those of Schreier for the normal case, are derived. (Received April 13, 1940.)

360. D. W. Hall: *On arc and tree preserving transformations.*

It is shown that every single valued transformation $T(A)=B$, where A and B are cyclic locally connected continua, which is arc preserving and tree preserving is necessarily a homeomorphism. No assumptions are made as to the continuity of the transformation T. (Received April 26, 1940.)

361. E. R. Lorch: *The integral representation of weakly almost-periodic transformations in reflexive vector spaces.*

As announced previously (abstract 45-11-401), a weakly almost-periodic transformation V in a reflexive space $\mathfrak{B}$ is characterized by the relation $\|V^n f\|\leq K\|f\|$, $n=0, \pm 1, \pm 2, \cdots$. For such a w.a.p. V, an integral representation is given which is the exact counterpart of the known representation $U=\int e^{i\lambda}dE(\lambda)$ for a unitary transformation U in a Hilbert space. Specifically it is proved: I. There exists a family of pairs of closed linear manifolds $\{\mathfrak{E}_\lambda, \mathfrak{F}_\lambda\}$, $-\infty<\lambda<\infty$, with the following properties: (a) $\mathfrak{E}_\lambda$ and $\mathfrak{F}_\lambda$ have only 0 in common; together they span $\mathfrak{B}$. (b) The pair $\{\mathfrak{E}_\lambda, \mathfrak{F}_\lambda\}$ reduces V. (c) $\mathfrak{E}_\lambda\supset\mathfrak{E}_\mu$ for $\lambda>\mu$; $\mathfrak{F}_\lambda\subset\mathfrak{F}_\mu$ for $\lambda>\mu$. (d) $\mathfrak{E}_{-\pi}=0$, $\mathfrak{F}_{-\pi}=\mathfrak{B}$, $\mathfrak{E}_\pi=\mathfrak{B}$, $\mathfrak{F}_\pi=0$. II. Given an $\epsilon>0$, there exists a set of numbers $\lambda_1, \lambda_2, \cdots, \lambda_n$ and closed linear manifolds $\mathfrak{A}_{\lambda_j}$, $j=1, \cdots, n$, such that (a) $\mathfrak{A}_{\lambda_j}$ lies in $\mathfrak{E}_{\lambda_j+\epsilon}$ and in $\mathfrak{F}_{\lambda_j-\epsilon}$, (b) on $\mathfrak{A}_{\lambda_j}$, $\|(V-e^{i\lambda})f\|\leq\epsilon\|f\|$, (c) $\mathfrak{B}$ is spanned by the $\mathfrak{A}_{\lambda_j}$, $j=1, \cdots, n$. III. The spectrum of V is competely characterized by the behavior of $\{\mathfrak{E}_\lambda, \mathfrak{F}_\lambda\}$. That is, for $-\pi<\lambda<\pi$, $e^{i\lambda}$ is in the point spectrum, the continuous spectrum, or the resolvent set of V if and only if λ is a point of discontinuity, of continuity, or of constancy, respectively, of $\{\mathfrak{E}_\lambda, \mathfrak{F}_\lambda\}$. The integral representation and spectral properties of the transformation $H=-i(V-I)(V+I)^{-1}$ are obtained. (Received April 25, 1940.)

362. A. N. Lowan and Jack Laderman: *On the distribution of errors in the computation of tables by differences.*

The values of a function may often be computed conveniently from the leading differences up to the order $r-1$, and the rth differences (which are not assumed to be constant) for all the required arguments. The values obtained by building up the table from the above differences are equal to those obtained from the formula $u_n=u_0+C_{n,1}\Delta u_0+C_{n,2}\Delta^2 u_0+\cdots+C_{n,r-1}\Delta^{r-1}u_0+\sum_{i=0}^{n-r}C_{n-i-1,r-1}\Delta^r u_i$. Expressions are given for the maximum possible error in u_n, for the distribution of the error, and for the variance of the distribution when the differences have been rounded in the kth decimal place. The above results were obtained in the course of work done by the Project for the Computation of Mathematical Tables, Work Projects Administration, New York City. (Received April 10, 1940.)

363. A. N. Lowan and Hyman Serbin: *On upper bounds for the remainder in the evaluation of differences.*

The nth difference of a function $f(x)$ defined over the interval from x_0 to x_0+nh may be expressed in the elegant form $\Delta^n f(x) = \int_0^h \int_0^h \cdots \int_0^h f^{(n)}(x+\xi_1+\xi_2+\cdots+\xi_n)d\xi_1 d\xi_2 \cdots d\xi_n = h^n \int_0^1 \int_0^1 \cdots \int_0^1 f^{(n)}(x+\xi_1+\xi_2+\cdots+\xi_n)d\xi_1 d\xi_2 \cdots d\xi_n$. This leads to $\Delta^n f(x) = h^n f^{(n)}(x) + \sum_{k=1}^{p} (k!)^{-1} h^{n+k} f^{(n+k)}(x) \int_0^1 \cdots \int_0^1 (\xi_1 + \cdots + \xi_n)^k d\xi_1 \cdots d\xi_n + [h^{n+p+1}/(p+1)!] \int_0^1 \cdots \int_0^1 (\xi_1 + \cdots + \xi_n)^{p+1} f^{(n+p+1)}(x+\theta(\xi_1+\cdots+\xi_n))d\xi_1 \cdots d\xi_n$. If the expression for $\Delta^n f(x)$ is truncated after the term involving the derivative of order $n+p$, an upper bound for the truncating error is given by $[h^{n+p+1}/(p+1)!]\{f^{(n+p+1)}\} \int_0^1 \cdots \int_0^1 (\xi_1+\cdots+\xi_n)^{p+1} d\xi_1 \cdots d\xi_n$ where $\{f^{(n+p+1)}\}$ is an upper bound of $f^{(n+p+1)}(x)$ in the given interval. An upper bound of the last integral is given by n^{p+1}. The application of this formula to x^{n+m} (where m is a positive integer) leads to an expression of Milne-Thomson's Bernoulli numbers $B^{(-n)}$ in terms of the above repeated integrals. The above results were obtained in the course of work done by the Project for the Computation of Mathematical Tables, Work Projects Administration, New York City. (Received April 6, 1940.)

364. M. H. Martin: *Quasi-Lagrangian systems.* Preliminary report.

Non-conservative dynamical systems $L_{\dot{q}_r} - L_{q_r} + k(t)L_{\dot{q}_r} = 0$ are studied. Such systems have properties allied to Lagrangian systems ($k=0$) and are termed *quasi-Lagrangian* systems. The restricted problem of three bodies involving the motion of an infinitesimal mass attracted by two finite masses moving in accordance with a preassigned solution of the two body problem leads to a quasi-Lagrangian system, as do certain dynamical systems containing frictional forces. The equations of motion may be put in the variational form $\delta\int_{t_0}^{t_1} e^l L dt = 0$, $(\dot{l} = k)$ and in the canonical form $\dot{q}_r = H_{p_r}$, $\dot{p}_r = -H_{q_r} - kp_r$ with the partial differential equation $S_t + kS + H(t, q_r, S_{q_r}) = 0$ playing the rôle of the Hamilton-Jacobi partial differential equation. The flow in the phase space is not incompressible, the volume V of a region being given in terms of its volume V_0 at $t=t_0$ by the relation $V = e^{-n(l-l_0)}V_0$. When $L_t \equiv 0$ and k is constant there exist generalizations of the energy integral and of the principle of Mapertuis. At least half the characteristic exponents of an equilibrium solution have negative (positive) real parts if $k>0$ $(k<0)$. If L contains no linear terms in $\dot{}_r$, there are no periodic solutions and positively (negatively) stable motions tend uniformly towards equilibrium points. The nature of flow in the neighborhood of equilibrium points is investigated. (Received April 12, 1940.)

365. H. J. Miser: *Regions and their "patterns" in conformal mapping.*

Let the circle $|z|<1$ be mapped by $w=f(z)$, $(f(0)=0)$, on a plane region S, the map of $|z|<r<1$ being S_r. Let $w_0 = T(w)$ be analytic in S, and let $T(0)=0$. If w_0 is in S for all w in S, then S is said to have the property T. It has been proved (L. R. Ford, Duke Mathematical Journal, vol. 1 (1935), pp. 103–104) that if S has the property T, so also does S_r. This paper considers the function $w_0 = tw$, and studies the set of values Σ in the t-plane, which is called the "pattern" of S, for which S has the property T. The set Σ contains 0 and 1, lies in $|t| \leq 1$, is closed, and is contained in the pattern Σ_r of S_r. (Received April 5, 1940.)

366. I. E. Segal: *Certain topological rings.*

The rings considered are Banach spaces which are rings, with the norm having the property $|ab| \leq |a| \cdot |b|$, for example, the ring of all Fourier-Stieltjes transforms. Theorems are derived which connect the properties of an element modulo the closed maximal ideals of the ring with the existence of an inverse to the element, or the existence of any given analytic function of the element. Some special cases are treated. (Received March 29, 1940.)

367. E. W. Titt: *On the characteristic theory of a first order partial differential equation.*

This paper can be regarded as a continuation of a previous study of the relation between the $(n-1)$-dimensional and the one-dimensional characteristic manifolds of a first order partial differential equation (Duke Mathematical Journal, vol. 3 (1937), p. 740). The method is as follows: first, derive the differential equations of an $(n-1)$-dimensional characteristic manifold as a set of consistency conditions which must be satisfied over exceptional manifolds, that is, manifolds over which the determinant of the left member of a system of linear equations vanishes; regard this set of consistency conditions as a new system of differential equations to be solved; repeat the characteristic process on this system of equations to obtain characteristic sub-manifolds of $(n-2)$ dimensions. Continuing this step-by-step process we finally arrive at characteristic sub-manifolds of one dimension. These are shown to be the classical characteristic strips of Cauchy and Lie. (Received March 29, 1940.)

368. A. D. Wallace: *On separation spaces.*

The purpose of this paper is to give a new system of axioms for topological spaces in which the notion of "mutual separation" is taken as undefined. It is shown that a set of separation axioms suffice to characterize a T_1-space, but examples indicate that a weaker topology would be more natural. Separation spaces seem to be well adapted to the study of analytic topology and a version of the Whyburn-Eilenberg factor theorem is given under very general conditions and without the use of upper semi-continuous decompositions. (Received April 26, 1940.)

369. J. L. Walsh and W. E. Sewell: *Degree of polynomial approximation to analytic functions. Problem β.*

If $f(z)$ is analytic for $|z|<1$ and satisfies the inequality $|f(z)| \leq M_1(1-|z|)^{p+\alpha}$, where p is a negative integer and $0<\alpha<1$, $f(z)$ is said to be of class $L(p, \alpha)$. If $f^{(p)}(z)$ is analytic for $|z|<1$, continuous for $|z| \leq 1$, and satisfies a Lipschitz condition on $|z|=1$ of order α, where p is a nonnegative integer and $0<\alpha<1$, $f(z)$ is said to be of class $L(p, \alpha)$. If $f(z)$ is of class $L(p, \alpha)$, there exist polynomials of respective degrees n such that $|f(z)-p_n(z)| \leq M\rho^{-n}n^{-p-\alpha}$ on the circle $|z|=1/\rho<1$. Reciprocally if there exist polynomials of respective degrees n such that $|f(z)-p_n(z)| \leq M\rho^{-n}n^{-p-\alpha-1}$ on the circle $|z|=1/\rho$, then $f(z)$ is of class $L(p, \alpha)$. These results extend to approximation on an arbitrary analytic Jordan curve. (Received April 15, 1940.)

370. J. V. Wehausen: *A space isomorphic to Hilbert space.*

A Banach space is constructed which is isomorphic to Hilbert space but which contains no euclidean spaces of dimension equal to or greater than 2. It is constructed as follows. Let $x=(x_1, x_2, \cdots)$ and $x^{(k)}=(x_1, x_2, \cdots, x_k, 0, \cdots)$. Define $\|x^{(1)}\|=|x_1|$, $\|x^{(k+1)}\|^{p_k}=\|x^{(k)}\|^{p_k}+|x_{k+1}|^{p_k}$, $(p_k\geqq 1)$, and $\|x\|=\lim_{k\to\infty}\|x^{(k)}\|$. The set of sequences for which $\|x\|$ is finite is the required space if p_k converges to 2 sufficiently fast ($p_k=2-1/k$ is adequate). (Received April 10, 1940.)

371. G. T. Whyburn: *On multicoherence.*

Consider the number $r(X)$ defined by Eilenberg for continua X as l.u.b. $[r(X_1, X_2)]$, where $X=X_1+X_2$ is any decomposition of X into continua X_1 and X_2 and $r(X_1, X_2)+1$ is the number of components of $X_1\cdot X_2$. It is shown that this number is cyclicly additive, that is, $r(M)=\sum r(E_i)$, where M is a locally connected continuum with true cyclic elements $[E_i]$. Also a new and direct proof (based on the simple fact that if a sum of k connected sets is connected, a connected partial sum consisting of any given number less than or equal to k of terms can be found) is given for the result due essentially to Eilenberg that if the transformation $f(A)=B$ is quasi-monotone in the sense of Wallace (that is, for each continuum K in B with a non-vacuous interior, $f^{-1}(K)$ consists of a finite number of components each mapping onto K), $r(B)\leqq r(A)$. This gives $r(B)\leqq r(A)$ for monotone transformations on continua A and for interior transformations on locally connected continua A. (Received April 27, 1940.)

372. G. T. Whyburn: *On singularities and inversion under interior transformations.*

A subset K is said to locally separate a set M at $x\in K$ provided that for every sufficiently small neighborhood V of x in M, $V-V\cdot K$ is disconnected. It is shown that if A is locally compact, $f(A)=B$ is continuous and interior on $A-E$ where $E\subset A$, x is any point of $E\cdot\overline{A-E}$ which is a component of $f^{-1}f(x)$, and U is any neighborhood of x, then $f(E\cdot U)$ locally separates B at $f(x)$. Thus, in particular, if E is not open in A and $f(E)$ does not locally separate B at any point, f is interior on A. (This includes and extends results of Stoilow and the author on singularities of interior transformations.) It is also shown that, if A is a locally connected continuum and $f(A)=B$ is interior and light, (1) if Y is a locally connected continuum in B whose interior is dense in Y, $f^{-1}(Y)$ is locally connected, (2) if A is on a 2-manifold, and p is any point of B of order greater than 2, $f^{-1}(p)$ is a finite set of points. These two results form the basis for a greatly simplified proof for the theorem that, under the conditions of (2), if K is a locally connected continuum in B, $f^{-1}(K)$ is locally connected. (Received April 27, 1940.)

373. G. T. Whyburn: *Transformations on boundary curves.*

For any subset $\underline{X}$ of a continuum M let $\alpha(\underline{X})$ denote the number of components of $M-\underline{X}$. If A is a boundary curve (equal to a Peano continuum every true cyclic element of which is a simple closed curve) and $\underline{P}=p_1+p_2+\cdots+p_m$ is any finite subset of M, it is shown that $\alpha(\underline{P})\leqq\alpha(p_1)+\cdots+\alpha(p_m)$ and the equality holds if, for each i and j, $\underline{P}$ contains all points of A separating p_i and p_j. This is used to prove that if $f(A)=B$ is non-alternating and light, for each $y\in B$, $\alpha(y)=\sum\alpha(x)$, $x\in f^{-1}(y)$. The converse holds if B is assumed to be a boundary curve or if $\alpha(y)$ is assumed finite for every $y\in B$. It results from this that if A, B and C are boundary curves and the transformations $f_1(A)=B$ and $f_2(B)=C$ are non-alternating and light, so also is the composite transformation $f(A)=f_2f_1(A)=C$. (Received April 27, 1940.)

374. Paul Alexandroff: *General combinatorial topology.*

The fundamental duality theorem which correlates the r-dimensional cohomology group of a closed set with the $(r+1)$-dimensional cohomology group of the residual set is here obtained by a systematic development of homology theory based on the inverse and direct spectra. The same development leads also to a simple definition of the cocycle ring. Thus the theory of the spectrum, a standard topological and group-theoretical concept, is now made to include results which have hitherto been obtained only by the special methods of Alexander and of Kolmogoroff. (Received May 8, 1940.)

375. J. P. Ballantine: *Proof of Poincaré's geometric ring theorem.*

Referring to the proof of Poincaré's geometric ring theorem (Transactions of this Society, vol. 14 (1913)) by G. D. Birkhoff, the author has attempted to supplement the reasoning given there by two lemmas. It is assumed that T is a continuous one-to-one transformation having no invariant point. The index of a closed curve C relative to T is defined as the number of revolutions advanced by the vector from P to $T(P)$ as P moves around C. Lemma 1: If there exists a circuit of index different from zero relative to T, there is one of index different from zero which can be inscribed in a circle of radius ϵ. Lemma 2: If there exists a circuit C of index different from zero inscribed in a circle of radius ϵ, and if $2\epsilon<\delta$, where δ is the least distance moved by any point under T, then the index of C is ± 1, $T(C)$ surrounds the circle of radius ϵ, and the area of $T(C)$ exceeds that of C. It follows that T is not area preserving. (Received June 24, 1940.)

376. E. F. Beckenbach: *An integral analogue of Laplace's equation.*

It is shown that if the real function $x(u, v)$ is continuous in a finite simply connected domain D then a necessary and sufficient condition that $x(u, v)$ be harmonic in D is that $\int_{C(u_0, v_0; r)} [\int_{C(u_0+s, v_0+t; \rho)} x(u_0+s+\xi, v_0+t+\eta)(d\xi+id\eta)](ds-idt)=0$ hold for all circles $C(u_0, v_0; r+\rho)$ in D. (Received April 27, 1940.)

377. R. S. Burington and J. M. Dobbie: *A new family of wing profiles.*

Piercy, Piper, and Preston (Philosophical Magazine, (7), vol. 24 (1937), pp. 425–444, [P]), using conformal mapping, have recently advanced a new family of wing profiles F which are thought to have certain advantages over earlier theoretical wing shapes. The present paper reconsiders this work from a rigorous mathematical point of view, with the purpose of definitely exhibiting a transformation which fulfills all the requirements as to analyticity and boundary conditions essential to the theory (Burington, *On the use of conformal mapping in shaping wing profiles*, to appear in the American Mathematical Monthly). Special attention is given to the singularities of the transformations used. A direct by-product of this study leads to a new and more general family B of airfoils, a subfamily of which is the family F. The sequence of transformations advanced herein, which maps a circle into an airfoil of family B (and F), seems much simpler than that used in the paper [P]. (Received May 15, 1940.)

378. J. H. Curtiss: *Necessary conditions in the theory of interpolation in the complex domain.*

Let $f(z)$ be a function which is analytic on a closed limited region R whose boundary B is the boundary of an infinite region K. Let $L_n(z; f)$ be the polynomial of degree at most $n-1$ which coincides with $f(z)$ in the points $\alpha_1^{(n)}, \alpha_2^{(n)}, \cdots, \alpha_n^{(n)}$ chosen on B. If at a single interior point z_0 of R, $\limsup_{n\to\infty} |L_n[z_0; 1/(t-z_0)]|^{1/n} \leqq 1$, all t in K, it follows that (i) $|\omega_n(z)|^{1/n} \to \Delta$ uniformly on any closed point set of the interior R, where Δ is the transfinite diameter of R and $\omega_n(z) = \prod_1^n (z-\alpha_k^{(n)})$, (ii) $L_n(z; f) \to f(z)$ uniformly, z in R, for each function $f(z)$ analytic on R, (iii) if B is a Jordan curve, the points $\alpha_k^{(n)}$ are uniformly densely distributed on B. In establishing this result, several auxiliary results are developed in some detail, including some new non-trivial sufficient conditions for convergence of the sequence $\{L_n\}$ and an inequality which promises to be useful in the study of the polynomials $\omega_n(z)$. (Received May 31, 1940.)

379. E. L. Dodd: *The probabilities for certain inequalities among the values of a moving average of chance variables.*

Let independent chance variables x_i be subject to the same law of distribution. Let $z_r = \sum x_i$, $i = r-n+1$ to r. Let p be the probability that $z_r \leqq z_{r+1} \leqq \cdots \leqq z_{r+k}$ or, more generally, that among $k+1$ consecutive z's a specified ordering should obtain. Then for every $n \geqq k$, this value of p is a constant. If, further, the x's are normally distributed, and s and t are whole numbers, the probability that $z_{r-s} \leqq z_r \geqq z_{r+t}$ is at least equal to $\frac{1}{4}$. The above statements remain valid if z_r denotes the moving average $\sum x_i/n$, instead of the moving sum. (Received May 20, 1940.)

380. H. E. Goheen: *On the primitive groups of the classes 4p and 5p.*

The author proves that there are no primitive groups of class $5p$, p a prime greater than 5, which contain a permutation of degree $5p$ and order p, and that with a few exceptions there are no primitive groups of class $4p$ which contain a permutation of order p and degree $4p$. The exceptions are $p=5$, in which case every such group is contained in the holomorph of the elementary abelian group of order 25, and $4p+1=q$, q a prime or a power of 5, in which case all the known groups of the required type are primitive subgroups of the triply transitive Mathieu group. The proof is based upon the examination of the normalizer of the cyclic group generated by the permutation of order p and minimum degree in the group in the light of a theorem of W. A. Manning (this Bulletin, vol. 13 (1906), p. 201), and the examination of the subgroup fixing one letter of a simply transitive primitive group in the light of a theorem of Weiss (this Bulletin, vol. 40 (1934), p. 402). Together with auxiliary considerations these dispose of every possibility except one, which is disposed of by means of the theory of group characters. (Received May 22, 1940.)

381. O. G. Harrold (National Research Fellow): *A note on strongly irreducible maps of an interval.*

A continuous mapping of the compact metric space A onto B has been called *strongly irreducible* provided no proper closed subset of A maps onto all of B. If M is any Peano space in which the nonlocal separating points are dense and P is any given countable dense set of nonlocal separating points, there exists a continuous mapping, $f(I) = M$, where I denotes the unit interval, such that $y \in P$ implies $f^{-1}(y)$ is a single point and $\overline{f^{-1}(P)} \supset I$. The set J^* of such strongly irreducible maps is a dense G_δ in the space J of maps of I onto M. These mappings are nowhere arc preserving in the sense that the only arcs which are transformed into arcs are the individual points. (Received May 21, 1940.)

382. L. B. Hedge: *Moment problem for a bounded region.*

A solution of the moment problem for a bounded interval, given by Hausdorff in 1923, is extended to any bounded region in euclidean n-space under certain conditions on polynomial expansions over the region. The resulting solution is shown to be valid for the n-dimensional sphere, and includes the Hausdorff case as well as known conditions on the "class" of Fourier and Fourier-Stieltjes series. (Received May 17, 1940.)

383. Ralph Hull: *The structure of certain Fuchsian groups.*

The units of a maximal order of a rational, indefinite, quaternion algebra form an infinite group which may be represented as a Fuchsian group, and whose structure is determined in a simple way by the fundamental number of the algebra (cf. American Journal of Mathematics, vol. 61 (1939), pp. 365–374). The group of units of a non-maximal order of such an algebra may similarly be represented as a Fuchsian group, and analogous methods may be applied to determine its structure, that is, the genus, and class-number of elliptic substitutions in the group. Groups thus studied include those which correspond to the totality of integral solutions of equations such as $x^2+ay^2-bu^2-abv^2=1$, where a and b are positive integers. (Received May 21, 1940.)

384. Bella Manel: *Conformal mapping of multiply connected domains on the basis of Plateau's problem.*

This paper is concerned with the conformal mapping of an arbitrary multiply connected plane domain G on the following normalized domains: B_1, the half-plane with finite slits radial to the origin one of which lies on the imaginary axis; B_2, the half-plane with elliptical holes having their axes mutually parallel and having a fixed eccentricity; B_3, the unit circle with concentric circular slits one of which has end points on the real axis. The method is to solve the Plateau problem for the contour bounding G with the desired conformally equivalent domains as the domains of representation. The solution yields the analytic function which performs the mapping. This connection between conformal mapping and the Plateau problem has been stressed by Douglas and by Courant. The procedure of Courant forms the basis of this investigation. For plane contours the existence of a solution to a minimum problem involving the Dirichlet integral is used. By a detailed analysis of the variational conditions it is shown that the solution to this minimum problem is a minimal surface for each of the normal domains B_1, B_2, B_3. (Received May 27, 1940.)

385. A. F. Moursund: *A note on Gibbs' phenomenon.*

The kernel for Nevanlinna's weak summation method, which is essentially the same as the Bosanquet-Linfoot zero order method, contains a constant whose value has no effect in determining the strength of the method in forcing convergence. In this note it is shown that this constant plays an important role in determining the power of the method to destroy Gibbs' phenomenon in Fourier series. Weak methods are exhibited which destroy Gibbs' phenomenon while stronger methods do not. (Received May 22, 1940.)

386. T. S. Peterson: *The Fredholm minors for Goursat's kernel.*

The purpose of this note is to give the explicit determinantal form for the Fredholm minors of a kernel having the form $\sum_{i=1}^{n} M_i(x)N_i(y)$. The result is obtained directly using theorems due to Platrier and Sylvester. (Received May 21, 1940.)

387. R. M. Robinson: *On the approximation of irrational numbers by fractions with odd or even terms.*

All fractions may be divided into four types, according as the numerator and denominator are odd or even. Let one or more of these types be selected as admissible. Using continued fractions, the largest value of μ is determined for which to every irrational ξ infinitely many admissible fractions A/B can be found satisfying $|\xi - A/B| < 1/\mu B^2$. A part of this problem was solved by W. T. Scott in the February issue of this Bulletin, using a different method. (Received May 29, 1940.)

388. P. M. Swingle: *Local connectedness and biconnected sets.*

In this paper it is proved for certain spaces that if a biconnected set B is connected im kleinen at a point p, then p is a dispersion point of B. The problem is generalized in several ways. (Received May 17, 1940.)

389. P. M. Swingle: *Non-transfinite connected sets.*

In this paper the definitions of finitely-containing, finitely-divisible, and finitely-convergeable connected sets are modified, substituting in place of connected subset either biconnected, widely connected, or punctiform connected subset. The relations between these new types of sets are studied, a solution being obtained of previously proposed problems (American Journal of Mathematics, vol. 53, pp. 373, 400, Problems 3 and last half 11). Also certain results are obtained concerning a modular connectivity which is defined. (Received May 17, 1940.)

390. C. W. Vickery: *On cyclically invariant graduation.* II.

The author has extended the results of an earlier paper (abstract 46-5-336, April 27, 1940). Suppose that $g(t)$ is a periodic function of period $2k$ having an expansion in Fourier series. Suppose that X is a random variable with characteristic function $f(w)$ such that, for every integer $n, f(n\pi/k) \neq 0$ and is an even function. Let $g^*(t) = g(t) - a_0/2$, where $a_0 = (1/k)\int_{-k}^{k} g(t)dt$. For every integer $m \geqq 0$, graduation operator G_m is defined as follows: $G_m\{g(t)\} = a_0/2 + E\{g^*(t+X)\}/f(m\pi/k)$. The terms of index m of the Fourier expansion of $g(t)$ are invariant with respect to the application of G_m. Necessary and sufficient conditions have been found that the application of G_m to any Fourier series produce a Fourier series. Sufficient conditions have been found that the application of G_m to any Fourier series produce a Fourier series that is convergent almost everywhere. (Received April 23, 1940.)

391. C. W. Vickery: *On systems of Fourier coefficients.*

If $\sum (a_n \cos nx + b_n \sin nx)$ is a Fourier series and $[\lambda_n]$ is a sequence of real numbers, then in order that $\sum \lambda_n(a_n \cos nx + b_n \sin nx)$ be a Fourier series it is necessary and sufficient that there exist a periodic function $g(x)$ of bounded variation and of period 2π such that, for each integer m, $f(m) = 1/\pi \int_{-\pi}^{\pi} e^{imx} dg(x)$ is an even function and such that $\lambda_n = f(n)$, n being positive. An example shows that it is not necessary that $g(x)$ be an odd function. If, furthermore, the second derivative of $g(x)$ exists for all values of x between $-\pi$ and π and is absolutely integrable, then $\sum\lambda_n(a_n \cos nx + b_n \sin nx)$ is a Fourier series which converges for all values of x except a set of measure 0. (Received April 23, 1940.)

392. H. S. Wall: *A continued fraction related to partition formulas.*

The author shows how four well known identities involving $\prod(1-ax^n)$ can be derived from a single continued fraction, namely, $1+rx/1+(1-r)wx/1+(1-w)wrx/1+(1-wr)w^2x/1+(1-w^2)w^2rx/1+\cdots$. (Received April 6, 1940.)

393. Y. K. Wong: *On biorthogonal matrices.*

Consider the basis $\mathfrak{A}$, $\mathfrak{P}^1$, $\mathfrak{P}^2$, ϵ^1, ϵ^2 (E. H. Moore, *General Analysis*, Part I, p. 16). Suppose that κ^{12} is by columns of $\mathfrak{M}(\epsilon^1)$. In a paper entitled *On non-modular matrices*, the author establishes the one-to-one correspondence between the class $\mathfrak{M}(\epsilon_\kappa^1\ \kappa^*\ \epsilon^2)$ of all vectors μ^1 modular as to ϵ^1 such that $J^1\kappa^{*21}\mu^1$ are in $\mathfrak{M}(\epsilon^2)$ and the class $\mathfrak{M}(\epsilon^2 \cap \epsilon_\kappa^2)$ of vectors μ^2 modular as to ϵ^2 and ϵ_κ^2. In the present paper, the author first obtains some properties on a pair of matrices κ^{12}, ψ^{21}, which are by columns of $\mathfrak{M}(\epsilon^1)$, $\mathfrak{M}(\epsilon^2)$ respectively, such that $J^2\kappa^{*12}\mu^2 = J^2\psi^{*12}\mu^2$ for every μ^2 in $\mathfrak{M}(\epsilon^2 \cap \epsilon_\kappa^2)$. Next, it is assumed that ϵ_0^1, ϵ_1^1 are idempotent as to ϵ^1, κ^{12} is by columns of $\mathfrak{M}(\epsilon_0^1)$, and ϕ^{21} is by rows conjugate of $\mathfrak{M}(\epsilon_1^1)$. Take any pair of vectors μ^1, ν^1 in $\mathfrak{M}(\epsilon_0^1\ \kappa^*\ \epsilon^2)$, $\mathfrak{M}(\epsilon_1^1\ \phi\ \epsilon^2)$ respectively; if $J^1\bar{\mu}^1\nu^1$ is equal to $J^2(J^1\bar{\mu}^1\kappa^{12}, J^1\phi^{21}\nu^1)$, then ϕ^{21}, κ^{12} are said to be biorthogonal as to $\epsilon^2\epsilon_1^1\epsilon_0^1$. The purpose of the paper is to investigate the properties of the biorthogonal pairs of matrices. (Received May 22, 1940.)

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THE ENGINEER GRAPPLES WITH NONLINEAR PROBLEMS[1]

THEODORE VON KÁRMÁN

I do not believe that one could connect justly the name of Gibbs with practical applications of applied mathematics, for his main interest was certainly centered on basic conceptions of mathematical physics. Nevertheless, for example, his beautiful work on graphical methods in thermodynamics is a brilliant example of the presentation of theoretical relations in a form which appeals to the engineer.

This lecture is intended as an effort to improve the convergence between the viewpoints of mathematics and engineering. Thus, I feel it is not inappropriate to dedicate it to the memory of Josiah Willard Gibbs.

Engineering mathematics is generally considered as a collection of mathematical methods adapted for the solution of relatively simple problems. These problems often might require lengthy numerical calculations or graphical constructions, but supposedly can be worked out without the use of advanced methods of mathematical analysis. This description was perhaps correct some decades ago; today a large group of scientific workers is engaged in applying various methods of classical and modern analysis to problems in electrical, civil, mechanical, aeronautical and also chemical engineering. It is not possible to give an exhaustive list of all types of problems which require the applications of advanced analytical methods. In the following table merely some of the most important engineering problems and the mathematical concepts and methods involved in their treatment are indicated:

ASSOCIATED TOPICS OF ENGINEERING AND MATHEMATICS

Mathematical topics	Engineering problems
Vector algebra, systems of linear equations. Tensors and matrices. Algebraic equations. Ordinary differential equations with given initial conditions. Elementary operational calculus.	Engine dynamics, vibration of systems with a finite number of degrees of freedom. Rotating electric machinery.
Ordinary differential equations and their boundary problems. Eigenvalues and eigenfunctions. Expansion in or-	Equilibrium, buckling and harmonic vibrations of beams. Critical frequencies and speeds. One-dimensional problems

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Mathematical topics	Engineering problems
thogonal functions. Calculus of variations of simple integrals.	of stationary flow of matter, ele tricity and heat.
Functions of complex variables. Conformal transformation. Logarithmic potential.	Two-dimensional velocity, electromagnetic and temperature fields.
Vector and tensor analysis. Partial differential equations of the elliptic type and their boundary problems. Expansion in orthogonal functions. Integral equations. Calculus of variations of multiple integrals.	Stress distribution in elastic bodies; equilibrium, buckling and harmonic vibration of plates and shells. Three-dimensional flow and field problems.
Partial differential equations of the parabolic and hyperbolic type. Advanced operational calculus.	Transient phenomena of heat conduction and wave propagation (variable flow in channels, water hammer in pipes, surges in cables, waves in solids and fluids, electromagnetic waves).

Though most of the various branches of mathematical analysis appear in this table, it does not follow that the engineer has to master, for example, the complete theory of linear differential and integral equations or the calculus of variations. He is mostly interested in methods of obtaining approximate solutions. However, it is interesting to note that the methods which are perhaps the most powerful tools of the mathematician for proofs of existence of solutions have been proved to be the most helpful in obtaining numerical approximations, for example, step-by-step integration, iteration, successive approximation, direct methods of finding maxima and minima. The iteration method is known to most engineers as the Stodola or Vianello method. The direct methods of the calculus of variations are known as the energy method, Rayleigh-Ritz method, Galerkin method, and so on.

The problems listed in the above table provide a happy hunting ground to the mathematician who is interested in applying his science for practical purposes. We know that not all mathematicians have this interest; a great British mathematician is credited with the remark that his field—the theory of numbers—is the only field of mathematics that, fortunately, has not been prostituted by being useful to some practical end. Others believe that the engineering applications of mathematics do not require ingenuity in the mathematical sense; however, there are many engineering problems that are far from being an easy mark for the hunter. Most of the problems compiled above are readily accessible to the methods of mathematical analysis because they are linear problems. In many cases the engineer can linearize his problem by means of simplifying assumptions and a mathematical text will easily supply him with all the help he needs. However, if the engineer has a real nonlinear problem, that is, one which loses its sense by linearization, very often he has to grapple with it by himself.

In this paper the attempt is made to give a review of nonlinear problems which are of practical interest and for whose solution methods have been worked out or at least suggested by mathematicians, physicists or engineers. The review is not exhaustive, but it was attempted to include examples of the most typical problems. The question of existence and uniqueness of solutions is only superficially touched; emphasis is laid on methods for obtaining approximate solutions. Many physicists and engineers have the feeling that so-called mathematical proofs of existence and uniqueness of solutions very often merely verify facts which are more or less evident from physical considerations. Unfortunately, in most nonlinear problems physical reasoning is not sufficient or not fully convincing, so that in these cases the questions of existence and uniqueness represent a real challenge to the mathematician. Certain investigations of H. Poincaré and T. Levi-Civita can be mentioned as examples in which mathematical reasoning has virtually extended the range of physical vision. The modern methods of function spaces are likely to make essential contributions to the problem of existence and uniqueness of solutions of nonlinear equations and their boundary problems. Promising steps in this direction have already been made. However, the writer is afraid that this aspect of the subject is beyond his own competency and the scope of this paper.

1. **Nonlinear oscillations. General remarks.** The theory of mechanical oscillations commonly used in engineering practice is based on the assumption that the restoring forces are proportional to the deflections and the damping forces to the velocities, that is, to the first derivatives of the deflections with respect to time. If we assume constant masses, the equations of motion of an oscillating system are linear, containing as coefficients constant inertia, damping and spring factors. We obtain analogous equations for the oscillations of an electric circuit by assuming constant impedances, that is, by assuming that the network contains only constant reactances, resistances and

capacities. The oscillations governed by linear equations may be called "linear oscillations."

Let us consider the oscillations of a single mass m assuming that the deflection x produces a restoring force $-kx$ and the damping force is equal to $-\beta\dot{x}$, where $\dot{x}=dx/dt=v$ is the velocity of the mass. Then the equation of motion for the free oscillations of this mass is

$$m\ddot{x} + kx + \beta\dot{x} = 0. \tag{1.1}$$

Introducing the natural frequency ω of the undamped oscillations, that is, the frequency for the case $\beta=0$, we have

$$\ddot{x} + \omega^2 x + \frac{\beta}{m}\dot{x} = 0. \tag{1.2}$$

We will generalize equation (1.2) in the form

$$\ddot{x} + \omega^2 x = f(x, \dot{x}). \tag{1.3}$$

This general form of the equation of free oscillation includes the cases of nonlinear damping and nonlinear restoring force. Often the deviation from linearity causes only a small deviation of the linear oscillation without changing the general character of the motion. For example, the simple harmonic oscillation, which we would obtain in the linear case, will be accompanied by harmonics of small amplitude if the deviation from linearity is taken into account. However, in some cases the whole character of the oscillatory motion changes. Slight negative damping in the range of small deflections may cause "self-excited periodic oscillations," whose period is quite different from the period of the undamped harmonic oscillation. Nonlinearity of the spring factor may fundamentally change the resonance phenomena, that is, the response of the system to external periodic forces. In the case of a pure linear system the response to a periodic force consists of a forced oscillation whose period is equal to that of the force and of free oscillations, whose period is the natural period of the system. If the two frequencies coincide, resonance occurs. One can easily visualize that in a nonlinear case the periodic force will excite also higher harmonics, that is, oscillations whose frequencies are multiples of its own frequency. It is more difficult to see offhand that in such a system an external force can also excite oscillations whose frequency is a fraction of its own frequency. This phenomenon is known as "subharmonic resonance." The mathematical analysis of the phenomena of self-excited oscillations and subharmonic resonance makes use of certain methods developed by H. Poincaré in connection with his astronomical investigations and shall be discussed briefly in the next two sections.

2. **Self-excited nonlinear oscillations (relaxation oscillations).** Let us assume that the function $f(x, \dot{x})$ in equation (1.3) has the form $f(x, \dot{x}) = \mu\phi(x)\dot{x}$, where μ is a positive constant parameter. It is seen that in this case the damping is proportional to the velocity $v = \dot{x}$, but the magnitude of the damping factor is in general a function of the deflection. The equation (1.3) in this case takes the form

$$\ddot{x} + \omega^2 x = \mu\phi(x)\dot{x}. \tag{2.1}$$

Equation (2.1) can be reduced to a differential equation of the first order by considering $\dot{x} = v$ as function of x. We obtain

$$\frac{dv}{dx} = -\omega^2\frac{x}{v} + \mu\phi(x). \tag{2.2}$$

This equation can be solved by the method of isoclines. Consider first the case $\mu = 0$. The integral curves of (2.2) are in this case concentric ellipses, given by

$$v^2 + \omega^2 x^2 = \text{const.} \tag{2.3}$$

One of these ellipses is shown in Fig. 1a. Since $dt = dx/v$, we obtain the x, v values corresponding to the actual motion by proceeding clockwise along the integral curves. The corresponding motion is a harmonic oscillation with constant amplitude.

Let us now assume that $\phi(x) = 1$, that is, the damping factor is constant. It is known that the resulting motion is a harmonic oscillation with decreasing amplitude if $\mu < 0$ and increasing amplitude if $\mu > 0$. We obtain logarithmic spirals, which in the stable case ($\mu < 0$) converge toward the origin (Fig. 1b) and in the unstable case ($\mu > 0$) diverge from the origin (Fig. 1c).

We shall now assume that $\phi(x)$ is variable and investigate the existence of periodic solutions. In the x, v plane a periodic solution appears as a closed curve. Let us multiply both sides of equation (2.2) by $v\,dx$ and integrate along the closed curve. Then we see that $\mu\oint\phi(x)v\,dx = \mu\oint\phi(x)v^2 dt = 0$. Hence $\phi(x)$ must change its sign during the motion. In other words, to obtain a periodic motion it is a necessary condition that the damping factor change its sign as function of the deflection.

Van der Pol studied in detail an interesting special case assuming that $\phi(x) = 1 - x^2$. It is seen that for $x < 1$ we have negative damping which becomes positive when $x>1$. Fig. 2 and Fig. 3 show integral curves in the x, v plane for $\mu=0.1$ and $\mu=1.0$ respectively. The integral curves near the origin have the shape of logarithmic spirals, but instead of diverging to infinity they converge to a closed curve. The integral curves originating at infinity converge to the same closed curve, which represents a periodic motion with constant amplitude. For small values of μ, for example, $\mu=0.1$, the closed curve is only slightly different from an ellipse, but its shape varies quite radically with increasing μ. The corresponding motions for $\mu=0.1$, 1.0 and 10 are shown in Fig. 4 where the deflection is plotted as function of time. It is seen that for $\mu=0.1$ the motion is very nearly harmonic, whereas for $\mu=10$ the oscillation is made up of sudden transitions between deflections of opposite sign. The period of the harmonic motion is de-

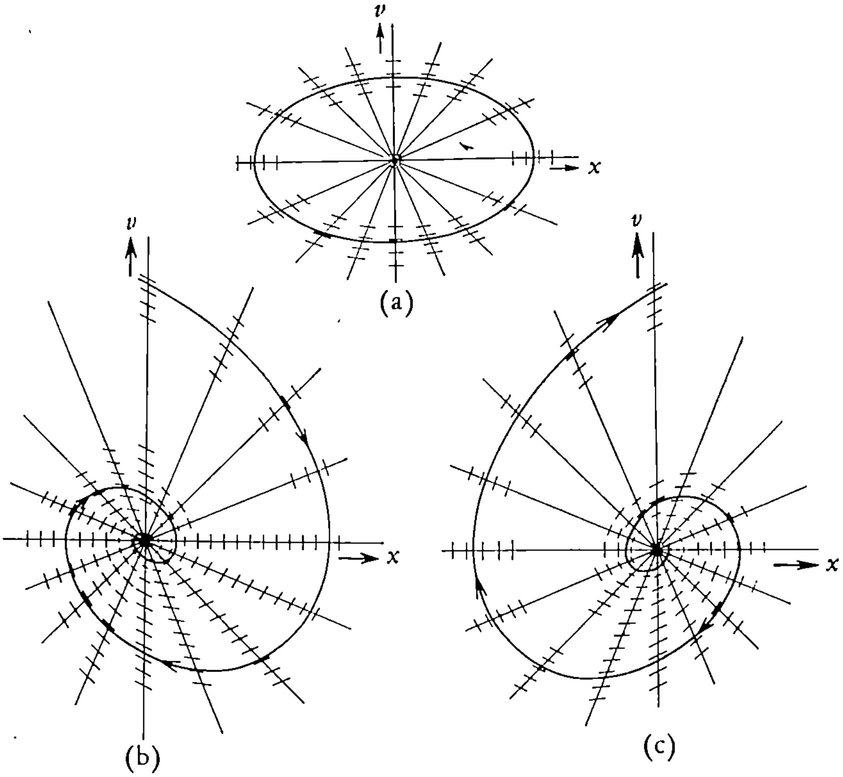
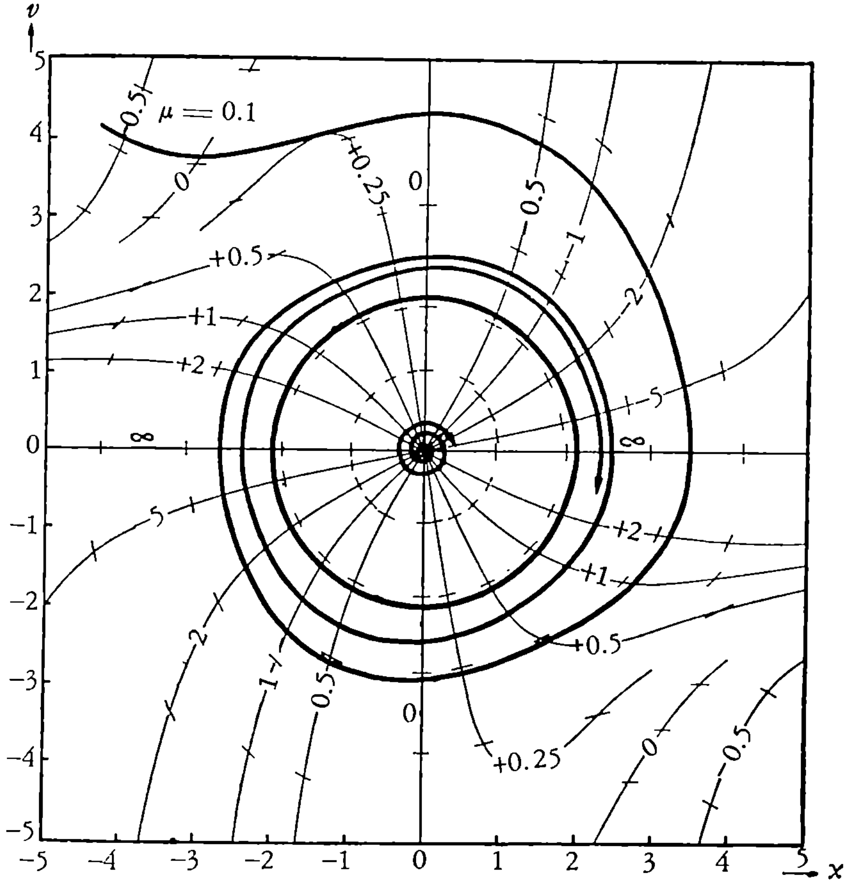


FIG. 1a, b, c

termined by the spring factor and the mass. The period of the motion corresponding to $\mu = 10$ is much larger and depends both on ω and μ.

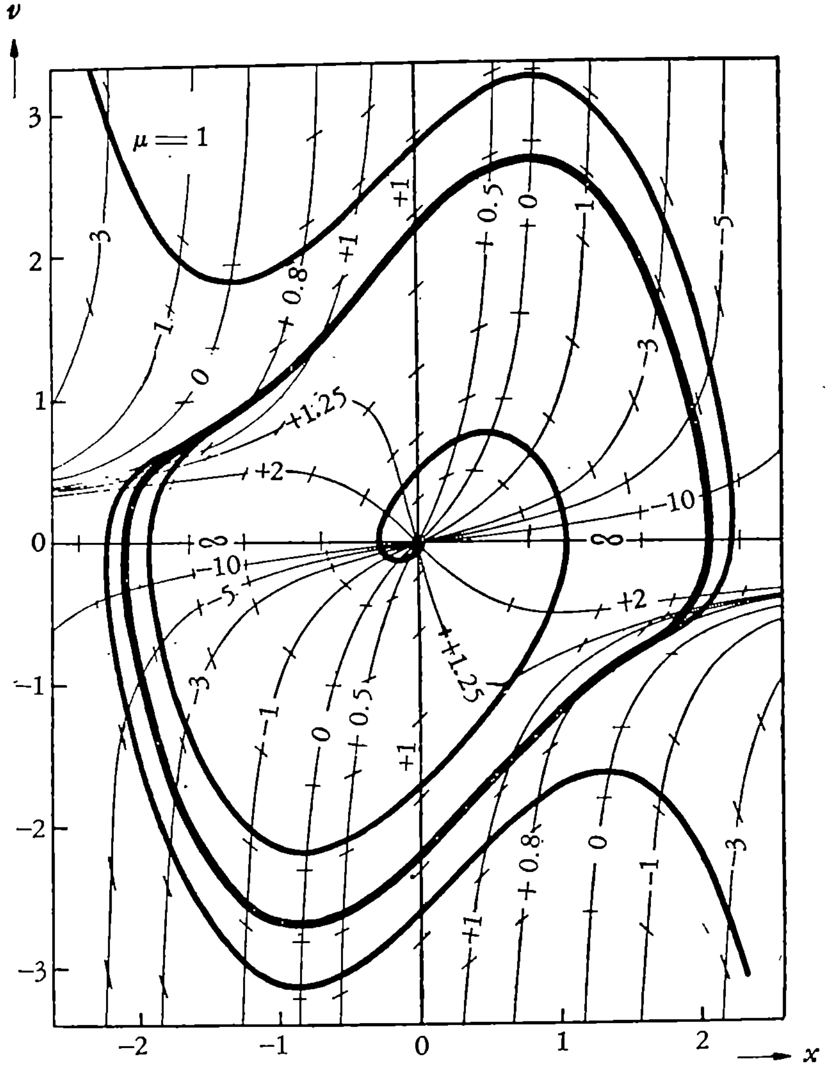


$$\frac{dv}{dx} = -\frac{x}{v} + \mu(1 - x^2) .$$

FIG. 2

From B. Van der Pol, Philosophical Magazine, vol. 2 (1926)

For this reason Van der Pol calls this type of motion a relaxation oscillation. The term self-excited oscillation is also often used. The physically interesting point is that we obtain a periodically undamped oscillation in a damped mechanical or electrical system because the damping coefficient is variable and for small deflections has a slight negative value.



$$\frac{dv}{dx} = -\frac{x}{v} + \mu(1 - x^2)$$

FIG. 3

From B. Van der Pol, Philosophical Magazine, vol. 2 (1926)

The stability of this type of motion can be investigated by the method of the "equations of variations." Let us assume that

$$v = \Phi(x,\ x_0,\ v_0) \tag{2.4}$$

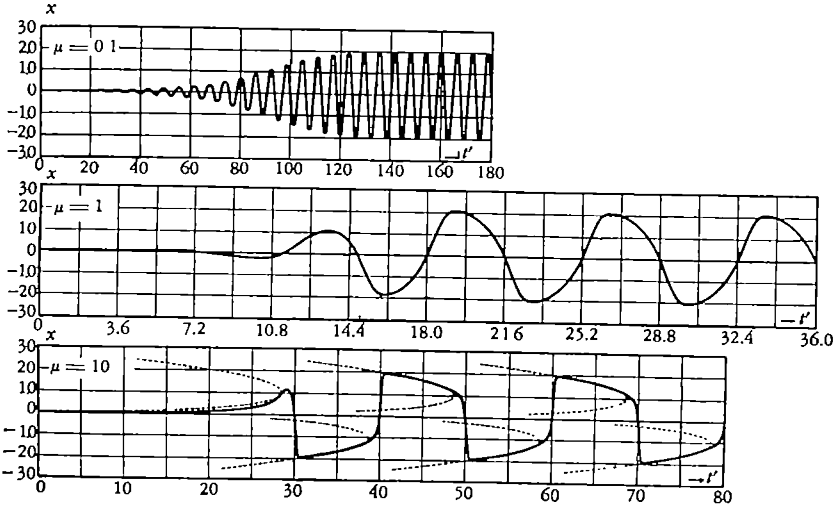


Fig. 4

From B. Van der Pol, Philosophical Magazine, vol. 2 (1926)

is a solution of (2.2) passing through the point x_0, v_0. Then we have

$$\frac{d}{dx}\left(\frac{\partial v}{\partial x_0}\right) = \frac{\partial}{\partial x_0}\left(\frac{dv}{dx}\right) = \frac{\partial}{\partial x_0}\left[-\,\omega^2\frac{x}{v} + \mu\phi(x)\right] \tag{2.5}$$

or

$$\frac{d}{dx}\left(\frac{\partial v}{\partial x_0}\right) = \frac{\omega^2 x}{v^2}\,\frac{\partial v}{\partial x_0} \tag{2.6}$$

where v is substituted from equation (2.4) as function of x, x_0 and v_0. It is seen that by integrating between the limits x_1 and x_2 we have

$$\left[\log\frac{\partial v}{\partial x_0}\right]_{x_1}^{x_2} = \int_{x_1}^{x_2}\frac{\omega^2 x}{v^2}\,dx = -\int_{x_1}^{x_2}\frac{1}{v}\,\frac{dv}{dx}\,dx + \mu\int_{x_1}^{x_2}\frac{\phi(x)}{v}\,dx.$$

Applying this equation to a closed cycle, we obtain

$$\log\left[\left(\frac{\partial v}{\partial x_0}\right)_2\Big/\left(\frac{\partial v}{\partial x_0}\right)_1\right] = \mu\oint\ \phi(x)dt. \tag{2.7}$$

In a similar way we obtain

$$\log\left[\left(\frac{\partial v}{\partial v_0}\right)_2 \Big/ \left(\frac{\partial v}{\partial v_0}\right)_1\right] = \mu \oint \phi(x)dt \tag{2.8}$$

or

$$\left(\frac{\partial v}{\partial v_0}\right)_2 = \left(\frac{\partial v}{\partial v_0}\right)_1 \exp\left\{\mu \oint \phi(x)dt\right\}. \tag{2.9}$$

Let us consider now an integral curve in the neighborhood of the closed cycle passing through the point x_0, $v_0+\delta v_1$, and denote the value of v obtained after completing one turn along the closed cycle by $v_0+\delta v_2$.

Then according to (2.9)

$$\delta v_2 = \delta v_1 \exp\left\{\mu \oint \phi(x)dt\right\}. \tag{2.10}$$

It is seen that if $\oint \phi(x)dt > 0$ the deviation of a neighboring integral curve from the closed cycle increases; if $\oint \phi(x)dt < 0$, decreases for each turn. We conclude that the motion corresponding to a periodic solution $x(t)$ is stable if $\oint \phi(x)dt < 0$.

The most important applications of the theory of oscillations of systems with nonlinear damping sketched in this section are in the field of radio. Both the theory and its applications have been treated by several authors. An extensive list of references is given in the paper of Van der Pol mentioned in the Bibliography at the end of this paper [5]. Van der Pol pointed out that in addition to oscillations in mechanical and electrical systems many other periodic phenomena have the character of relaxation oscillations. He found that differential equations similar to those of the relaxation oscillation describe with fair approximation various periodic biological processes, for example, the "struggle for life" of certain animal groups investigated mathematically by A. J. Lotka and Vito Volterra. Van der Pol believes that even periodic business cycles show a certain analogy to the relaxation oscillation of a physical system. The essential condition for such oscillations is negative damping for small deviations and a rather rapidly increasing positive damping for large deviations from the equilibrium position. The psychological response of certain groups of people to changing business conditions shows doubtless some analogy to the behaviour of mechanical systems capable of relaxation oscillations.

3. **Subharmonic resonance.** Let us now assume that the function $f(x, \dot{x})$ in equation (1.3) has the form $f(x, \dot{x}) = -\mu[\dot{x}+\psi(x)]$; that is, the system has linear damping but the elastic restoring force is a nonlinear function of the deflection. The center of interest is the response of the system to the action of a periodic force; as we mentioned, it was observed that in many cases the oscillations of the natural frequency ω of the system are excited by a periodic force whose frequency is a multiple of ω. This curious phenomenon is called "subharmonic resonance."

If we consider first the linear system corresponding to $\mu=0$ and assume that a periodic force $a \sin n\omega t$ is applied to the system, the general solution of the equation of motion is

$$x^{(0)} = c^{(0)} \sin (\omega t - \alpha^{(0)}) + \frac{a}{\omega^2(1 - n^2)} \sin n\omega t. \tag{3.1}$$

The first term on the right side represents a free oscillation of the system with an arbitrary amplitude $c^{(0)}$ and phase angle $\alpha^{(0)}$, and the second term a forced oscillation whose frequency is equal to the frequency of the impressed force. If we introduce slight constant damping [μ=const., $\psi(x)=0$] the free oscillation dies out and the forced vibration remains with a slightly modified amplitude. Let us now assume that $\psi(x)\neq 0$. Several authors reached the conclusion that in this case the solution consists of higher harmonics with the frequencies $2n\omega$, $3n\omega$, $\cdots$. In fact, if one would start from the forced oscillation

$$\bar{x}^{(0)} = \frac{a}{\omega^2(1 - n^2)} \sin n\omega t$$

and determine a next approximation by substituting $\bar{x}^{(0)}$ in $\psi(x)$, it is easily seen that one would obtain forced oscillations whose frequencies are multiples of $n\omega$. However, there is a fallacy in this conclusion because in order to obtain the most general type of periodic motion we have to start with the complete solution (3.1) and put

$$x = x^{(0)} + \mu x^{(1)} + \mu^2 x^{(2)} + \cdots. \tag{3.2}$$

The function $x^{(0)}$ has two undetermined constants c^0 and α^0. They must be chosen in such a way that x is a periodic function of t with the period $T=2\pi/\omega$. Substituting (3.2) in the differential equation

$$\ddot{x} + \omega^2 x = \mu f(x, \dot{x}) + a \sin n\omega t \tag{3.3}$$

and equating terms of the order μ on both sides of the equation, we obtain

$$\ddot{x}^{(1)} + \omega^2 x^{(1)} = f(x^{(0)}, \dot{x}^{(0)}). \tag{3.4}$$

The solution of (3.4) that satisfies the initial conditions $x^{(1)} = \dot{x}^{(1)} = 0$ for $t = 0$ is given by

$$x^{(1)} = \int_0^t f[x^{(0)}(\tau), \dot{x}^{(0)}(\tau)] \sin \omega(t - \tau) d\tau. \tag{3.5}$$

This solution corresponds to a periodic motion with the period $2\pi/\omega$, if

$$\begin{aligned} x^{(1)}(2\pi/\omega) &= -\int_0^{2\pi} f[x^{(0)}(\tau), \dot{x}^{(0)}(\tau)] \sin \omega\tau d\tau = 0, \\ \dot{x}^{(1)}(2\pi/\omega) &= \int_0^{2\pi} f[x^{(0)}(\tau), \dot{x}^{(0)}(\tau)] \cos \omega\tau d\tau = 0. \end{aligned} \tag{3.6}$$

Equations (3.6) are two equations for the undetermined parameters $c^{(0)}$ and $\alpha^{(0)}$. Since $x^{(0)}(t)$ and $\dot{x}^{(0)}(t)$ are periodic functions of t with the period $2\pi/\omega$, we can expand $f[x^{(0)}(t), \dot{x}^{(0)}(t)]$ in a Fourier series of the form $\sum_n a_n \sin n\omega t + \sum_n b_n \cos n\omega t$. According to equations (3.6) the coefficients a_1 and b_1 vanish. Therefore, the solution of equation (3.4) will have the form

$$\begin{aligned} x^{(1)} = c^{(1)} \sin [\omega t - \alpha^{(1)}] &+ \sum_{n=2}^{\infty} a_n \frac{\sin n\omega t}{\omega^2(1 - n^2)} + \frac{b_0}{\omega^2} \\ &+ \sum_{n=2}^{\infty} b_n \frac{\cos n\omega t}{\omega^2(1 - n^2)} \cdot \end{aligned} \tag{3.7}$$

Substituting $x = x^{(0)} + \mu x^{(1)}$ and $\dot{x} = \dot{x}^{(0)} + \mu \dot{x}^{(1)}$ into $f(x, \dot{x})$ in equation (3.3), we obtain an equation for $x^{(2)}$ and two equations analogous to equations (3.6). If we neglect higher powers of μ, we have two linear equations for $c^{(1)}$ and $\alpha^{(1)}$. Thus we are able to determine x as function of t to any desired accuracy by successive approximations.

It is seen that the final solution will contain a harmonic oscillation with the frequency ω. This oscillation is called the subharmonic resonance. It has a definite amplitude c and phase α, which can be determined by the method of successive approximations described above. Subharmonic resonance occurs if the equations (3.6) have real roots $c^{(0)}$ and $a^{(0)}$. Also it is necessary that the determinant of the linear equations for $c^{(1)}$ and $\alpha^{(1)}$ be different from zero.

The physical meaning of the conditions (3.6) is the following. The expression $\mu f(x^{(0)}, \dot{x}^{(0)})$ is an approximate expression for the force resulting from damping and elastic restraint. If the conditions (3.6) are satisfied, that is, the harmonic components of the force with the frequency ω are zero, the work done by this force on the deflection $c^{(0)} \sin (\omega t-\alpha^{(0)})$ vanishes.

The problem of subharmonic resonance was treated by a great number of authors. The most important papers are mentioned in the bibliography appended at the end of this lecture. Tuckerman also constructed a simple and ingenious model for the demonstration of subharmonic resonance.[2] His attention was called to the problem by the fact that certain parts of an airplane can be excited to violent oscillation by an engine running with a number of revolutions much larger than the natural frequency of the oscillating parts.

The method of this section can be applied to forced oscillations of systems with nonlinear damping treated in §2. It has been found that also in such systems a force of the frequency $n\omega$ can produce harmonic oscillations of the frequency ω. This procedure in the radio-technique is called "frequency demultiplication." Frequency demultiplication can occur in systems with positive or negative damping for small deflections, that is, also in such systems which are not capable of self-excited relaxation oscillations.

The mathematical problems outlined in §§2 and 3 represent simple cases of problems treated by H. Poincaré in his various researches. The mathematician will readily recognize in the closed curves shown in Figs. 2 and 3 the "cycles limites" studied by Poincaré in his paper, "Sur les courbes définies par une équation différentielle," *Oeuvres*, vol. 1. The method used in this section is analogous to the methods of the perturbation theory given in the *Méthodes nouvelles de la mécanique céleste.*

4. **Nonlinear problems in the theory of elasticity. Large deflections.** In the theory of elasticity usually two fundamental assumptions are made: small deflections and linear stress-strain relations (Hooke's law). Under these assumptions the problem of stress distribution in elastic systems is governed by linear partial differential equations. However, if we give up one or both of these assumptions, we obtain nonlinear equations. We consider in this section the case in which Hooke's law is assumed to be correct, that is, the stresses are linear functions of the strain components, but the deflections produced by external loads are so large that in the expressions for the strain components the squares and products of the deflections and their derivatives cannot be neglected. A classical example of this type is the problem of the "Elastica," that is, the bending and twisting of thin

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[2] The author is indebted to Dr. Tuckerman for an extensive list of references and for a film shown during the presentation of this lecture.

rods. Many of the great mathematicians of the eighteenth and nineteenth century, for example, James and Daniel Bernoulli, Euler, Lagrange and Kirchhoff, were interested in this problem.

Let us consider a simple case: two equal and opposite forces of magnitude P, acting along the x axis, are applied to the endpoints of a uniform strut. The endpoints can move freely along the x axis, but are restrained to this axis. The strut can be deflected in the x, y plane. Then we obtain for the deflection w in the y direction the nonlinear differential equation

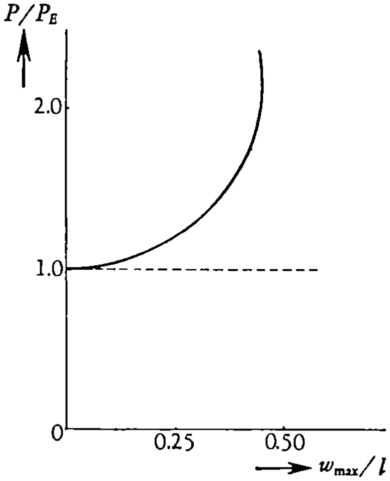

FIG. 5

$$\frac{d^2w}{ds^2} + \frac{P}{B}w\left[1 - \left(\frac{dw}{ds}\right)^2\right]^{1/2} = 0 \tag{4.1}$$

where s is the length of the arc measured along the center line of the strut and B is its bending stiffness. We assume that a positive value of P corresponds to compression in the strut.

For small deflections equation (4.1) becomes

$$\frac{d^2w}{ds^2} + \frac{P}{B}w = 0. \tag{4.2}$$

If we consider P as parameter in equations (4.1) and (4.2), the eigenvalues of (4.2) which comply with the boundary conditions $w=0$ for $x=0$ and $x=l$ (l the length of the strut) give us the points of bifurcation of the equilibrium (that is, the branch points of the solutions). These eigenvalues are equal to $P_n = Bn^2\pi^2/l^2$, where n is a positive integer. The first characteristic value ($n=1$) determines the so-called buckling load, or "Euler-load," $P_E = \pi^2 B/l^2$ of the strut. The eigenfunction corresponding to P_E gives the shape of the buckling column leaving the magnitude of the deflection undetermined. If we are interested in the force-deflection relation beyond the buckling load, we have to integrate equation (4.1), which can easily be done by use of elliptic functions. Fig. 5 shows the ratio between the force P and the buckling load P_E as function of the ratio between the deflection $w_{\max}$ of the midpoint and the length l of the beam.

The analogous but much more complicated problem of the finite deflection of plates loaded beyond their buckling limits has great importance for the determination of the strength of certain aircraft structures. The so-called stressed skin (or monocoque) method of construction widely employed for metal airplane wings and fuselages is based on the idea of loading the thin metal skin beyond its buckling limit. In fact the skin in the buckled or "wave" state is able to carry stresses which in many cases are large multiples of the buckling stress. Then the maximum load is determined by the ultimate strength of the buckled skin and the aim of theory is the determination of the stress distribution and the maximum stress occurring in the wave state.

To fix the idea let us assume that a plate of constant thickness t, which in the undeflected state is parallel to the x, y plane, is supported freely along two parallel edges, $y = \pm b$, and is subjected to uniformly distributed compressive forces acting normal to its other two edges, $x = \pm a$. The problem is much more complicated than the problem of the Elastica for the following reason: In the case of the deflected thin rod, the resultant of the normal stresses in an arbitrary cross section is determined statically by the end loads. In the case of the deflected plate, the distribution of the resultant stresses acting in the plate depends on the distribution of the normal deflection $w(x, y)$. Let us denote the components of the resultant stresses per unit length of the plate by σ_x, σ_y and τ_{xy}. It is known that these stresses are in equilibrium if they can be derived from a function $F(x, y)$—called Airy's function—by the relations

$$\sigma_x = \frac{\partial^2 F}{\partial y^2}, \qquad \sigma_y = \frac{\partial^2 F}{\partial x^2}, \qquad \tau_{xy} = -\frac{\partial^2 F}{\partial x \partial y}. \tag{4.3}$$

If the squares and products of the deflections and strain components are neglected, the function $F(x, y)$ satisfies the linear differential equation

$$\Delta\Delta F = 0. \tag{4.4}$$

On the other hand if no normal external load is applied to the plate, the normal deflection $w(x, y)$ obeys the equation

$$B\Delta\Delta w = 0 \tag{4.5}$$

where B is the bending stiffness of the plate per unit length. However, if we take into account the quadratic terms in the deflections and strain components, we obtain the following two simultaneous equations for the functions $F(x, y)$ and $w(x, y)$:

$$\begin{aligned} \Delta\Delta F &= E\left[\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}\right], \\ B\Delta\Delta w &= t\left[\frac{\partial^2 F}{\partial y^2}\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 F}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - 2\,\frac{\partial^2 F}{\partial x \partial y}\frac{\partial^2 w}{\partial x \partial y}\right]. \end{aligned} \tag{4.6}$$

These equations were first given by the present writer in 1910.

Let us assume that in the undeflected state

$$\sigma_x = -\Pi, \quad \sigma_y = \tau_{xy} = 0.$$

Then substituting these values in the second equation (4.6) we obtain the linearized equation for $w(x, y)$:

$$B\Delta\Delta w + \Pi t \frac{\partial^2 w}{\partial x^2} = 0. \tag{4.7}$$

This equation has solutions different from $w = 0$ which satisfy homogeneous boundary conditions for certain eigenvalues of the parameter Π.

For example, assuming hinged support at the edges $x = \pm a$, $y = \pm b$, the eigenvalues will be given by

$$\Pi = \frac{B}{t}\,\frac{\pi^2 a^2}{4m^2}\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)^2 \tag{4.8}$$

where m and n are integers. These eigenvalues represent the branch points of the equilibrium for the plate. The lowest eigenvalue gives the buckling load Π_B.

If the load Π is only slightly higher than this lowest critical value Π_B, the system of equations (4.6) can be solved by successive approximations. However, if the stress applied is many times larger than the "Euler stress" Π_B, this method converges much too slowly for practical purposes. A satisfactory solution of equations (4.6) for large values of Π/Π_B has not as yet been given. The present author in a joint paper with E. E. Sechler and L. H. Donnell attacked the problem with a somewhat crude approximation using the notion of the "effective width." The simplifying assumption was made that two narrow strips near the supported edges carry essentially the whole load. The computations of H. L. Cox, M. Yamamoto and K. Kondo, and K. Marguerre are more elaborate; however, they also are not devoid of arbitrary assumptions. Marguerre uses the Rayleigh-Ritz method for the approximate solution of equations (4.6). Since he retains only a few terms of a trigonometric series, his solution if Π/Π_E is very large does not approximate very well the shape of the deflected plate observed in the experiments.

Recently K. Friedrichs and J. Stoker attacked the problem of stress distribution in a circular plate beyond the first buckling, assuming axial symmetry for the deflection. Due to this assumption, the problem is simpler than that of the rectangular plate. They were able to integrate the system of equations (4.6) which in this case are reduced to ordinary differential equations. They investigated the solution for the whole range of values of the ratio Π/Π_E, where Π is the radial stress per unit area applied on the circumference of the plate and Π_E is the first critical value of Π. Below the buckling load, that is, for $\Pi < \Pi_E$, the circumferential stress σ_c is uniformly distributed over the whole plate and is equal to Π. If Π exceeds the buckling stress, σ_c increases near the edge of the plate and decreases in the center part. For large values of Π/Π_E, σ_c in the center part becomes negative; it changes from compression to tension, whereas the whole external load is practically carried by a narrow strip near the edge, which is analogous to the effective width introduced by the present writer in the theory of the rectangular plate mentioned above. In this limiting case the method of perturbations can be applied with success since the perturbation term contains the derivative of the highest order. This amounts to an asymptotic integration of the differential equations resulting from equations (4.6). Let us denote the average value of σ_c by p and the edge stress by p_e. Then Friedrichs and Stoker obtained the following asymptotic approximation for the relation between p and p_e

$$\frac{p}{E(t/a)^2} = .73\left[\frac{p_e}{E(t/a)^2}\right]^{2/3} \tag{4.9}$$

where a is the radius of the plate.

The same relation with a numerical factor of the same order of magnitude is easily obtained by the method which the present author used in the case of rectangular plates. Hence, the analysis of Friedrichs and Stoker can be considered as a verification of the authors' simplifying assumptions.

The theory of large deflections of plates subjected to lateral loads requires the integration of the nonhomogeneous system of equations

$$(4.10)\qquad \begin{aligned} \Delta\Delta F &= E\left[\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}\right], \\ F\Delta\Delta w - t\left[\frac{\partial^2 F}{\partial y^2}\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 F}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - 2\frac{\partial^2 F}{\partial x \partial y}\frac{\partial^2 w}{\partial x \partial y}\right] &= p(x, y) \end{aligned}$$

where $p(x, y)$ is the lateral load per unit area. This system of equations can be solved by successive approximations. Denoting the solution of the nonhomogeneous linear equation

$$(4.11)\qquad \Delta\Delta w = p/B$$

by $w_0(x, y)$, we substitute this function in the first of the equations (4.10) and determine a first approximation of $F_1(x, y)$; then, substituting the derivatives of F_1 in the second equation (4.10), we calculate the next approximation $w_1(x, y)$, and so on. The system of equations (4.10) was solved for certain simple contours and loading conditions by S. Timoshenko and S. Way.

The problem of large deflections of curved rods was treated by several authors. Thus R. Grammel investigated the deformation of a circular ring by uniformly distributed moments along the circular ring axis, which tend to turn the cross sections of the ring around their centers. He found that at a critical value of the moment the ring jumps over into a new equilibrium position.

Another interesting case is the deflection of a flat arch under lateral load where the deflection is of the same order of magnitude as the height of the arch. Let us assume, for example, that the two endpoints of a flat circular arch are pin-joined at two fixed points on the x axis and the arch is loaded by a uniform normal load p per unit length. If R is the radius of curvature of the arch, the uniform normal load produces a compressive force equal to pR and the first bifurcation point of the equilibrium is obtained approximately by Euler's equation for a straight strut:

$$(4.12)\qquad (pR)_E = \pi^2 \frac{B}{l^2}$$

where B is the bending stiffness, assumed to be constant along the length and l is the length of the arch. The nonlinear equation, which takes into account the large deflections, is analogous to the equation of the Elastica mentioned before. However, whereas in the case of the originally straight Elastica the thrust to be applied increases with the deflection, in the case of the originally curved arch, we find that the load p which is necessary to keep the rod in the deflected shape can be smaller than the critical load corresponding to equation (4.12). Thus, as Fig. 5 shows, a straight beam under axial load has only one equilibrium position for values of the parameter P, which are smaller than the first eigenvalue P_B, whereas the flat arch has several possible equilibrium positions for values $p < p_B$ as shown in Fig. 6. Hence the engineer is more interested in the smallest value of the parameter p, which is compatible with the existence of such equilibrium positions than in the value p_B corresponding to the branch point of these solutions. To be sure a continuous transition from the unloaded position to such an equilibrium state is only possible if the load p first reaches the critical value p_B. However, imperfectness of the structure and slight deviations from the uniform load distribution might cause a jump between equilibrium states along AB and BC, practically at any value p between $p_{\min}$ and p_e.

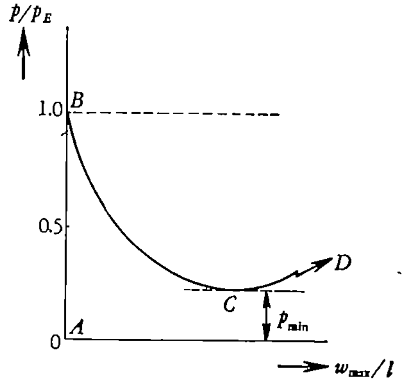

FIG. 6

It seems to the writer that these considerations are of great importance for the explanation of certain systematic discrepancies found between the breakdown loads of curved shells observed experimentally and computed analytically from the eigenvalues of linearized equations. For example, the differential equation for the small deflections of a thin-walled spherical shell of constant thickness subjected to uniform external pressure was carefully investigated by several authors, for example, R. Zoelly, E. Schwerin, and A. Van der Neut. The critical pressure corresponding to the smallest eigenvalue of the differential equation is found to be equal to

$$p = \frac{2}{[3(1-\nu^2)]^{1/2}} E\left(\frac{t}{R}\right)^2 \tag{4.13}$$

where E is Young's modulus, ν Poisson's ratio, t the wall thickness, and R the radius of the spherical shell. The experiments show that the spherical shell collapses under a pressure that is only one-quarter to one-third of this value. Also, the shape of the buckled shell is very different from the wave form predicted by the theoretical eigenfunction that corresponds to the smallest eigenvalue. In a paper published jointly with H. S. Tsien the writer has shown by approximate calculation of the large deflections the existence of more than one equilibrium state at values of the parameter p which are much smaller than the value given by equation (4.13) and are of the order of the values observed experimentally. A similar discrepancy between the prediction of the linear theory and the experimental results in the case of the buckling of cylindrical shells under compression applied parallel to the cylinder axis probably can also be explained by analogous considerations.

Let us finally consider the problem of large deformations of a three-dimensional isotropic elastic continuum. This problem, although treated in great detail in the past by several authors, was more or less neglected until interest was stimulated by the possible application of tensor calculus as shown by recent publications of E. Trefftz, F. D. Murnaghan, and M. A. Biot. The first treatment is due to G. Kirchhoff (1852); later J. Boussinesq (1872), M. Brillouin (1891) and J. Finger (1894) were interested in the problem. An excellent presentation of the subject can be found in a review of E. and F. Cosserat (1896). The analysis of all these authors is based on the fundamental assumption that the state of the matter at a point P, whose coordinates before the deformation are x_i, is determined by the deformation of the infinitesimal neighborhood of P, that is, by the coefficients of the linear transformation

$$d\bar{x}_i = d(x_i + \xi_i) = \frac{\partial \bar{x}_i}{\partial x_k} dx_k \tag{4.14}$$

where x_i and $\bar{x}_i$ are the coordinates of P before and after deformation. The vector whose components are $\bar{x}_i - x_i = \xi_i$ is the displacement vector. The transformation (4.14) has nine coefficients. However, a rigid rotation does not produce stresses, and therefore the stresses can only depend on the coefficients of the square of the line element $d\bar{x}_i$. We write

$$d\bar{x}_i^2 = g_{ik}dx_i dx_k \tag{4.15}$$

where

$$g_{ik} = \alpha_{il}\alpha_{lk} \tag{4.16}$$

and the α's are the coefficients of the transformation (4.14). In the case of rigid rotation, $g_{ik} = \delta_i^k$ (that is, $g_{ik} = 1$ if $i = k$ and $g_{ik} = 0$ if $i \neq k$). Hence, we conclude that the six quantities $g_{ik} - \delta_i^k$ determine the pure deformation of an element. E. and F. Cosserat, and later E. Trefftz, defined these quantities as the components of the strain ϵ_{ik} and concluded that the stress-strain energy of an isotropic body must be a function of the invariants which can be constructed of the quantities ϵ_{ik}. If the form of the energy function is assumed, the components of the stress tensor are obtained by variation of the energy and application of the principle of virtual displacements.

If the squares and products of the derivatives of the ξ_i's are neglected, $g_{ik} = \delta_i^k + \partial\xi_i/\partial x_k + \partial\xi_k/\partial x_i$. Hence, in this case the transformation (4.14) can be split into two parts:

$$d\xi_i = \frac{1}{2}\left(\frac{\partial\xi_i}{\partial x_k} + \frac{\partial\xi_k}{\partial x_i}\right)dx_k + \frac{1}{2}\left(\frac{\partial\xi_i}{\partial x_k} - \frac{\partial\xi_k}{\partial x_i}\right)dx_k. \tag{4.17}$$

The first term represents an infinitesimal pure deformation, the second term an infinitesimal rigid rotation. The quantities $\epsilon_{ik} = \frac{1}{2}(\partial\xi_i/\partial x_k + \partial\xi_k/\partial x_i)$ are known as components of the strain tensor. If Hooke's law is valid, the stress-strain energy is a homogeneous quadratic form of the ϵ_{ik}'s, and the components of the stress tensor are derivatives of this quadratic form with respect to the ϵ_{ik}'s.

If higher terms in the derivatives of the ξ_i's are taken into account, two difficulties arise:

(a) The separation of the pure deformation from the rotation and the definition of the strain tensor becomes somewhat arbitrary.

(b) One must differentiate between stresses referred to the surface elements of the undeformed and deformed system.

M. A. Biot suggested the following definition of the rotation and the strain tensor: The linear transformation

$$\bar{x}_i = x_i + \frac{\partial\xi_i}{\partial x_k}x_k \tag{4.18}$$

can be replaced by two successively applied linear transformations, where the first one:

$$x_i' = \beta_{ik} x_k \tag{4.19}$$

is an orthogonal transformation representing a finite rigid rotation and the coefficients of the second one:

$$\bar{x}_i = \gamma_{ik} x_k' \tag{4.20}$$

constitute a symmetrical matrix, that is, $\gamma_{ik} = \gamma_{ki}$. Then the first transformation defines in a unique way the rotation and the second one the pure deformation. The strain tensor is defined by

$$\epsilon_{ik} = \tfrac{1}{2}(\gamma_{ik} - \delta_i^k). \tag{4.21}$$

We notice that the γ_{ik}'s are equal to the g_{ik}'s only in the first approximation; they differ if higher terms in the derivatives of the ξ_i's are taken into account.

F. D. Murnaghan's definition of the strain tensor is based on the inverse transformation

$$dx_i = \frac{\partial x_i}{\partial \bar{x}_k} d\bar{x}_k. \tag{4.22}$$

If we express the square of the line element ds_i by $d\bar{s}_i$, we obtain an expression of the form

$$ds_i^2 = h_{ik} d\bar{x}_i d\bar{x}_k. \tag{4.23}$$

Then we can consider

$$\epsilon_{ik} = \tfrac{1}{2}(\delta_i^k - h_{ik}) \tag{4.24}$$

as the components of the strain tensor.

Using different conceptions for the strain tensor, Hooke's generalized law appears in different forms in the publications mentioned. Fortunately the question is of academic rather than practical importance. With the exception of a few materials with large elastic extension, such as rubber, the strain components are very small within the limit of elasticity, and the usual first approximation is entirely satisfactory. However, if at least one dimension of the body is small compared to the others, as in the case of plates, shells, and beams, it is possible that some of the nine derivatives $\partial\xi_i/\partial x_k$ are large, although the six components of the strain tensor are small. This takes place in some of the problems mentioned before in this section. It appears that the most important task of the general transformation theory of finite deformations should be a systematic and exact verification of the differential equations for large deflection of beams, plates and shells. The present deduction of these equations is based on intuitive assumptions rather than on an exact application of the general equations of elasticity for a three-dimensional continuum.

5. **Stress distribution in plastic and noncoherent masses.** Let us drop now the assumption that the material obeys Hooke's law. Two types of nonlinear strain-stress relations have practical importance. First, in some materials, although there is a unique and reversible relation between stress and strain, the linear law does not approximate this relation with a sufficient accuracy. Second, we find that beyond a certain stress limit the deformation is no longer *purely elastic*; that is, after removing the load, we obtain a so-called *permanent set*. The nonreversible part of the deformation is known as *plastic deformation* or *plastic flow*; the stress limit beyond which plastic flow is obtained is called the *yield point*. In general if plastic flow takes place the yield point is raised, that is, additional plastic flow can only be produced by further increase of the load. This effect is called *cold hardening* of the material. Stress distribution in bodies with nonlinear elastic behaviour and in materials undergoing cold hardening were treated analytically only incidentally. However, the limiting case in which cold hardening can be neglected is readily accessible to analytical treatment and has been the subject of quite a large number of ingenious investigations.

A material that is capable of plastic deformation without cold hardening is called *perfectly plastic*. The experiments show that the yield condition of a perfectly plastic material is given by a constant difference between the largest and smallest principal stress. This condition must be satisfied in the whole domain in which plastic flow takes place. Let us restrict ourselves to two-dimensional problems, assuming that the material flows in planes parallel to the x, y plane. In this case the value of the third principal stress, which is parallel to the z axis, necessarily lies between the values of the two other principal stresses. If we derive the stresses σ_x, σ_y, τ_{xy} from the stress function $F(x, y)$ by means of the relations

$$(5.1) \qquad \sigma_x = \frac{\partial^2 F}{\partial y^2} = t, \qquad \sigma_y = \frac{\partial^2 F}{\partial x^2} = r, \qquad \sigma_{xy} = -\frac{\partial^2 F}{\partial x \partial y} = -s,$$

the square of the difference of the principal stresses σ_1 and σ_2 is equal to

$$(5.2)\qquad (\sigma_1 - \sigma_2)^2 = (r + t)^2 - 4(rt - s^2).$$

Hence, the yield condition can be written in the form

$$(5.3)\qquad (r + t)^2 - 4(rt - s^2) = K^2,$$

where K is a characteristic constant for the material.

Equation (5.3) is a nonlinear partial differential equation of the second order of the hyperbolic type. The equation of the characteristics is

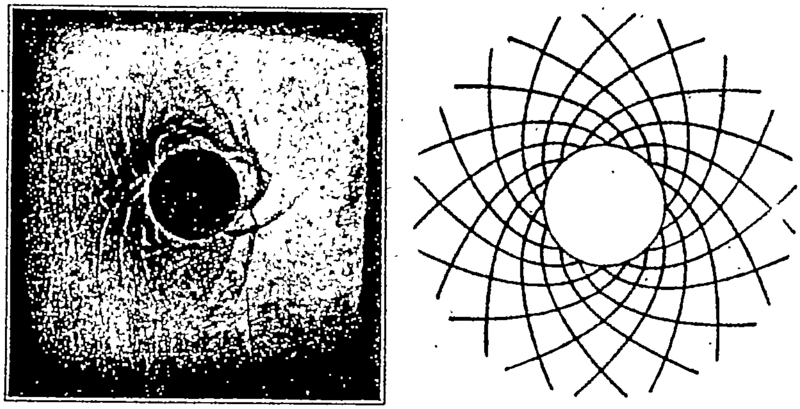

FIG. 7
From A. Nadai, *Plasticity*

$$(5.4)\qquad (r - t)\left[\left(\frac{dy}{dx}\right)^2 - 1\right] - 4s\,\frac{dy}{dx} = 0$$

or

$$(5.5)\qquad 2\frac{dy}{dx}\Big/\left[1 - \left(\frac{dy}{dx}\right)^2\right] = \frac{r - t}{2s}\,.$$

Let us denote in an arbitrary point the angle between the tangents to the characteristic curves passing through this point and the positive x axis by θ_1 and θ_2; then according to equation (5.5) θ_1 and θ_2 satisfy the equation

$$(5.6)\qquad \tan 2\theta = \frac{r - t}{2s}\,.$$

Now the angles ϕ_1 and ϕ_2 between the directions of the principal stresses σ_1 and σ_2 and the x axis are the roots of the equation

$$\tan 2\phi = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} = \frac{2s}{r - t} \cdot \tag{5.7}$$

From (5.6) and (5.7) we obtain $\tan 2\theta \tan 2\phi = 1$. The curves which follow the direction of the principal stresses are known as isostatics. Hence, the characteristics are a system of orthogonal curves that intersect the isostatics at 45° and are tangential to the direction of the maximum shearing stress. Now the plastic flow consists essentially of a gliding along the planes in which the maximum shearing stresses occur. Therefore the characteristics are the traces of the *gliding surfaces* in the x, y plane. Sometimes they are visible on the surface of the body subjected to plastic flow without any artifice (Fig. 7); in other cases they can be brought out by etching (Fig. 8).

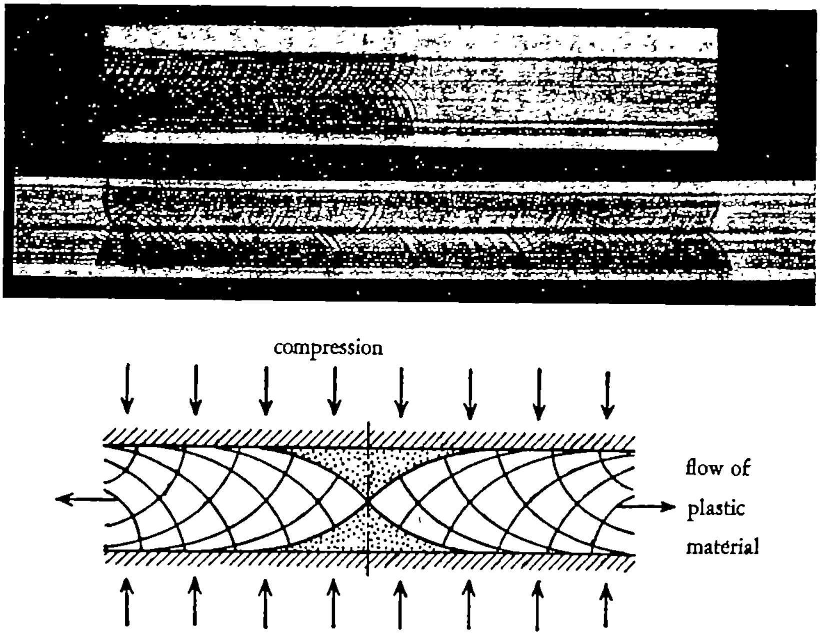

FIG. 8
From A. Nadai, *Plasticity*

The geometrical properties of the characteristics given by equation (5.5) and the integration of the differential equation (5.3) by means of Riemann's method were studied by H. Hencky, L. Prandtl, C. Carathéodory and E. Schmidt. Let us introduce the parameters of the characteristics α and β as independent variables, the average normal stess $p=\frac{1}{2}(r+t)$, and the coefficients A and B of the line element

$$ds^2 = Ad\alpha^2 + Bd\beta^2 \tag{5.8}$$

as unknown functions. Then we obtain the equations

$$\frac{\partial^2 p}{\partial\alpha\partial\beta} = \frac{K}{2} \tag{5.9}$$

and

$$\frac{1}{B}\frac{\partial A}{\partial\beta} = f(\alpha), \qquad \frac{1}{A}\frac{\partial B}{\partial\alpha} = g(\beta) \tag{5.10}$$

where $f(\alpha)$ and $g(\beta)$ are arbitrary functions of the parameters α and β. To integrate the equations (5.9) and (5.10) we start from two given characteristic curves $\alpha=0$ and $\beta=0$ and choose the values of $A(\alpha, 0)$ and $B(0, \beta)$ along these curves, putting, for example, $A(\alpha, 0)=B(0, \beta)=1$. Since the characteristic curves are orthogonal,

$$-\frac{1}{B}\frac{\partial A}{\partial\beta} = \frac{\partial\theta}{\partial\alpha},$$

where θ is the angle between the x axis and the tangent of a curve $\beta=\text{const.}$, the tangent being drawn in the direction of increasing α; similarly

$$\frac{1}{A}\frac{\partial B}{\partial\alpha} = \frac{\partial\theta}{\partial\beta}\cdot$$

Hence, the functions $f(\alpha)$ and $g(\beta)$ are known and the system of equations (5.10) can be integrated by Riemann's method. If the network of characteristic curves is so determined, equation (5.9) gives the value of p. Then the magnitudes of the principal stresses can be calculated from $\frac{1}{2}(\sigma_1+\sigma_2)=p$ and $|\sigma_1-\sigma_2|=K$. The stresses are directed at every point parallel to the bisectors of the characteristic curves.

The stress distribution in a mass of sand which is in the limiting state of equilibrium, that is, just about to collapse, can be determined in a similar manner. If the sand is under a large external load and the influence of its own weight can be neglected, the stress function $F(x, y)$ satisfies the equation

$$(r+t)^2 - \frac{4}{1-f^2}(rt - s^2) = 0 \tag{5.11}$$

where f is the coefficient of friction between the sand particles. This equation is identical to the fundamental equation of the theory of earth pressure developed by Rankine and can be applied to various problems in soil mechanics. However, in many applications gravity has to be taken into account. The differential equation of the stress function in this case is no longer homogeneous, but contains also terms which are linear in r, t and s. Fortunately, in a few cases of practical importance the partial differential equation can be reduced to an ordinary equation; such problems have been treated by several authors.

Both in the theory of plastic and noncoherent masses, so far we have assumed that the stress distribution is two-dimensional. If we try to extend the theory to the three-dimensional case, the difficulty arises that the gliding conditions (5.3) and (5.11) involve only two principal stresses leaving the third principal stress undetermined. There are two ways of attacking the problem. First, one can consider the equilibrium of the whole system as a variation problem: the potential energy of the system has to be a minimum under the restriction that the yield condition is not violated at any point. Hence, we have to solve a variation problem, where the extremals satisfy an inequality. The solution of this variation problem leads to the equation (5.3) or (5.11) for the largest and smallest principal stresses and to an additional equation for the median principal stress.

The second method of dealing with the problems of three-dimensional stress distribution is to introduce certain empirical stress and rate of strain relations for the plastic deformation. One can assume, for example, proportionality between the stress tensor and the derivative of the strain tensor with respect to time. This assumption replaces Hooke's law and furnishes the necessary compatibility equations for the determination of the third principal stress. The same two methods can be applied to the practically quite important case in which one part of the body is in the elastic and another part in the plastic state. Both methods furnish a sufficient number of boundary conditions for the surface separating the elastic and plastic domains. However, this surface is not a priori given and, therefore, the solution of the problem is in general very difficult. It has been worked out only in a few geometrically simple cases.

6. **Hydrodynamics of ideal incompressible fluids.** In this and the following sections we will consider nonlinear problems occurring in fluid mechanics. In §§6 to 8 we assume a nonviscous and incompressible fluid; in §9 we shall take into account viscosity, and in §10 compressibility.

The velocity distribution in an ideal incompressible fluid can be determined by the so-called Eulerian equations of motion. Denoting the components of the velocity by u_i, the density of the fluid by ρ, the pressure by p and the components of the volume forces per unit mass by F_i, we have three equations for the dynamic equilibrium:

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + F_i, \tag{6.1}$$

and the kinematic equation of continuity:

$$\frac{\partial u_i}{\partial x_i} = 0. \tag{6.2}$$

The equations (6.1) are nonlinear in the u_i's. However, if we assume that the flow is without vorticity, that is, the u_i's are derivatives of a velocity potential ϕ, and that the forces F_i are derivatives of a force potential V, the three equations (6.1) are satisfied by their first integral

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}\frac{\partial \phi}{\partial x_i}\frac{\partial \phi}{\partial x_i} + \frac{p}{\rho} - V = \text{const.}, \tag{6.3}$$

and the equation of continuity becomes

$$\Delta\phi = 0. \tag{6.4}$$

Hence, in the case of irrotational motion the problem of determining the velocity distribution is reduced to finding solutions of Laplace's equation (6.4) that satisfy certain boundary conditions. These boundary conditions are in certain cases linear. For example, if the flow takes place between solid boundaries, the component of the velocity normal to these boundaries vanishes, and, therefore, $\partial\phi/\partial n = 0$. Hence, in this case the problem of finding the velocity distribution is reduced to a well known linear boundary problem of the potential theory.

If the fluid is bounded by a free surface, that is, by another fluid of negligible specific gravity which is at rest, the boundary condition has to express that the pressure p is constant along the free boundary. Hence, we have from equation (6.3)

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}\frac{\partial \phi}{\partial x_i}\frac{\partial \phi}{\partial x_i} - V = \text{const.}$$

Assuming that gravity is the only force acting on the fluid, we substitute $V = -gx_3$, where g is the acceleration of gravity directed parallel to the negative x_3 axis, and obtain

(6.5) $$\frac{\partial\phi}{\partial t}+\frac{1}{2}\frac{\partial\phi}{\partial x_i}\frac{\partial\phi}{\partial x_i}+gx_3 = \text{const.}$$

The difficulty of solving flow problems with free boundary arises not only from the nonlinear character of the boundary condition (6.5), but also from the fact that the boundary itself is unknown. Three-dimensional problems of this type have been solved only in a few cases; hence, we shall restrict ourselves to two-dimensional flow in the x_1, x_3 plane. We also assume that the flow is stationary, that is, ϕ is independent of time and therefore $\partial\phi/\partial t=0$.

Replacing the coordinates x_1 and x_3 by x and y, we have at a free surface the boundary condition

(6.6) $$\frac{1}{2}\left[\left(\frac{\partial\phi}{\partial x}\right)^2+\left(\frac{\partial\phi}{\partial y}\right)^2\right]+gy = \text{const.}$$

Let us introduce the complex variable $z=x+iy$ and two analytical functions of z, namely, $f=\phi+i\psi$, where ϕ is the velocity potential and ψ the stream function, and $\omega=\theta+i\log q$, where θ is the angle between the velocity vector and the x axis, and q the magnitude of the velocity vector. The components of the velocity of flow in the x and y direction are equal to $u=\partial\phi/\partial x$ and $v=\partial\phi/\partial y$ respectively. Hence, $u-iv=df/dz$ and, therefore, $\omega=i\log(u-iv)$. Then $\theta=\tan^{-1} v/u$ and

$$\log q = \log |u-iv| = \tfrac{1}{2}\log\left[\left(\frac{\partial\phi}{\partial x}\right)^2+\left(\frac{\partial\phi}{\partial y}\right)^2\right]$$

are harmonic functions of x and y and also of ϕ and ψ. Putting $\tau=\log q$, we write the boundary condition (6.6) in the form

(6.7) $$\tfrac{1}{2}e^{2\tau}+gy = \text{const.}$$

or, by differentiating with respect to the arc along the free boundary,

(6.8) $$e^{2\tau}\frac{\partial\tau}{\partial s}+g\frac{\partial y}{\partial s} = 0.$$

Taking into account that the free boundary is a streamline ($\psi=\text{const.}$)

$$\frac{\partial\tau}{\partial s}=\frac{\partial\tau}{\partial\phi}q=\frac{\partial\tau}{\partial\phi}e^{\tau}.$$

On the other hand, $\partial y/\partial s=\sin\theta$ and, therefore, we obtain from (6.8)

(6.9) $$\frac{1}{3}\frac{\partial}{\partial\phi}(e^{3\tau}) = -g\sin\theta.$$

We now consider ϕ and ψ as independent variables; then we have to satisfy the boundary (6.9) condition along a horizontal straight line $\psi=\text{const.}$ in the ϕ, ψ plane, where $\Delta\theta=0$ and $\Delta\tau=0$. We remember that θ and τ are the real and imaginary parts of an analytical function of $\phi+i\psi$.

The nonlinear boundary condition (6.9) can be linearized in the following two cases:

(a) If the effect of gravity is negligible and the fixed boundaries are plane surfaces, then we obtain for the free boundary, from (6.9),

$$\frac{\partial\tau}{\partial\phi} = 0. \tag{6.10}$$

The fixed boundaries being plane surfaces, their traces in the x, y plane are straight lines, that is, along such boundaries θ is constant, and we have therefore

$$\frac{\partial\theta}{\partial\phi} = 0. \tag{6.11}$$

This case was treated in great detail by G. Kirchhoff, H. Helmholtz, Lord Rayleigh and others, and has been applied to many practical problems such as the *vena contracta*, the flow of water over a weir, the air resistance of an inclined flat plate, the planing of a flat plate over a water surface, and so on.

(b) If gravity is taken into account and the motion is only slightly different from a uniform parallel flow with the velocity U, we can assume that θ is small and $\tau=T+u/U$, where $T=\log U$. Neglecting the squares and products of θ, u and v, we have from equation (6.9)

$$U^2\frac{\partial u}{\partial\phi} = -g\theta \tag{6.12}$$

or

$$U^3\frac{\partial\theta}{\partial\psi} + g\theta = 0. \tag{6.13}$$

Let us assume that $\psi=0$ corresponds to the free surface, $\psi=-\infty$ to the infinite depth of the water mass. We also assume that θ, u and v disappear at infinite depth. Then

$$\theta = \theta_0 e^{-\lambda\psi}\sin\lambda\phi \tag{6.14}$$

satisfies the equation $\Delta\theta=0$ and the boundary condition for $\psi=-\infty$.

We still have to satisfy the boundary condition (6.13) for $\psi=0$. Substituting (6.14) into equation (6.13), we have

$$-U^2\lambda + g = 0 \tag{6.15}$$

and, therefore,

$$\theta = \theta_0 e^{-g\psi/U^3} \sin g\phi/U^3. \tag{6.16}$$

Let us denote the equation of the free surface by $\eta=\eta(x)$. With the

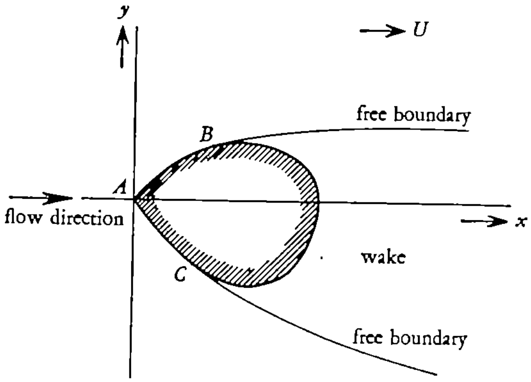

FIG. 9

approximation used here we can replace ϕ by Ux and $d\eta/dx$ by θ. Hence, we obtain

$$\eta = \eta_0 \cos \frac{xg}{U^2} \tag{6.17}$$

where $\eta_0=-\theta_0 U^2/g$.

If we consider the flow relative to a coordinate system moving from right to left with the velocity U, the surface (6.17) represents a wave progressing with the velocity U on the surface of a water mass which is at rest at infinite depth. The wave length L is equal to $2\pi U^2/g$.

The solution of waves of small amplitude outlined here is the classical solution of Airy and Stokes. In the next two sections we drop the restrictions that made the linearization of the boundary conditions possible and consider the resulting nonlinear problems.

7. **Fluid jets.** T. Levi-Civita was the first to extend the theory of Kirchhoff and Helmholtz to jets bounded by curved survaces. In a paper published in the Rendiconti del Circolo Matematico di Palermo in 1907, he treated the problem of the wake behind a blunt cylindrical body (Fig. 9). Let us assume that the fluid follows the curved surfaces

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AB and AC, and that behind the body a dead fluid region, the so-called *wake*, is bounded by free surfaces which extend from B and C to infinity. If we consider ϕ and ψ as independent variables, and θ and $\tau = \log q$ as two harmonic functions of ϕ and ψ, the boundary condition for the free surfaces is $\tau = \text{const}$. To find the boundary condition for a fixed curved surface it is convenient to use so-called natural equations. For example, we can express the inclination θ as function of the length of arc s. Then the curvature $d\theta/ds$ is also a known function of θ, say $d\theta/ds = kK(\theta)$, where k is a numerical parameter. Now the trace of the fixed surface is a streamline, and, therefore,

$$\frac{\partial\theta}{\partial s} = \frac{\partial\theta}{\partial\phi}\frac{\partial\phi}{\partial s} = e^{\tau}\frac{\partial\theta}{\partial\phi}.$$

Hence, we obtain the boundary condition

$$\frac{\partial\theta}{\partial\phi} = ke^{-\tau}K(\theta). \tag{7.1}$$

It is seen that for plane surfaces $K(\theta) = 0$. Then we have the special case treated by Kirchhoff, Helmholtz and Lord Rayleigh and equation (7.1) is reduced to equation (6.11).

Levi-Civita, by use of convenient transformations of the variables, reduced the problem to the following:

To find an analytic function $\theta + i\tau$ of a complex variable $\zeta = \xi + i\eta$, which is zero for $\zeta = 0$, regular in the half of the unit circle above the real axis, and whose real and imaginary parts satisfy the following boundary conditions:

(a) on the circumference of the unit half circle

$$\frac{d\theta}{d\sigma} = lf(\sigma)e^{-\tau}K(\theta)$$

where σ is the argument of ζ on the unit circle and $f(\sigma)$ is a known function of σ.

(b) on the diameter along the real axis $\tau = 0$.

The problem can be solved by successive approximations or by expansion in a series of powers of the parameter k.

The theory of Levi-Civita was further developed by S. Brodetzky, C. Schmieden, S. Bergmann, V. Valcovici, R. Thirry, L. Rosenhead and others. An interesting aspect of this theory is that the *point of separation*, that is, the point at which the free boundary begins, is undetermined between certain limiting values. These limiting values are determined by the conditions that the free boundaries shall not intersect either the fixed boundaries or each other. The question of existence and uniqueness of the solution for a given separation point was discussed by A. Weinstein, G. Hamel, H. Weyl and K. Friedrichs.

If we take gravity into account and the fixed surfaces are curved, the boundary conditions are nonlinear both at the free and the fixed boundaries. The theory of such "heavy" jets has many important practical applications in the design of weirs, spillways, and so on. A spillway (Fig. 10) is a steep channel at the downstream face of a dam, designed for the smooth discharge of large quantities of water. The engineer has to design the spillway in such a way that the pressure does not sink below the atmospheric pressure at any point on the surface where the water is in contact with the spillway. If we compare the pressure at the points P and Q—that is, on the opposite sides of a cross section normal to the average flow direction—we see that the height difference between P and Q causes the pressure to be higher at Q whereas the centrifugal force causes the pressure to be higher at P. A correct design requires an approximate balance between these two opposite effects. If the pressure at Q is sub-atmospheric, air enters between the jet and the wall and might cause an unstable jumping of the jet between two possible flow patterns; one of them is a jet with two free boundaries and the other one a jet with one free and one fixed boundary. Cases are known in which the instability of the jet has caused such violent oscillations of the air that when the spillway was in use windows were rattled within a radius of several miles around the damsite. Many engineers have "grappled" with the mathematical solution of the spillway problem; however, the mathematical

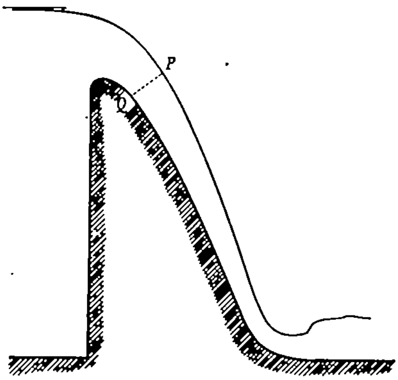

FIG. 10

analysis has not furnished sufficiently dependable information, so that spillway design is based mostly on the results of model experiments.

8. **Waves with finite amplitude.** The theory of waves with infinitestimally small amplitude sketched in §6 is due to G. B. Airy (1845) and G. G. Stokes (1847). Stokes himself, and later Lord Rayleigh, T. H. Havelock, A. G. M. Michell and several other authors gave further approximations to the exact shape of a wave of finite amplitude progressing without change (so-called wave of permanent type). However, the systematic treatment of the problem, including proofs for the existence and convergence of solutions, has been given only recently by T. Levi-Civita and his collaborators, D. J. Struik and A. Weinstein.

The analytical problem involved in the theory of waves of finite amplitude progressing on the surface of a water mass of infinite depth can be formulated in the following way:

To find all functions $\theta + i\tau$ of a complex variable $\zeta = \rho e^{i\sigma}$ that are holomorphic in the interior of the unit circle $|\zeta| = 1$, have the value zero at $\zeta = 0$, and satisfy at the circumference of the unit circle the condition

$$\frac{d\tau}{d\sigma} = p e^{-3\tau} \sin \theta \tag{8.1}$$

where $p = gL/2\pi U^2$ is a positive undetermined parameter. The symbols θ, g, L and U have the meaning given in §6 and $\tau = \log g/U$. The variable ζ is connected with the complex potential function $f = \phi + i\psi$ used in §6 by the relation

$$\zeta = e^{2\pi i f/UL}. \tag{8.2}$$

If τ and θ are small, we have approximately $dt/d\sigma = p\theta$ or

$$\rho \frac{\partial \theta}{\partial \rho} = p\theta. \tag{8.3}$$

The problem is in this case a linear eigenvalue problem; the solution is

$$\theta = \theta_0 \rho^p \sin p(\sigma - \sigma_0) \tag{8.4}$$

where p is an integer and θ_0 and σ_0 are arbitrary constants. However, it can be shown that physically the solutions corresponding to different integral values of p are all identical. If we choose, for example, $p = n$, this only means that we consider a group of n identical waves with the wave length L as one wave of the wave length nL. Hence, it is sufficient to consider the case $p=1$, which gives the relation

$$U^2 = \frac{gL}{2\pi} \tag{8.5}$$

between the velocity of propagation U and the wave length L, in accordance with the result given in §6. The factor in equation (8.4) and, therefore, the height of the waves in this approximation remain undetermined.

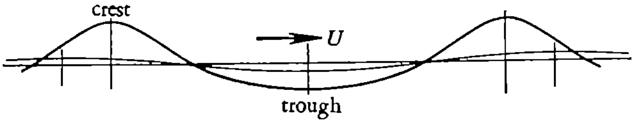

Fig. 11

Levi-Civita has shown that if $p<1$ but $1-p$ is sufficiently small, a certain wave shape of definite height/length ratio corresponds to every value of p. He also obtained convergent series for the calculation of the wave shape. He uses as a first approximation the solution of the linear problem for $p=1$, namely,

$$\theta + i\tau = -i\mu\zeta = -i\mu\rho\cos\sigma + \mu\rho\sin\sigma \tag{8.6}$$

where μ is a real constant. Then the solution which satisfies the exact boundary condition (8.1) can be written in the form

$$\theta + i\tau = \sum_{n=1}^{\infty} \omega_n(\zeta)\mu^n \tag{8.7}$$

where the $\omega_n(\zeta)$'s are polynomials of the nth degree in ζ. Levi-Civita calculated numerically the polynomials $\omega_n(\zeta)$ up to $n=5$. Fig. 11 shows an example of the wave shape resulting from his computations compared with the shape of an infinitesimally small wave of the same velocity of propagation. The limiting smallest value of p and the corresponding shape of the highest wave has not as yet been exactly determined. G. Stokes believed that the highest wave has a sharp edge at the crest with a vertex angle of 120°.

D. J. Struik extended the investigations of Levi-Civita to waves of finite height in canals of finite depth. A. Weinstein investigated the interesting problem of the so-called "solitary wave," that is, a single elevation of the water level travelling with constant velocity and preserving the same shape along a canal of finite depth. This phenomenon was studied as early as 1844 by J. Scott Russel, who observed disturbances of this type generated by the motion of large barges in relatively narrow canals. He followed them several miles on horseback and measured in this way their velocity of propagation.

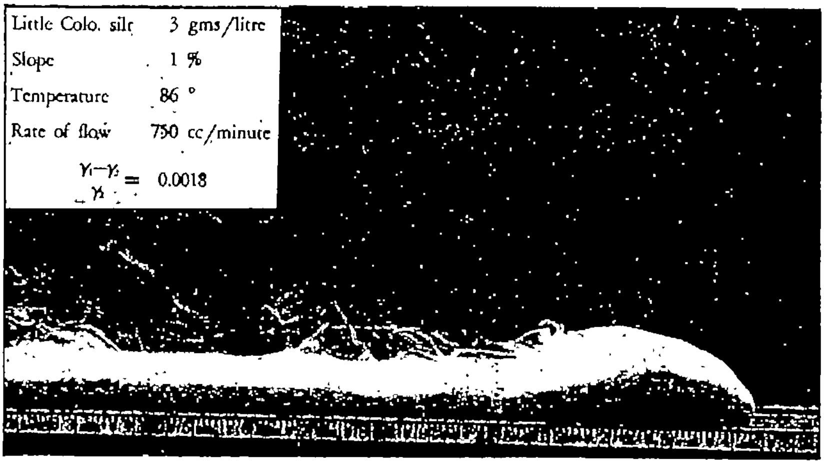
Little Colo. silt 3 gms/litre
Slope 1 %
Temperature 86 °
Rate of flow 750 cc/minute
$\frac{\gamma_1-\gamma_2}{\gamma_2}$ = 0.0018

FIG. 12
Courtesy of H. S. Bell, Soil
Conservation Service

Nonlinear problems of fluid motion in rivers and artificial channels were treated in the "hydraulic manner" by J. Boussinesq. The "hydraulic manner" of dealing with flow problems consists essentially in replacing the partial differential equations for the velocity distribution by an ordinary differential equation for the mean velocity averaged over a cross section. By combining this dynamical equation with the equation of continuity, one obtains an ordinary nonlinear differential equation for the height of the water level. A great number of problems treated in this way can be found, for example, in P. Forchheimer's book *Hydraulik*, Teubner, 1930.

Interesting nonlinear problems related analytically and physically to the problem of surface waves arise in the analysis of the flow in stratified fluid media. The stability of heavy fluids of different densities flowing one above the other with different velocities was investigated by means of the method of small oscillations by H. Helmholtz; V. Bjerknes considered the small perturbations occurring along plane surfaces of discontinuity between fluids of different densities for several combinations of density and velocity distribution. However, the exact shape of separating surfaces with large curvature has not yet been thoroughly investigated.

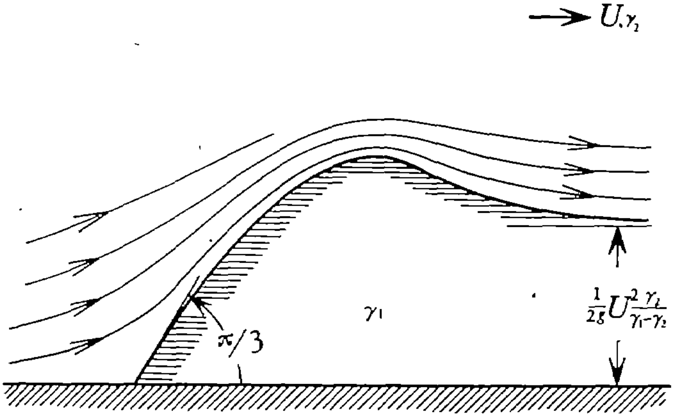

FIG. 13

A case of this type is, for example, the intrusion of a heavy fluid mass moving with constant velocity into a mass of lighter fluid, as illustrated by the meteorological phenomenon of a progressing *cold front* or by the intrusion of muddy water into a reservoir filled with clear water, or salt water into a mass of fresh water. Fig. 12 shows the shape of the surface separating muddy and clear water. The front of muddy water progresses with approximately constant speed and keeps its permanent shape. Hence, if we consider the flow relative to a coordinate system moving with the heavier fluid, the latter can be assumed to be at rest and the lighter fluid to be in stationary motion (Fig. 13). The pressure p must be equal on both sides of the discontinuity surface. Then, if the specific gravity of the heavier fluid is γ_1 and that of the lighter fluid γ_2, the equilibrium condition for the heavy fluid yields the condition $p/\gamma_1+y=\text{const.}$, and Bernoulli's equation applied to the lighter fluid leads to $p/\gamma_2+y+q^2/(2g)=\text{const.}$, where y is the height measured from the bottom and q is the velocity of the lighter fluid along the discontinuity surface. Taking into account that, at the intersection of the discontinuity surface and the bottom, $y=0$ and $q=0$, we obtain the condition for the discontinuity surface:

$$\frac{q^2}{2g} - \frac{\gamma_1 - \gamma_2}{\gamma_2} y = 0. \tag{8.8}$$

It is seen that (8.8) is a nonlinear boundary condition of the type of equation (6.6). The main interest is directed toward the determination of the shape of the discontinuity surface. We can show—using a reasoning similar to that employed by G. G. Stokes for the determination of the steepest slope occurring in waves of finite height—that if frictional effects are neglected the slope of the discontinuity surface at the intersection with the bottom must be equal to $\pi/3$. The complex velocity potential for the flow of the lighter fluid in the neighborhood of the intersection $z=x+iy=0$ has the form

$$f = Cz^\alpha \tag{8.9}$$

where α and C are real, positive constants. Hence, the magnitude of the velocity is given by $|df/dz| = C\alpha|z^{\alpha-1}|$, and, therefore, q is proportional to $y^{\alpha-1}$ and q^2 to $y^{2(\alpha-1)}$. Since, according to equation (8.8), q^2 is proportional to y, we must have $2\alpha-2=1$ or $\alpha=\frac{3}{2}$. Hence, $f=Cz^{3/2}$ or $z=(f/C)^{2/3}$. Since the streamline $\psi=0$ consists of the bottom line and the trace of the discontinuity surface, and $f=0$ corresponds to the point of intersection, the angle between the two branches of the streamline $\psi=0$ is equal to the change in the value of the imaginary part of $\log z=\frac{2}{3}\log(f/C)$, when we pass from $\phi<0$ to $\phi>0$, that is, equal to $2\pi/3=120°$.

In an actual experiment (Fig. 12) the friction on the bottom causes a certain distortion of the discontinuity surface; however, if this friction is diminished by introducing a thin fluid layer at rest under the progressing muddy water, the slope obtained experimentally is in fair agreement with the calculated value. It can also be shown analytically that the front of the heavy fluid must have a so-called *head*, whose peak is considerably higher than the mean thickness of the heavy layer, which is equal to

$$y = \frac{\gamma_2}{\gamma_1 - \gamma_2}\frac{U^2}{2g},$$

where U is the velocity of progression of the front. However, the shape of the discontinuity surface has not yet been determined; it is not known whether a mathematical analysis would lead to a surface approaching the horizontal level asymptotically or to waves with decreasing or constant amplitudes.

Also, the nonlinear boundary problems resulting from the analysis of the motion of planing surfaces have not yet been attacked seriously. The shape of the free water surface behind a planing surface has been determined by H. Wagner, but he neglected the effect of gravity. The investigators who have taken into account gravity[3] assumed small elevations and depressions of the water surface. The gap between these two first approximations ought to be filled by further investigations.

9. **Viscous fluids.** The dynamical equations for the flow of a viscous incompressible fluid have the form:

$$\frac{\partial u_i}{\partial t} + u_k \frac{\partial u_i}{\partial x_k} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu \Delta u_i \tag{9.1}$$

where $i = 1, 2, 3$, and ν is the kinematic viscosity of the fluid; the continuity equation is, in accordance with equation (6.2)

$$\frac{\partial u_i}{\partial x_i} = 0. \tag{9.2}$$

Equations (9.1) are known as the equations of Navier and Stokes.

In the case of viscous fluids we have to assume that at fixed boundaries not only the normal but also the tangential components of the velocity vanish. The equations (9.1) and (9.2) are to be integrated with these boundary conditions.

There are a few cases in which this system of nonlinear equations can be exactly solved by known functions:

(a) In some cases the solutions which satisfy the linear part of the equation make the nonlinear terms vanish identically. Thus the problem is reduced to the integration of a system of linear equations. For example, if only one component of the velocity, for example, the component u_1, is different from zero, from equation (9.2) it follows that u_1 is a function of x_2 and x_3 only. Let us consider stationary flow; then we have to conclude from equations (9.1) that $\partial p/\partial x_2 = \partial p/\partial x_3 = 0$ and

$$\nu\left(\frac{\partial^2 u_1}{\partial x_2^2} + \frac{\partial^2 u_1}{\partial x_3^2}\right) = \frac{1}{\rho}\frac{dp}{dx_1} = \text{const.} \tag{9.3}$$

The solution of this equation furnishes the velocity distribution in a cylindrical pipe. The boundary condition is $u_1 = 0$ at the wall.

(b) Certain solutions of the equations for irrotational flow of nonviscous fluids lead to exact solutions of the system of equations (9.1) and (9.2) if we assume that the streamlines remain unaltered, while the magnitude of the velocity changes. A typical flow of this type is the two-dimensional radial flow between two divergent or convergent walls. Writing equations (9.1) and (9.2) in polar coordinates and denoting the radial velocity by u_r, we obtain the dynamical equations for stationary flow in the form

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[3] Cf., for example, Lamb's *Hydrodynamics*, pp. 375–383.

$$(9.4)\qquad \begin{aligned} u_r \frac{\partial u_r}{\partial r} &= -\frac{1}{\rho}\frac{\partial p}{\partial r_1} + \nu\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} - \frac{1}{r^2}\right)u_r, \\ 0 &= -\frac{1}{\rho}\frac{1}{r}\frac{\partial p}{\partial \theta} + \frac{2\nu}{r^2}\frac{\partial u_r}{\partial \theta} \end{aligned}$$

while the continuity equation is solved by

$$(9.5)\qquad u_r r = f(\theta).$$

We introduce $f(\theta)$ as the unknown function, substituting $u_r = f(\theta)/r$ in equations (9.4), and eliminate the pressure p from these two equations. Then we obtain for $f(\theta)$ the ordinary differential equation:

$$(9.6)\qquad f'' + 4f + \frac{1}{\nu} f^2 = \text{const.},$$

which can be integrated by use of elliptic functions. The boundary conditions are $u_r = 0$ for $\theta = \pm\alpha$, where α is half of the angle of convergence or divergence. The interesting feature of this problem is that if the flow is convergent, equation (9.6) has only one solution which satisfies the boundary conditions at the walls; however, if the flow is divergent and the total amount of fluid flowing through a cross section $Q = r\int_{-\alpha}^{+\alpha} u_r d\theta$ exceeds a certain value, the solution is no longer unique. For example, for the same value of Q, we may obtain a symmetrical and two asymmetrical solutions. The symmetrical solution represents a flow in which the fluid shoots with high velocity through the center of the diverging channel and back flow takes place at both sides of the jet. The two asymmetrical solutions represent flow patterns with positive velocity near one wall and backflow in the neighborhood of the other wall. For very large values of Q the sign of the velocity changes many times between $\theta = -\alpha$ and $\theta = \alpha$. This solution was found and discussed by the present writer in collaboration with one of his Ph.D. candidates in 1913. The young man was killed at the beginning of the first world war and the investigation remained unpublished. Shortly afterward the same solution was found by G. Hamel as a special case of a wider class of exact solutions of the Navier-Stokes equations.

Methods for approximate integration of the differential equations

(9.1) are known for two limiting cases: namely, for very large and very small values of the parameter ν.

(A) If ν is large, we obtain a first approximation by neglecting the quadratic terms, so that we have

$$\frac{\partial u_i}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu\Delta u_i. \tag{9.7}$$

Eliminating p and using equation (9.2), we obtain three linear differential equations for u_1, u_2 and u_3. These equations were integrated for certain geometrically simple cases by G. G. Stokes. The second approximation should be obtained by substituting the first approximation in the quadratic terms. It is difficult to find a satisfactory justification for this method of approximation. As a matter of fact, it was shown by C. W. Oseen that in the case of a solid moving in a fluid, the solution of (9.7) represents a first approximation only in the neighborhood of the solid, no matter how large the parameter ν may be.

(B) From an engineering point of view the limiting case of small viscosity is more important. This topic is known as the theory of the laminar boundary layer in the engineering literature. The basic mathematical idea of this theory is the approximate solution of the hydrodynamic equations by asymptotic integration. L. Prandtl's fundamental paper *On the theory of fluids with small viscosity* was presented at the First International Congress for Mathematics in 1904 in Heidelberg and drew very little attention from the mathematicians attending this congress who were much more concerned with the problems of George Cantor's *Mengenlehre* and other subjects of pure mathematics, on which interest was focused in those days, than in the speculations of a young professor of applied mechanics, whose presentation perhaps was not very clear from a purely mathematical point of view. Several years passed before the method was correctly understood. Nowadays it is widely used in the theory of skin friction and many other hydrodynamical, aerodynamical and meteorological phenomena in which inertia forces and viscous forces are of the same order of magnitude.

Let us consider the following simple problem: An incompressible viscous fluid flows along a flat plate that extends along the positive x axis from $x=0$ to $x=\infty$ (Fig. 14). (We shall use the coordinates x, y, z instead of x_1, x_2, x_3.) No other boundaries exist; the velocity of flow at $x=-\infty$ is directed parallel to the x axis and is equal to U. We assume two-dimensional stationary motion and introduce the stream function $\psi(x, y)$ defined by the relations

$$u = \frac{\partial\psi}{\partial y_1}, \qquad v = -\frac{\partial\psi}{\partial x}$$

which satisfy identically the continuity equation. Then the dynamical equations yield the following differential equation of fourth order for ψ:

$$\frac{\partial\psi}{\partial y}\frac{\partial\Delta\psi}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial\Delta\psi}{\partial y} = \nu\Delta\Delta\psi. \tag{9.8}$$

If $\nu=0$, the presence of the plate has no influence on the motion and

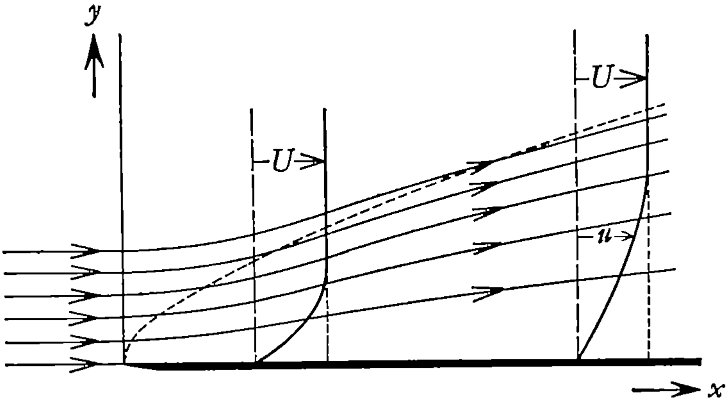

FIG. 14

we have $u=U$ or $\psi_0=Uy$; the only boundary condition at $y=0$ is $\nu=0$. However, if $\nu\neq 0$, no matter how small ν may be, we have the additional boundary condition $u=0$ for $y=0$.

To obtain an approximate solution for small values of ν, we use the method of perturbations, starting from the solution for $\nu=0$. We notice that the perturbation term contains higher derivatives of ψ than the differential equation (9.8) with $\nu=0$. It is known that in such a case we have to change the independent variable by a substitution which contains the parameter ν so that we write

$$\psi = Uy + \nu^{\alpha}\psi_1(x, \eta) \tag{9.9}$$

where $\eta = y\nu^{\beta}$ and α is positive. The exponents α and β are to be determined in such a way that $u=\partial\psi/\partial y$ is of the order of U and the largest terms on the left and right sides of equation (9.8) are of the same order. The first condition leads to $\alpha+\beta=0$. Substituting the expression (9.9) into equation (9.8) and putting $\beta=-\alpha$, we find that the largest terms on the left side are

$$\frac{\partial\psi}{\partial y}\frac{\partial^3\psi}{\partial x\partial y^2}=\nu^{-\alpha}\left(U+\frac{\partial\psi_1}{\partial\eta}\right)\frac{\partial^3\psi_1}{\partial x\partial\eta^2}$$

and

$$-\frac{\partial\psi}{\partial x}\frac{\partial^3\psi}{\partial y^3}=-\nu^{-\alpha}\frac{\partial\psi_1}{\partial x}\frac{\partial^3\psi_1}{\partial\eta^3};$$

the largest term on the right side is $\nu\,\partial^4\psi/\partial y^4=\nu^{1-4\alpha}\partial^4\psi_1/\partial\eta^4$. These terms are of the same order if $\alpha=\frac{1}{2}$. Hence equation (9.9) becomes

$$\psi=\nu^{1/2}U\eta+\nu^{1/2}\psi_1(x,\eta) \tag{9.10}$$

where $\eta=y/\nu^{1/2}$.

With this substitution, equation (9.8) takes the form

$$\left(U+\frac{\partial\psi_1}{\partial\eta}\right)\frac{\partial^3\psi_1}{\partial x\partial\eta^2}-\frac{\partial\psi_1}{\partial x}\frac{\partial^3\psi_1}{\partial\eta^3}=\frac{\partial^4\psi_1}{\partial\eta^4}. \tag{9.11}$$

This equation can be integrated once with respect to η:

$$\left(U+\frac{\partial\psi_1}{\partial\eta}\right)\frac{\partial^2\psi_1}{\partial x\partial\eta}-\frac{\partial\psi_1}{\partial x}\frac{\partial^2\psi_1}{\partial\eta^2}=\frac{\partial^3\psi_1}{\partial\eta^3}+f(x). \tag{9.12}$$

Since $\partial\psi_1/\partial\eta$ and $\partial^2\psi_1/\partial\eta^2$ are zero for $\eta=\infty$, the arbitrary function $f(x)$ must be zero. We notice that the substitution $\psi_1(x,\eta)=x^{1/2}f(\eta/x^{1/2})$ reduces equation (9.12) to an ordinary differential equation for the function f where $\zeta=\eta/x^{1/2}$. In fact, we obtain

$$-\tfrac{1}{2}(U\zeta+f)f''=f'''. \tag{9.13}$$

This differential equation has been integrated numerically for the boundary conditions $f=0$, $f'=-U$ at $\zeta=0$ and $f'=0$ at $\zeta=\infty$, by H. Blasius (1908) and more accurately by C. Töpfer (1912). The region near the wall where f or the stream function of the perturbation, $\psi_1(x,y)$, is materially different from zero is called the "boundary layer." The order of magnitude of the extension of this region normal to the wall is $(\nu x/U)^{1/2}$; we say in engineering language that we have a boundary layer of the thickness δ, where δ is of the order of $(\nu x/U)^{1/2}$. To fix ideas, δ can be defined by the relation $\nu^{1/2}\int_\infty^0\partial\psi_1/\partial\eta\,d\eta=U\delta$. This value of δ is called the displacement thickness, and is given by $\delta=1.72/(\nu x/U)^{1/2}$. The velocity distribution in the boundary layer is given by $u=U+f'$. The calculated values of u/U are plotted in

Fig. 15 and compared with experiment. The solution of equation (9.13) determines also the skin friction, that is, the shearing stress

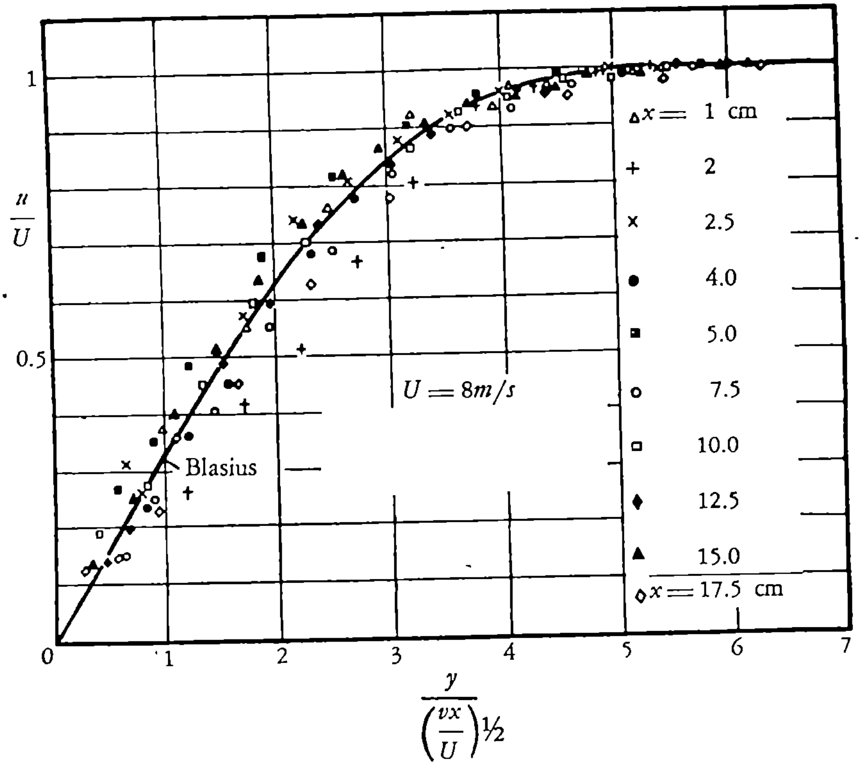

FIG. 15

From Hansen, Aachen Abhandlungen, vol. 8

at the surface; its value per unit area of the plate is variable with x, that is, with the distance from the leading edge of the plate. At a distance x, we have

$$\tau = \rho\nu\left(\frac{\partial u}{\partial y}\right)_{y=0} = 0.66414\frac{\rho U^2}{2}\frac{1}{(Ux/\nu)^{1/2}}. \tag{9.14}$$

The dimensionless quantity Ux/ν is called the Reynolds number of the plate extending from $x=0$ to $x=x$.

We have carried through this calculation in the manner of the calculus of perturbations. Prandtl considered the hydrodynamic equations in the form given by (9.1) and (9.2) and determined the relative orders of magnitude of the terms in these equations, assuming that the velocity component u changes from its value U to zero in a small range near the wall. Denoting the order of magnitude of this range by δ, he concludes that δ must be of the order $\nu^{1/2}$ and $\partial u/\partial y$, $\partial^2 u/\partial y^2$ of the order $1/\nu^{1/2}$, $1/\nu$ respectively while $\partial u/\partial x$ is of standard order. Then from equations (9.1) it follows that the derivative of the pressure with respect to a coordinate normal to the wall is of the order $\nu^{1/2}$, hence negligible in the approximation used in this theory. In this way we obtain, for example, in the case of uniform flow along a flat plate, the simplified equations:

$$(9.15)\qquad u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\nu\frac{\partial^2 u}{\partial y^2},\qquad \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0.$$

The boundary conditions are $u=v=0$ at the wall and $u=U$ at $y=\infty$, where U is the velocity "outside of the boundary layer," that is, the velocity resulting for $y=0$ from the solution of the equation $\Delta\psi_0=0$ for nonviscous fluids.

If we have two-dimensional flow along a wall but the velocity U is not constant, equations (9.15) take the form:

$$(9.16)\qquad u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=U\frac{dU}{dx}+\nu\frac{\partial^2 u}{\partial y^2},\qquad \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0.$$

Prandtl has shown that these equations can also be applied to the flow along a curved surface, provided the lines $x=\text{const.}$ are normal to the surface, x is the length of arc measured at the surface and y is the normal distance from the surface. Equations (9.16) furnish a fair approximation if $K\delta$ and $(dK/dx)\delta^2$ are sufficiently small, K the curvature of the wall and δ the thickness of the boundary layer.

The integration of the system of equations (9.16) has great importance for many applications. Various attempts have been made to obtain approximative solutions. An excellent review of the methods applied is given in *Modern Developments in Fluid Dynamics*, edited by S. Goldstein. Here only a short account will be given:

(a) S. Goldstein showed that equations (9.16) can be reduced to an ordinary differential equation if $U(x)$ has the form const. x^n or const. $e^{\lambda x}$. The case $n=0$ leads to the case treated above; $n=1$ that is, $U=\text{const.}\ x$ corresponds to a boundary layer starting from a stagnation point of the main flow. This problem has been treated by Blasius, Hiemenz, Howarth and Bickley. The equation for arbitrary n has been studied by Falkner and Skan and by Hartree. Special cases were discussed by R. Mills.

(b) Let us assume that $U(x)$ is given in the form of a power series

$$U = \beta_1 x + \beta_2 x^2 + \beta_3 x^3 + \cdots . \tag{9.17}$$

Then we put $u = \partial\psi/\partial y$, $v = -\partial\psi/\partial x$ and

$$\psi = F_1 x + F_2 x^2 + F_3 x^3 + \cdots . \tag{9.18}$$

By substituting (9.17) and (9.18) in equations (9.16) and equating the terms with the same power of x, we obtain systems of ordinary differential equations for the functions $F_1, F_2, \cdots$ of y. This method was used by Hiemenz, Howarth, Bickley and others.

(c) We might start from an "initial velocity profile" at $x=0$, assuming, for example,

$$u(0, y) = \alpha_1 y + \alpha_2 \frac{y^2}{2!} + \alpha_3 \frac{y^3}{3!} + \cdots \tag{9.19}$$

and calculate the "next profile," that is, $u(\Delta x, y)$. However, in general, that is, for arbitrary values of $\alpha_1, \alpha_2, \cdots$, we obtain a singularity at $x=y=0$. If the function $U(dU/dx)$ has the form

$$U \frac{dU}{dx} = \gamma_1 x + \gamma_2 x^2 + \cdots , \tag{9.20}$$

the function $\psi(x, y)$ is regular only if certain relations between the coefficients $\alpha_1, \alpha_2, \cdots$ and $\gamma_1, \gamma_2, \cdots$ are satisfied. Hence, the application of a step-by-step method progressing in the x-direction is rather difficult.

(d) A method of obtaining approximate values of the thickness of the boundary layer and the skin friction as function of x has been proposed by the present writer and worked out in detail by Pohlhausen. By integrating the first equation of (9.16) between $y=0$ and an arbitrary fixed value of y and using the condition of continuity, we obtain

$$\frac{d}{dx}\int_0^y u^2 dy + \left[uv\right]_0^y = \nu\left[\frac{\partial u}{\partial y}\right]_0^y + \frac{d}{dx}\int_0^y \frac{U^2}{2} dy. \tag{9.21}$$

Taking into account the second equation (9.16), the second term on the left side of (9.20) becomes $-U(d/dx)\int_0^y u dx$ and we obtain

$$\frac{d}{dx}\int_0^y (U^2 - u^2) dy - U\frac{d}{dx}\int_0^y (U - u) dy = -\nu\left[\frac{\partial u}{dy}\right]_0^y . \tag{9.22}$$

Extending the integral to a value $y=\delta$, where $U-u$ is negligible, that is, over the whole cross section of the boundary layer, we can assume that $\partial u/\partial y = 0$ for $y = \delta$ and we have

$$\frac{d}{dx}\int_0^\delta (U^2 - u^2)dy - U\frac{d}{dx}\int_0^\delta (U - u)dy = \nu\left(\frac{\partial u}{\partial y}\right)_0 . \tag{9.23}$$

This equation, given first by the present writer, is an expression for the conservation of the momentum of the fluid involved in the boundary layer. Let us assume that the velocity distribution between $y = 0$ and $y = \delta$ can be expressed by a polynomial of the form

$$U - u = U\left[1 - a\frac{y}{\delta} + b\left(\frac{y}{\delta}\right)^2 + c\left(\frac{y}{\delta}\right)^3\right] \tag{9.24}$$

where a, b and c are functions of x. Then from $U - u = 0$ and $\partial u/\partial y = 0$ at $y = \delta$ follows: $1 - a + b + c = 0$ and $-a + 2b + 3c = 0$. For $y = 0$ we have, from the first equation of (9.16), $\nu(\partial^2 u/\partial y^2) + U(dU/dx) = 0$; hence

$$\frac{2b\nu}{\delta^2} + \frac{dU}{dx} = 0.$$

It is seen that a, b and c are expressed by δ and dU/dx. Therefore, if we substitute (9.24) into equation (9.23), we obtain an ordinary nonlinear differential equation of first order for δ.

This method was applied with success in many cases and gave fair approximations for engineering purposes. It has the disadvantage that it is difficult to estimate the degree of approximation. If the actual shape of the velocity distribution profile is very different from the assumed polynomial, the result can be misleading.

(e) R. von Mises suggested a transformation of coordinates for equation (9.16), introducing x and ψ as independent variables and $Z = U^2 - u^2$ as the unknown function. Then $u(\partial/\partial x + (v/u)\partial/\partial y)$ has to be replaced by $u\partial/\partial x$ and $\partial/\partial y$ by $u\partial/\partial\psi$. We obtain in this way

$$\frac{\partial Z}{\partial x} = \nu u \frac{\partial^2 Z}{\partial \psi^2} \tag{9.25}$$

where $u = (U^2 - Z^2)^{1/2}$ and U is a given function of x. If we compare this equation with equation (9.12) it is seen that its order is reduced from 3 to 2. However, an additional quadrature is necessary to obtain the velocity as function of x and y. von Mises suggested a step-by-step method to solve equation (9.25). The difficulty caused by a singularity at $x = \psi = 0$ mentioned under item (c) occurs also in this case.

(f) If we introduce instead of x the variable $\phi = \int_0^x U dx$ (that is, the value of the potential function obtained by solving the equation $\Delta\psi_0=0$ for the nonviscous case), equation (9.25) can be written in the form

$$\frac{\partial Z}{\partial \phi} = \nu \frac{u}{U} \frac{\partial^2 Z}{\partial \psi^2}\cdot \tag{9.26}$$

In the outer portion of the boundary layer, u/U is not very different from unity. Then we find that Z satisfies approximately the heat conduction equation:

$$\frac{\partial Z}{\partial \phi} = \nu \frac{\partial^2 Z}{\partial \psi^2}\cdot \tag{9.27}$$

The boundary condition is $Z=[U(x)]^2$ for $\psi=\infty$. The solution satisfying equation (9.27) and this boundary condition might be called the "outer solution." We shall now calculate the "inner solution," that is, approximate values of Z for small values of ϕ, and the two solutions shall be joined. We choose as the point of junction the value of $\psi=\psi_j$ for which

$$\frac{\partial^2 u}{\partial y^2} = \frac{u}{2} \frac{\partial^2(u^2)}{\partial \psi^2} = 0$$

(inflexion point of the velocity profile). Since $\partial^2 Z/\partial\psi^2 = -\partial^2(u^2)/\partial\psi^2$, we obtain from equation (9.26)

$$u\frac{\partial^2(u^2)}{\partial \psi^2} = \frac{U}{\nu} \frac{\partial Z}{\partial \phi}\cdot$$

To obtain a first approximation we will replace $\partial Z/\partial\phi$ by a function of ϕ and u; then we have to solve an ordinary differential equation for u, where the coefficients are functions of ϕ. For $\psi=0$

$$\frac{\partial Z}{\partial \phi} = \frac{\partial}{\partial \phi}\left[U^2 - u^2\right] = 2U\frac{dU}{d\phi};$$

for the point of junction, $\partial Z/\partial\phi=0$. Hence, we replace $\partial Z/\partial\phi$ by

$$2U\frac{dU}{d\phi}\left(1-\frac{u}{u_j}\right),$$

where u_j is the value of $u=(-Z^2+U^2)^{1/2}$ obtained from the outer solution.

This method was given by the writer and C. B. Millikan in a joint paper and applied in detail to the case in which U is a linear function of x, that is, $U = U_0 + U_1 x$, where U_0 and U_1 are constants.

Other more or less laborious methods for obtaining approximate solutions of the boundary layer equations have been suggested by L. Howarth, Green, Falkner and Skan. The equations for nonstationary and for three-dimensional motion have been studied by several authors and integrated in certain special cases.

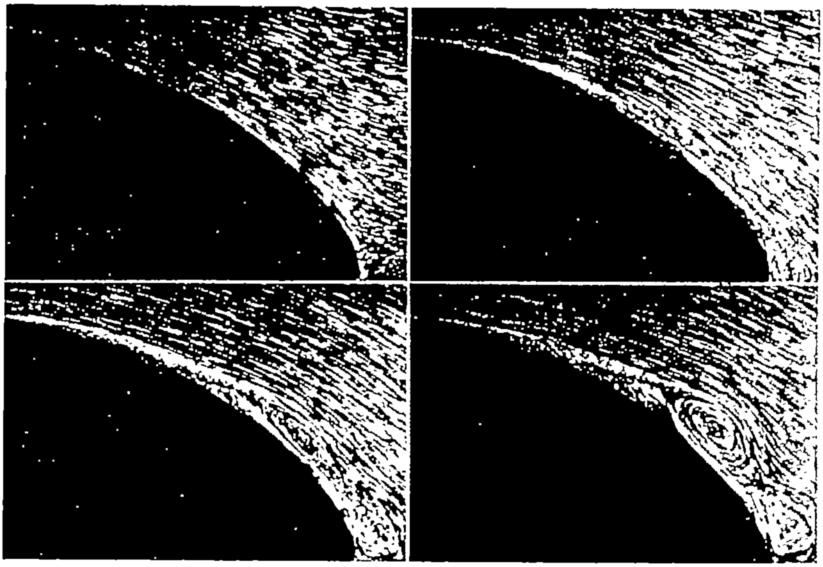

FIG. 16
From Prandtl-Tietjens, *Hydro and Aeromechanics*, vol. 2

One result of general importance is shown by all these investigations: The flow pattern depends largely on the sign of the function $U(dU/dx)$. If $U(dU/dx) > 0$, which means that $dp/dx < 0$, the thickness of the boundary layer decreases or increases slowly, and the solution of equations (9.16) can be continued without difficulty for the whole range in which $U(x)$ is given. However, if $U(dU/dx) < 0$, that is, $dp/dx > 0$, we soon come to a point at the surface where $\partial u/\partial y = 0$. Beyond this point the streamline $\psi = 0$ has two branches; the flow separates from the surface and back flow takes place between the two branches of $\psi = 0$. This point is known as the *separation point*. It can be clearly recognized in the flow pictures shown in Fig. 16 which represent successive stages of an accelerated flow around a cylindrical body. Unfortunately, practically all the methods mentioned above converge very slowly, if at all, when we approach the separation point, and, therefore, it is very difficult to discuss the mathematical character of the solution of the boundary layer equations in the neighborhood of this point. An additional difficulty is caused by the fact that by the separation of the flow from the surface vorticity is transported into the free stream and, therefore, the assumption of irrotational motion outside the boundary is no more valid. Hence, in such cases the perturbation method fails.

Due to this failure of the method we do not get an answer for one of the fundamental questions of the hydrodynamics of real fluids, that is: What is the flow pattern of a real fluid around a submerged body in the limiting case $\nu \to 0$? As a matter of fact this problem is still not solved. Consider, for example, two-dimensional flow around a circular cylinder. We are not able to decide whether the flow pattern for $\nu \to 0$ approaches the potential flow of a nonviscous fluid or a stationary flow pattern consisting of a vortex-free region and a wake with continuously distributed vorticity, as suggested by Oseen, or a nonstationary flow pattern with concentrated vortex columns of alternating circulation, a flow pattern treated by the present author. It seems that we have here an example in which the analytical methods are not sufficient, at least at the present time, to solve a problem of purely analytical character.[4]

10. **Compressible fluids.** In the previous sections the fluid was assumed to be incompressible, that is, the equation of continuity or conservation of matter was used in the form

$$\frac{\partial u_i}{\partial x_i} = 0. \tag{10.1}$$

If we drop this restriction and denote the variable density of the fluid by ρ, we have

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u_i)}{\partial x_i} = 0. \tag{10.2}$$

---

[4] The reader may wonder why the turbulence problem has not been included in this review. This omission is due to the fact that the study of turbulent motion has not, as yet, led to a clear mathematical formulation of the fundamental laws for the mean motion. The problem of transition from laminar to turbulent motion has been discussed mathematically by several authors by means of linearized equations without reaching a satisfactory agreement with experiment. The adequate treatment of the nonlinear equations is bound to contribute essentially to the solution of this important problem.

In this case we have five unknown quantities: the three velocity components, the density, and the pressure. To solve the problem properly we have to introduce thermodynamical concepts. We have to add the temperature as an additional unknown variable and two equations: the equation of state and the equation of the conservation of energy (mechanical plus thermal). However, in many cases the problem can be simplified by assuming a definite thermodynamic process, for example, adiabatic or isothermal change of state. This amounts to a known relation between p and ρ. Let us assume, for example, an ideal fluid undergoing an adiabatic change of state, then we have the relation $p = \text{const.}\ \rho^{\kappa}$. Introducing the velocity of sound c defined by $c^2 = dp/d\rho$, we obtain the following system of equations

$$(10.3) \qquad \frac{\partial u_i}{\partial t} + u_k \frac{\partial u_i}{\partial x_k} = -\frac{c^2}{\rho}\frac{\partial p}{\partial x_i}, \qquad \frac{\partial \rho}{\partial t} + \frac{\partial(\rho u_i)}{\partial x_i} = 0.$$

These nonlinear equations can be linearized in the following cases:

(a) Assume that the velocity components u_i and the deviation of ρ from average value ρ_0 are small. In this case we obtain

$$(10.4) \qquad \frac{\partial u_i}{\partial t} = -\frac{c^2}{\rho_0}\frac{\partial \rho}{\partial x_i}, \qquad \frac{\partial \rho}{\partial t} + \rho_0 \frac{\partial u_i}{\partial x_i} = 0.$$

The velocity of sound can be considered as constant. Eliminating the velocity components u_i, we obtain for ρ the wave equation:

$$(10.5) \qquad \frac{1}{c^2}\frac{\partial^2 \rho}{\partial t^2} = \Delta\rho.$$

(b) If the velocity components u_2 and u_3 are small whereas u_1 might be large but differs only slightly from a mean value U, it follows again that the deviation of the density ρ from a mean value ρ_0 is small. Then putting $u_1 = U + u_1'$, $u_2 = u_2'$, $u_3 = u_3'$ we obtain

$$(10.6) \qquad \frac{\partial u_i'}{\partial t} + U\frac{\partial u_i'}{\partial x_1} = -\frac{c^2}{\rho_0}\frac{\partial \rho}{\partial x_i}, \qquad \frac{\partial \rho}{\partial t} + U\frac{\partial \rho}{\partial x_1} + \rho_0\frac{\partial u_i'}{\partial x_i} = 0.$$

The velocity of sound can again be considered constant, so that eliminating the velocity components u_i' we have

$$(10.7) \qquad \frac{1}{c^2}\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x_1}\right)^2 \rho = \Delta\rho.$$

Let us consider the stationary case, that is, $\partial\rho/\partial t = 0$. Then we have

$$(10.8)\qquad \left(1-\frac{U^2}{c^2}\right)\frac{\partial^2 p}{\partial x_1^2}+\frac{\partial^2 p}{\partial x_2^2}+\frac{\partial^2 p}{\partial x_3^2}=0.$$

It is evident that the linear differential equation (10.8) is of the elliptic type if $U<c$ (subsonic flow) and of the hyperbolic type, if $U>c$ (supersonic flow). In the first case equation (10.8) is reduced to Laplace's equation by the affine transformation

$$(10.9)\qquad x_1'=\frac{x_1}{(1-U^2/c^2)^{1/2}},\qquad x_2'=x_2,\qquad x_3'=x_3.$$

If $U>c$, equation (10.8) is of the type of the wave equation and can be integrated by use of well known methods developed for the solution of this equation.

Equations (10.5) and (10.8) are often used for approximate solution of the system of equations (10.3). However, in many cases these approximate solutions do not describe the physical picture closely enough and we are forced to search for methods of solving the nonlinear equations.

Let us consider stationary irrotational two-dimensional flow, using the symbols x and y for the coordinates, and u and v for the velocity components. Then denoting the potential function by $\phi(x, y)$ we obtain, by putting $\partial\rho/\partial t=\partial u/\partial t=\partial v/\partial t=0$ and eliminating ρ from equations (10.3),

$$(10.10)\qquad \left(1-\frac{u^2}{c^2}\right)\frac{\partial^2\phi}{\partial x^2}-2\frac{uv}{c^2}\frac{\partial^2\phi}{\partial x\partial y}+\left(1-\frac{v^2}{c^2}\right)\frac{\partial^2\phi}{\partial y^2}=0.$$

In this equation c^2 is a given function of $q^2=u^2+v^2$. Namely, according to the definition of the velocity of sound c^2 is a given function of ρ; on the other hand, it follows from (10.3) that

$$(10.11)\qquad d(u^2+v^2)+\frac{c^2}{\rho}d\rho=0.$$

Let us apply Legendre's transformation introducing the velocity components u and v as independent variables and $X=xu+yv-\phi$ as the unknown function. We obtain

$$(10.12)\qquad \left(1-\frac{u^2}{c^2}\right)\frac{\partial^2 X}{\partial v^2}+2\frac{uv}{c^2}\frac{\partial^2 X}{\partial u\partial v}+\left(1-\frac{v^2}{c^2}\right)\frac{\partial^2 X}{\partial u^2}=0.$$

If in the whole domain occupied by the fluid $u^2+v^2>c^2$, equation (10.12) is of the hyperbolic type and has fixed real characteristics in the u, v plane. Therefore, in this case the integration of equation

(10.12) can be achieved by the methods of characteristics. For the construction of the characteristics it is convenient to use the polar coordinates $q=(u^2+v^2)^{1/2}$ and $\theta=\tan^{-1} v/u$ in the u, v plane. In these coordinates the differential equation of the characteristic curves becomes

$$\frac{1}{q}\frac{dq}{d\theta}=\frac{c}{(q^2-c^2)^{1/2}}. \tag{10.13}$$

Since c is a function of q only, equation (10.13) is a differential equation of the first order for $q(\theta)$. Its solution consists of two families

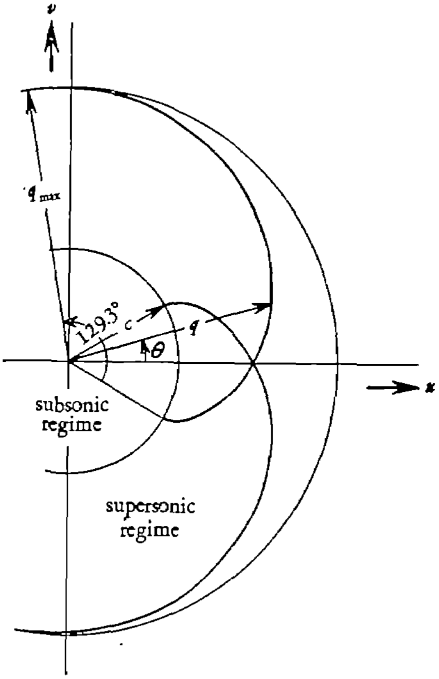

FIG. 17

of curves: one family corresponds to the positive, the other to the negative sign of the square root. If the isothermal process is assumed, $c=$const. and the characteristics are two families of involutes of the circle of the radius c covering the whole domain of the u, v plane outside of this circle. If, according to the so-called polytropic law, p is proportional to ρ^k, we obtain epicycloids which touch the two circles whose radii are equal to the minimum value of c and the maximum value of q respectively. The characteristic curves are indicated in Fig. 17. If the characteristics of (10.12) in the u, v plane are known, the next step consists of constructing the corresponding characteristic curves of (10.10) in the x, y plane. L. Prandtl and A. Busemann have worked out graphical methods for this purpose.

It is a characteristic property of equation (10.10) that in the supersonic range it can have discontinuous solutions. Both density and velocity change discontinuously at certain surfaces. Such discontinuities occurring in supersonic flow are known as shock waves; they also can be considered as waves of finite amplitude if we look at the phenomenon from the standpoint of an observer moving with the fluid. B. Riemann was the first who pointed out (in 1860) that in a compressible fluid waves of finite amplitude can produce discontinuities. Later W. J. M. Rankine, H. Hugoniot, J. Hadamard, E. Jouguet, G. I. Taylor and others, worked out in detail the theory of the discontinuous solutions. Engineering applications of the theory of shock waves were made by A. Stodola, L. Prandtl, and many other authors. Applications to the ballistic problem were made by E. Mach, C. Cranz, G. I. Taylor, J. W. Maccoll, and P. S. Epstein.

If in the whole domain $u^2+v^2=q^2<c^2$, the problem of integration of equation (10.10) can be attacked by the method of successive approximations. Let us consider the classical example of the two-dimensional flow of a compressible fluid around a circular cylinder. We assume that the velocity of the flow at large distance from the obstacle is parallel to the x axis and has the magnitude U. We denote the velocity of sound corresponding to the density and pressure prevailing in the undisturbed flow by c_0 and the exponent of the polytropic law by κ. Then the velocity of sound at an arbitrary point, where the magnitude of the velocity is equal to q, is given by

$$c^2 = c_0^2 + \frac{\kappa - 1}{2}(U^2 - q^2). \tag{10.14}$$

Introducing this relation into equation (10.10) and using polar coordinates r and θ, we obtain

$$\begin{aligned}\left[1 - \tfrac{1}{2}(\kappa - 1)M^2\left(\frac{q^2}{U^2} - 1\right)\right]\Delta\phi \\ = \frac{1}{2}\,\frac{M^2}{U^2}\left(\frac{\partial\phi}{\partial r}\,\frac{\partial q^2}{\partial r} + \frac{1}{r^2}\,\frac{\partial\phi}{\partial\theta}\,\frac{\partial q^2}{\partial\theta}\right)\end{aligned} \tag{10.15}$$

where $M = U/c_0$ is called Mach's number and

$$q^2 = \left(\frac{\partial\phi}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial\phi}{\partial\theta}\right)^2.$$

We assume that ϕ can be developed in a series of ascending powers of M^2:

$$\phi = \phi_0 + M^2\phi_1 + M^4\phi_2 + \cdots . \tag{10.16}$$

Obviously, we obtain from equation (10.15)

$$\Delta\phi_0 = 0. \tag{10.17}$$

It is seen that $\phi_0(r, \theta)$ is the solution for incompressible fluid. We put the radius of the circle equal to unity. Then the boundary condition is $\partial\phi_0/\partial r = 0$ for $r = 1$, and we obtain:

$$\phi_0 = (r + 1/r)\cos\theta. \tag{10.18}$$

We also develop q^2 in ascending powers of M^2 by writing

$$q^2 = q_0^2 + M^2 q_1^2 + M^2 q_2^4 + \cdots . \tag{10.19}$$

Substituting the expressions (10.16) and (10.19) in equation (10.15) and equating the terms multiplied by the same power of M, we have the following relations for $\phi_1, \phi_2, \cdots$

$$\begin{aligned} \Delta\phi_1 &= \frac{1}{2U^2}\left(\frac{\partial\phi_0}{\partial r}\frac{\partial q_0^2}{\partial r} + \frac{1}{r^2}\frac{\partial\phi_0}{\partial\theta}\frac{\partial q_0^2}{\partial\theta}\right), \\ \Delta\phi_2 &= \frac{1}{2U^2}(k-1)(q_0^2 - 1)\Delta\phi_1 \\ &\quad + \frac{1}{2U^2}\left(\frac{\partial\phi_0}{\partial r}\frac{\partial q_1^2}{\partial r} + \frac{1}{r^2}\frac{\partial\phi_0}{\partial\theta}\frac{\partial q_1^2}{\partial\theta}\right) \\ &\quad + \frac{1}{2U^2}\left(\frac{\partial\phi_1}{\partial r}\frac{\partial q_0^2}{\partial r} + \frac{1}{r^2}\frac{\partial\phi_1}{\partial\theta}\frac{\partial q_0^2}{\partial\theta}\right). \end{aligned} \tag{10.20}$$

The boundary conditions are $\partial\phi/\partial r = 0$ for $r = 1$ and $\partial\phi_1/\partial r = \partial\phi_2/\partial r = 0$ for $r = \infty$.

The method sketched here is essentially identical to that used by Lord Rayleigh, O. Janzen, L. Poggi, C. Kaplan, I. Ismai, K. Tamada and Y. Saito. The calculations of the Japanese authors are the most reliable ones; they corrected some numerical errors which occurred in previous publications. G. I. Taylor worked out an experimental method for solving the same problem by successive approximations. He employed an electric device analogous to that used for the experimental solution of Laplace's equation.

Another method for solving problems of subsonic flow in compressible fluids was suggested by P. Molenbroek and A. Chaplygin. These authors use a transformation of independent variables which is analogous to the Legendre transformation mentioned before in this section. However, instead of using the potential $\phi(x, y)$ and its Legendre transform $X(u, v)$ as unknown functions they use $\phi(x, y)$ and the stream function $\psi(x, y)$ which satisfies identically the continuity equation for stationary flow:

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} = 0. \tag{10.21}$$

We write $\rho u = \partial\psi/\partial y$ and $\rho v = -\partial\psi/\partial x$. Then we have

$$d\phi = udx + vdy, \qquad d\psi = -\rho vdx + \rho udy. \tag{10.22}$$

We resolve these equations for dx and dy and introduce as independent variables polar coordinates in the u, v plane, namely, $q = (u^2+v^2)^{1/2}$ and $\theta = \tan^{-1} v/u$. Substituting

$$d\phi = \frac{\partial\phi}{\partial q} dq + \frac{\partial\psi}{\partial\theta} d\theta, \quad d\psi = \frac{\partial\psi}{\partial q} dq + \frac{\partial\psi}{\partial\theta} d\theta, \quad u = q \cos\theta, \quad v = q \sin\theta$$

into equations (10.22) and resolving them for dx and dy, we obtain

$$\begin{aligned} dx &= \left(\frac{\cos\theta}{q}\frac{\partial\phi}{\partial q} + \frac{\sin\theta}{q\rho}\frac{\partial\psi}{\partial q}\right) dq + \left(\frac{\cos\theta}{q}\frac{\partial\phi}{\partial\theta} + \frac{\sin\theta}{q\rho}\frac{\partial\psi}{\partial\theta}\right) d\theta, \\ dy &= \left(-\frac{\sin\theta}{q}\frac{\partial\phi}{\partial q} + \frac{\cos\theta}{q\rho}\frac{\partial\psi}{\partial q}\right) dq - \left(\frac{\sin\theta}{q}\frac{\partial\phi}{\partial\theta} + \frac{\cos\theta}{q\rho}\frac{\partial\psi}{\partial\theta}\right) d\theta. \end{aligned} \tag{10.23}$$

The relations

$$\frac{\partial}{\partial\theta}\left(\frac{\partial x}{\partial q}\right) = \frac{\partial}{\partial q}\left(\frac{\partial x}{\partial\theta}\right), \qquad \frac{\partial}{\partial\theta}\left(\frac{\partial y}{\partial q}\right) = \frac{\partial}{\partial q}\left(\frac{\partial y}{\partial\theta}\right)$$

furnish the following two relations between the derivatives $\partial\phi/\partial q$, $\partial\phi/\partial\theta$ and $\partial\psi/\partial q$, $\partial\psi/\partial\theta$:

$$\begin{aligned} -\sin\theta\frac{\partial\phi}{\partial q} + \frac{1}{\rho}\cos\theta\frac{\partial\psi}{\partial q} &= -\frac{\cos\theta}{q}\frac{\partial\phi}{\partial\theta} - \frac{\sin\theta}{q\rho}\left(1 + \frac{q}{\rho}\frac{d\rho}{dq}\right)\frac{\partial\psi}{\partial\theta}, \\ -\cos\theta\frac{\partial\phi}{\partial q} - \frac{1}{\rho}\sin\theta\frac{\partial\psi}{\partial q} &= \frac{\sin\theta}{q}\frac{\partial\phi}{\partial\theta} - \frac{\cos\theta}{q\rho}\left(1 + \frac{q}{\rho}\frac{d\rho}{dq}\right)\frac{\partial\psi}{\partial\theta}. \end{aligned} \tag{10.24}$$

Multiplying the first equation of (10.24) by $\cos\theta$ and the second by $\sin\theta$, we have by subtraction

$$\frac{1}{\rho}\frac{\partial\psi}{\partial q} = -\frac{1}{q}\frac{\partial\phi}{\partial\theta}. \tag{10.25}$$

Multiplying the first equation by $\sin\theta$ and the second by $\cos\theta$, we obtain by addition

$$\frac{\partial\phi}{\partial q} = \frac{1}{q\rho}\left(1 + \frac{q}{\rho}\frac{d\rho}{dq}\right)\frac{\partial\psi}{\partial\theta}. \tag{10.26}$$

From equation (10.11) follows $(q/\rho)d\rho/dq = -q^2/c^2$; hence, equation (10.26) becomes

$$\frac{\partial\phi}{\partial q} = \frac{1}{q\rho}\left(1 - \frac{q^2}{c^2}\right)\frac{\partial\psi}{\partial\theta}. \tag{10.27}$$

Eliminating, for example, ϕ from equations (10.25) and (10.27), we obtain the following linear differential equation for ψ:

$$\frac{q\rho}{1 - q^2/c^2}\frac{\partial}{\partial q}\left(\frac{q}{\rho}\frac{\partial\psi}{\partial q}\right) + \frac{\partial^2\psi}{\partial\theta^2} = 0, \tag{10.28}$$

where c is given as function of q by means of the relations $c^2 = dp/d\rho$, $dp/\rho + qdq = 0$ and $p = f(\rho)$. The last relation depends on the law of the change of state of the fluid.

Chaplygin has shown that the differential equation (10.29) can be reduced to Laplace's equation by an appropriate choice of the relation $p = f(\rho)$. Laplace's equation for the polar coordinates q and θ has the form

$$q\frac{\partial\psi}{\partial q}\left(q\frac{\partial\psi}{\partial q}\right) + \frac{\partial^2\psi}{\partial\theta^2} = 0. \tag{10.29}$$

Equation (10.29) is reduced to the form (10.29) if

$$\frac{dq}{q} = \frac{1}{a}\frac{\rho dq}{q} = a\frac{(1 - q^2/c^2)}{q\rho}dq \tag{10.30}$$

where a is an arbitrary constant. Hence, we have to choose $p = f(\rho)$ such that

$$\rho^2 = a^2(1 - q^2/c^2)$$

or

$$q^2 = c^2 - \frac{\rho^2c^2}{a^2}. \tag{10.31}$$

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Since $d(q^2)/d\rho = -2c^2/\rho$, we obtain by differentiation of (10.31) the relation

$$\frac{d(c^2)}{d\rho} = -2\frac{c^2}{\rho}. \tag{10.32}$$

From equation (10.32) follows $c^2 = dp/d\rho = B/\rho^2$ and

$$p = A - \frac{B}{\rho}. \tag{10.33}$$

Introducing the specific volume $V = 1/\rho$, we can write equation (10.33) in the form

$$p - p_0 = \text{const.}\ (V_0 - V). \tag{10.34}$$

It is seen that we obtain Laplace's equation for ψ (and also for ϕ) if the polytropic curve in the p, V plane is replaced by a straight line.

It is obvious that if the density change is small the adiabatic or other polytropic curves can be approximated by straight lines. A. Chaplygin, D. Riabouchinsky, B. Demtschenko, J. Caius and A. Busemann applied this method of approximation to the slow motion of a compressible fluid. It seems to the writer that an application to the flow of a compressible fluid with a velocity which is comparable to the velocity of sound is of greater practical interest. Let us assume that the undisturbed velocity of the fluid is U (where $U < c$) and the density corresponding to the velocity is $\rho_0 = 1/V_0$. If we assume that the change of the density is sufficiently small so that the pressure can be expanded as function of $V - V_0$ and approximated by a linear term, it is possible to determine possible flow patterns, velocity and pressure distributions in compressible fluids by use of the methods of conformal transformation. By means of successive approximations, which require considerably less labor than Rayleigh's method mentioned before in this section, it is also possible to determine the velocity and pressure distribution around a given body. H. S. Tsien worked out this application of Chaplygin's method upon suggestion of the present writer.

The problem of flow of a compressible fluid around a cylindrical body or a body of axial symmetry is of great practical importance in view of the increasing speed of aircraft, which already have flown faster than 2/3 of the velocity of sound. Experimental evidence seems to show that if the velocity at any point of the wing surface reaches the velocity of sound, a shock wave appears. However, G. I. Taylor calculated numerically with great patience an analogous flow problem, namely, the flow of a compressible fluid through a nozzle, whose cross section decreases to a minimum value and again increases. He obtained a flow pattern, showing near the walls a velocity greater than that of sound, but no discontinuity. Another important case is the motion of a blunt-shaped body at supersonic velocity. Observations show that a curved discontinuity surface is created in front of the body; between this surface and the blunt nose of the body the flow is subsonic and the velocity becomes zero at the nose of the body. The same phenomenon occurs also in the case of a body with a pointed nose or leading edge at velocities slightly higher than the velocity of sound (Fig. 18a), whereas if the speed is sufficiently large, the shock wave starts from the pointed edge, or in the case of a body of rotation from the apex (Fig. 18b). This problem is important in ballistics; for example, the so-called terminal velocity of aerial bombs

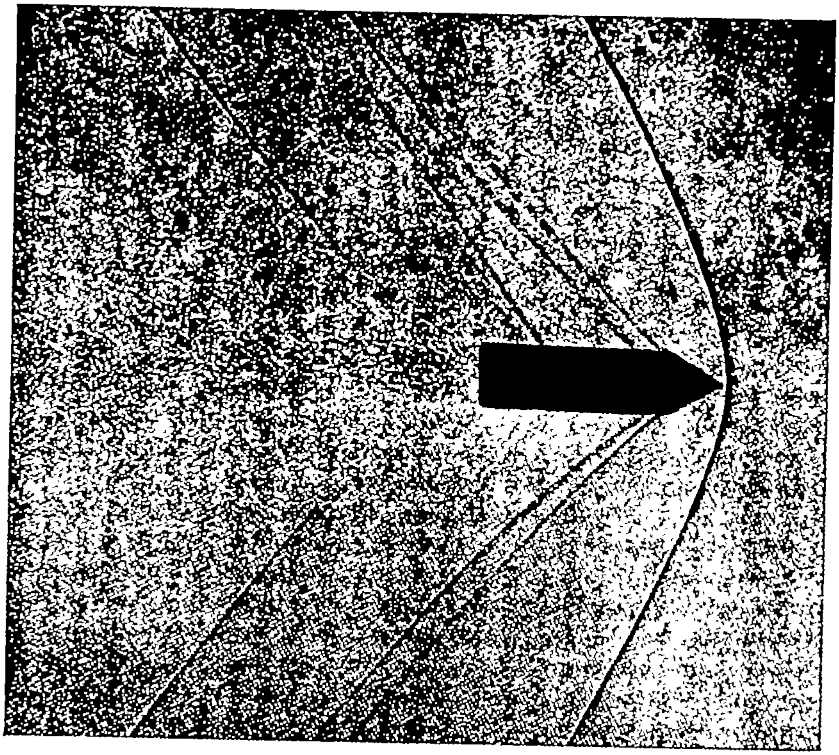

Fig. 18a. Photograph of a bullet in flight. From G. I. Taylor and J. W. Maccoll, *Aerodynamic Theory*, vol. 3

dropped from large heights is in this range. Unfortunately, no method as yet is available for solving such mixed subsonic-supersonic problems. The difficulty is that the differential equation for the potential or the stream function is of the elliptic type in one part of the domain and of the hyperbolic type in the other. It seems to the writer that

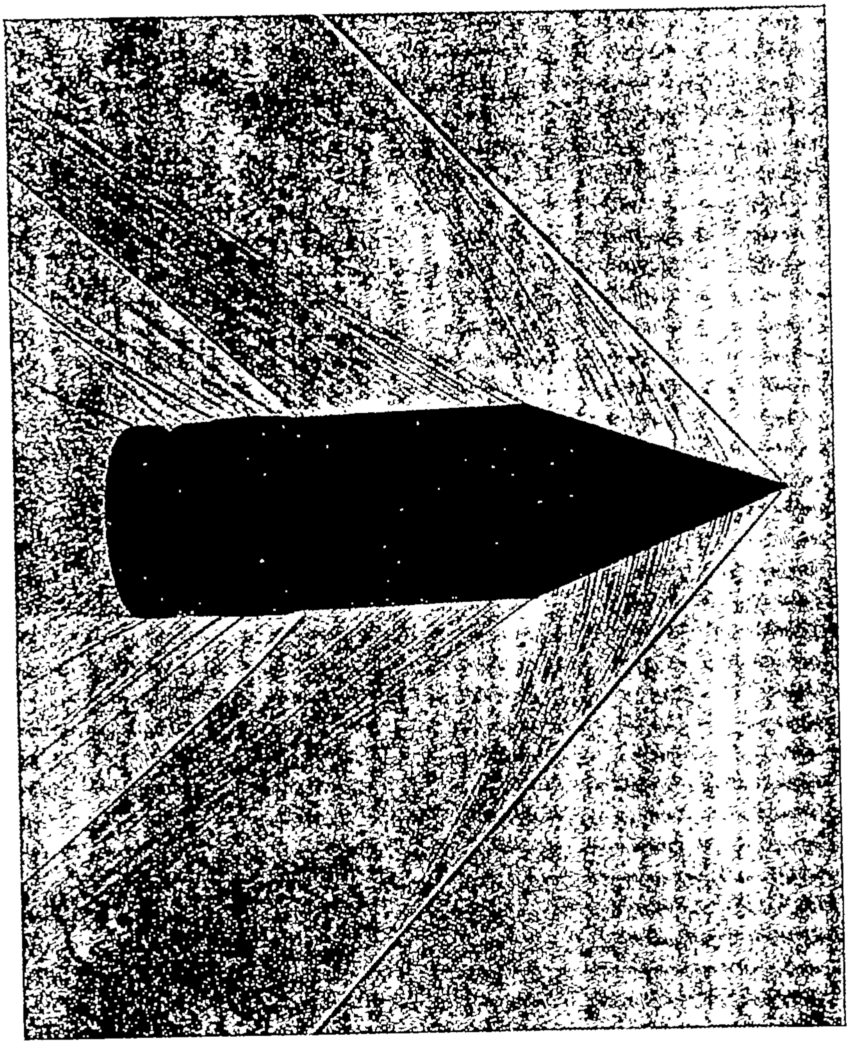

FIG. 18b. Photograph of a conical headed projectile in flight. Photograph by T. Harris, British Ordnance Committee

the invention of methods for the solution of such mixed problems would be an achievement both from a practical and a mathematical point of view.

An attempt was made in this paper to show the applications of analytical methods available for the solution of certain nonlinear problems in which the engineer is interested. Some gaps were shown and frontiers indicated beyond which the safe guidance of the mathematical analysis is for the time being lacking. I would consider this lecture a great success if it would induce mathematicians to volunteer for pioneering work in pushing those frontiers "farther west."

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SULLE TRASFORMAZIONI CREMONIANE CHE POSSEGGONO PER CURVA DI PUNTI UNITI UNA SESTICA CON DIECI PUNTI DOPPI

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In una precedente memoria *Sulla trasformazioni Cremoniane del piano che posseggono una curva di punti uniti*[1] ho messo in evidenza che tra i tipi possibili di siffatte trasformazioni si presentano quelle che hanno per curva di punti uniti una sestica con 10 punti doppi; per provare l'effettiva esistenza di tali trasformazioni ne ho dato un esempio (num. 21) che però è risultato essere insussistente perchè, come ha recentemente osservato A. B. Coble,[2] la trasformazione che si ottiene non è altro che l'identità. Perciò do quì un altro esempio.

La generica sestica razionale, con dieci punti doppi $P_1, P_2, \cdots, P_7, A, B, C$ di cui gli ultimi tre nei vertici del triangolo fondamentale $A(0, 1, 0)$, $B(1, 0, 0)$, $C(0, 0, 1)$, ha equazione parametrica:

$$\begin{aligned} x &= t(t-\alpha_1)(t-\alpha_2)(t-\gamma_1)(t-\gamma_2), \\ y &= t(t-\beta_1)(t-\beta_2)(t-\gamma_3)(t-\gamma_4), \\ z &= (t-\alpha_1)(t-\alpha_2)(t-\beta_1)(t-\beta_2)(t-\gamma_5)(t-\gamma_6). \end{aligned}$$

Imponiamo a questa sestica di soddisfare alla seguente condizione: *le due involuzioni individuate dalle coppie che cadono nei punti doppi A, C e B, C devono essere permutabili.* Questo porta che sia $\alpha_1\alpha_2 = -\beta_1\beta_2$ o meglio $\beta_2 = -\alpha_1\alpha_2/\beta_1$; cioè si ha un solo legame tra i coefficienti e la condizione, come è naturale, risulta perfettamente simmetrica rispetto ai punti doppi A e B. In questo caso l'equazione della nostra sestica S diventa:

$$\begin{aligned} x &= t(t-\alpha_1)(t-\alpha_2)(t-\gamma_1)(t-\gamma_2), \\ y &= t(t-\beta_1)(t+\alpha_1\alpha_2/\beta_1)(t-\gamma_3)(t-\gamma_4), \\ z &= (t-\alpha_1)(t-\alpha_2)(t-\beta_1)(t+\alpha_1\alpha_2/\beta_1)(t-\gamma_5)(t-\gamma_6), \end{aligned}$$

ed un semplice esame di questa equazione parametrica ci assicura che nell'intorno dei tre punti doppi A, B, C non esistono altri punti multipli di S.

Ma ancora con una sola condizione si può imporre che la sestica abbia:

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[1] Rendiconti del Seminario Matematico della Università di Roma, (4), vol. 2 (1938).

[2] This Bulletin, vol. 45 (1939).

I. un punto triplo in luogo di tre punti doppi, o
II. un tacnodo, o
III. tre punti doppi su una retta, o
IV. sei punti doppi su una conica,

gli altri casi, come vedremo, non ci interessano perchè oltre che $P_1, \cdots, P_7$ fanno intervenire il punto doppio C ovvero, per la simmetria, A e B contemporaneamente.

Sorge così il dubbio che la condizione imposta equivalga a qualcuna di quelle sopra elencate.

Il primo caso si esclude subito. Supponiamo infatti che la nostra sestica abbia un punto triplo P in corrispondenza ai valori η_1, η_2, η_3 del parametro t; in tal caso le rette per P segano ulteriomente S secondo una g_3^1 che sul parametro t è espressa da:

$$(t-\alpha_1)(t-\alpha_2)(t-\delta_1)+\lambda(t-\beta_1)(t+\alpha_1\alpha_2/\beta_1)(t-\delta_2)=0\ ;$$

siccome poi questa g_3^1 deve contenere anche un gruppo del tipo $(0, \infty, \delta_2)$ segue che $\delta_1=-\delta_2\ (=\delta)$. Si osservi ora che le rette per A e B segano ulteriormente S secondo due g_4^1 espresse, sul parametro t dalle equazioni:

$$t(t-\gamma_1)(t-\gamma_2)+\lambda(t-\beta_1)(t+\alpha_1\alpha_2/\beta_1)(t-\gamma_5)(t-\gamma_6)=0$$
$$t(t-\gamma_3)(t-\gamma_4)+\mu(t-\alpha_1)(t-\alpha_2)(t-\gamma_5)(t-\gamma_6)=0$$

e queste due g_4^1 devono possedere rispettivamente due gruppi del tipo $(\delta, \eta_1, \eta_2, \eta_3)$, $(-\delta, \eta_1, \eta_2, \eta_3)$ il che però porta un'ulteriore condizione sui coefficienti; possiamo quindi concludere che in generale la nostra sestica possiede 10 punti doppi e non un punto triplo e sette punti doppi.

Gli altri tre casi non sono riuscito ad escluderli. Preciserò perciò che cosa diventano nei vari casi le trasformazioni involutorie del quarto tipo di Bertini[8] che hanno per fondamentali gli otto punti $(P_1, \cdots, P_7, A)$, $(P_1, \cdots, P_7, B)$.

Se non si presenta nessuno dei casi del numero precedente, ovvero se la sestica S ha un tacnodo si hanno due trasformazioni del 17° ordine con i punti fondamentali: $(P_1^6, \cdots, P_7^6, A^6)$, $(P_1^6, \cdots, P_7^6, B^6)$.

Se P_1, P_2, P_3 sono in linea retta si hanno due trasformazioni del 16° ordine con i punti fondamentali: $(P_1^5, P_2^5, P_3^5, P_4^6, \cdots, P_7^6, A^6)$, $(P_1^5, P_2^5, P_8^5, P_4^6, \cdots, P_7^6, B^6)$.

Se $P_1, \cdots, P_6$ sono su una conica si hanno due trasformazioni del 13° ordine con i punti fondamentali: $(P_1^4, \cdots, P_6^4, P_7^6, A^6)$, $(P_1^4, \cdots, P_6^4, P_7^6, B^6)$.

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[8] V. Snyder, *The involutorial birational transformation of the plane, of order* 17, American Journal of Mathematics, vol. 33 (1911).

Cioè indicando con b_A la trasformazione del quarto tipo di Bertini che ha i punti fondamentali $(P_1, \cdots, P_7, A)$ e con b_B quella con i punti fondamentali $(P_1, \cdots, P_7, B)$ possiamo dire che le due trasformazioni b_A e b_B hanno rispettivamente i punti A e B come fondamentali e mutano i loro intorni in sestiche che passano doppiamente per $P_1, \cdots, P_7$ mentre hanno un punto triplo l'una in A e l'altra in B.

Data la natura della sestica S le due trasformazioni involutorie b_A e b_B sono permutabili su essa mentre, come dimostreremo, non lo sono sul piano; se infatti fosse $b_Ab_B = b_Bb_A$ siccome B è unito per b_A mentre è mutato da b_B in una sestica irriducibile $D_6\,(P_1^2, \cdots, P_7^2, B^3)$ seguirebbe che questa sestica D_6 sarebbe unita per b_A in contraddizione con il lemma VI (num. 10) della mia memoria precedentemente citata. Da ciò segue che la trasformazione $Q = b_Ab_Bb_Ab_B$ non è identica ed ha la sestica S come luogo di punti uniti.

Questa trasformazione Q non ha però tutti e dieci i punti doppi di S come fondamentali; per averne una che soddisfi anche a questo requisito, basta prendere una trasformazione cremoniana T la quale oltre che mutare in sè la sestica abbia per fondamentali quei punti doppi di S che non lo sono per Q, e quindi costruire la: $R = T^{-1}QTQ^{-1}$.

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THE SOLUTION OF BOUNDARY VALUE PROBLEMS BY A DOUBLE LAPLACE TRANSFORMATION

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1. **Introduction.** The Laplace transformation has most frequently been used to transform a linear partial differential equation with two independent variables and constant coefficients into an ordinary differential equation, the "subsidiary equation," from the solution of which that of the original equation is deduced. If it is applied to an equation with more than two independent variables, the subsidiary equation is also a partial differential equation and is usually solved by classical methods. The object of this note is to point out that partial differential equations in which the range of two (or more) independent variables is $(0, \infty)$ may easily be handled by simultaneous Laplace transformations in these variables. The point of view is that of previous papers[1] in which the Laplace transformation method has been regarded as purely formal, and the solution as subject to verification. The method is related to that of Doetsch[2] for equations of elliptic type but assumes no theoretical basis. In §3 a simple two variable problem is discussed to illustrate the method and in §§4 and 5 are given two new three variable problems of a type to which it is well adapted.

2. **The general method.** We consider a function $v(x, t)$ in the range $x>0$, $t>0$.

Let

$$u_0(x) = v(x, 0), \qquad u_1(x) = \left[\frac{\partial v}{\partial t}\right]_{t=0},$$

$$w_0(t) = v(0, t), \qquad w_1(t) = \left[\frac{\partial v}{\partial x}\right]_{x=0}.$$

The Laplace transform with respect to t will be denoted by a "bar," thus

$$\bar{f}(p) = \int_0^\infty e^{-pt} f(t)\,dt,$$

and that with respect to x by a capital letter, thus

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[1] Carslaw and Jaeger, Proceedings of the Cambridge Philosophical Society, vol. 35 (1939), p. 394; Proceedings of the London Mathematical Society, in press; this Bulletin, vol. 45 (1939), p. 407.

[2] Doetsch, *Theorie und Anwendungen der Laplace-Transformation*, Berlin, 1937, chap. 22.

$$G(p') = \int_0^\infty e^{-p'x} g(x)dx,$$

and the double Laplace transform by both:

$$\overline{V}(p', p) = \int_0^\infty \int_0^\infty e^{-pt-p'x} v(x, t)dxdt;$$

p and p' are supposed to have real parts sufficiently large to ensure convergence. This notation indicates compactly the variables occurring; if there is another independent variable y, it will occur throughout.

Then subject to suitable restrictions on v we have

$$(1) \qquad \int_0^\infty \int_0^\infty e^{-pt-p'x} \frac{\partial v}{\partial t} dxdt = p\overline{V} - U_0,$$

$$(2) \qquad \int_0^\infty \int_0^\infty e^{-pt-p'x} \frac{\partial^2 v}{\partial t^2} dxdt = p^2\overline{V} - pU_0 - U_1,$$

$$(3) \qquad \int_0^\infty \int_0^\infty e^{-pt-p'x} \frac{\partial v}{\partial x} dxdt = p'\overline{V} - \bar{w}_0,$$

$$(4) \qquad \int_0^\infty \int_0^\infty e^{-pt-p'x} \frac{\partial^2 v}{\partial x^2} dxdt = p'^2\overline{V} - p'\bar{w}_0 - \bar{w}_1.$$

From a given differential equation and boundary conditions we obtain a subsidiary equation and boundary conditions by multiplying by exp $[-pt-p'x]$, integrating with respect to x and t from 0 to ∞, and using (1) to (4). Solving this equation we obtain $\overline{V}(p', p)$.

To derive $v(x, t)$ from $\overline{V}(p', p)$ we assume an inversion theorem which may be derived formally from Fourier's integral theorem in several variables:

$$(5) \qquad v(x, t) = -\frac{1}{4\pi^2} \int_{\gamma-i\infty}^{\gamma+i\infty} d\lambda \int_{\gamma'-i\infty}^{\gamma'+i\infty} e^{\lambda t+\lambda' x} \overline{V}(\lambda', \lambda)d\lambda',$$

where $\pi > \arg \lambda > -\pi$, and $\pi > \arg \lambda' > -\pi$, provided that:

$$(6) \qquad \overline{V}(\lambda', \lambda) \text{ is bounded in some half-planes } R(\lambda') > \alpha',\ R(\lambda) > \alpha, \text{ and } \gamma > \alpha,\ \gamma' > \alpha'.$$

Other conditions in addition to (6) will of course be necessary for the truth of the inversion theorem (5), but neither these nor the assumptions involved in (1) to (4) need be discussed since the whole process of determining $v(x, t)$ as a double contour integral of type (5) is regarded as purely formal. It has then to be verified that this solution does satisfy the differential equation and boundary conditions. This is done in the case of the equation of conduction of heat by transforming the paths of integration in (5) to paths beginning in the third and ending in the second quadrant, and the verification is then performed on these new integrals. The whole process follows the lines of the one-variable case[3] and will not be given here. The contour integrals of type (5) are evaluated by a deformation of the contour precisely as in the one-variable case, this will be done without comment, the method and justification having been given in the papers referred to.

3. **Heat conduction in the region $x>0$; the initial temperature zero; the end $x=0$ kept at unit temperature for $t>0$.** We have to solve

$$\frac{\partial^2 v}{\partial x^2} - \frac{1}{\kappa}\frac{\partial v}{\partial t} = 0, \qquad t > 0,\ x > 0, \tag{7}$$

with[4]

$$u_0(x) = 0, \qquad x > 0, \tag{8}$$

$$w_0(t) = 1, \quad t > 0, \qquad w_1(t) \text{ unknown}. \tag{9}$$

By (1) and (4) the subsidiary equation is

$$\left(p'^2 - \frac{p}{\kappa}\right)\bar{V} = \frac{p'}{p} + \bar{w}_1,$$

and so

$$\bar{V}(p', p) = \frac{p' + p\bar{w}_1(p)}{p(p'^2 - p/\kappa)}.$$

For $\bar{V}(\lambda', \lambda)$ to satisfy (6) we must have[5] $\bar{w}_1(\lambda) = -(\lambda\kappa)^{-1/2}$ and thus

$$\bar{V}(\lambda', \lambda) = \frac{1}{\lambda[\lambda' + (\lambda/\kappa)^{1/2}]}.$$

Hence by the inversion theorem (5)

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[3] Carslaw and Jaeger, loc. cit.

[4] Boundary conditions stated shortly as in (8) and (9) are to be understood as:

$$\lim_{t\to 0} v(x, t) = 0, \qquad \text{for fixed } x > 0, \tag{8'}$$

$$\lim_{x\to 0} v(x, t) = 1, \qquad \text{for fixed } t > 0. \tag{9'}$$

[5] From this result the flux of heat at the origin may be obtained directly.

$$v(x, t) = -\frac{1}{4\pi^2}\int_{\gamma-i\infty}^{\gamma+i\infty} d\lambda \int_{\gamma'-i\infty}^{\gamma'+i\infty} \frac{e^{\lambda t+\lambda' x}d\lambda'}{\lambda[\lambda' + (\lambda/\kappa)^{1/2}]}$$

$$(10) \qquad = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{\lambda t - x(\lambda/\kappa)^{1/2}} \frac{d\lambda}{\lambda}$$

$$= 1 - \frac{2}{\pi^{1/2}}\int_0^{x/2(\kappa t)^{1/2}} e^{-\xi^2}d\xi.$$

4. **Heat conduction in the region $x > 0$, $0 < y < b$; $y = b$ maintained at unit temperature for $t > 0$; $y = 0$ and $x = 0$ maintained at zero for $t > 0$; the initial temperature zero.** We have to solve

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} - \frac{1}{\kappa}\frac{\partial v}{\partial t} = 0, \qquad t > 0,\ x > 0,\ 0 < y < b,$$

with

$$u_0(y, x) = 0, \qquad x > 0,\ 0 < y < b,$$

$$w_0(y, t) = 0, \qquad t > 0,\ 0 < y < b,$$

$$w_1(y, t) \text{ unknown},$$

$$v(y, x, t) = 0, \qquad \text{when } y = 0,\ x > 0,\ t > 0,$$

$$v(y, x, t) = 1, \qquad \text{when } y = b,\ x > 0,\ t > 0.$$

The subsidiary equation is

$$(11) \qquad \frac{d^2\overline{V}}{dy^2} + q^2\overline{V} = \bar{w}_1(y),$$

where $q^2 = p'^2 - p/\kappa$. This is to be solved with $\overline{V} = 1/pp'$ when $y = b$ and $\overline{V} = 0$ when $y = 0$. The solution is

$$(12) \qquad \begin{aligned} \overline{V}(y, p', p) = {} & \frac{\sin qy}{pp' \sin qb} - \frac{\sin qy}{q \sin qb}\int_y^b \sin q(b - y')\bar{w}_1(y')dy' \\ & - \frac{\sin q(b - y)}{q \sin qb}\int_0^y \sin qy'\bar{w}_1(y')dy'. \end{aligned}$$

Now $\sin b(\lambda'^2 - \lambda/\kappa)^{1/2}$, *qua* function of λ', has zeros at $\lambda' = \pm(n^2\pi^2/b^2 + \lambda/\kappa)^{1/2}$, $n = 0, 1, \cdots$, so to satisfy (6) the numerator of $\overline{V}(y, \lambda', \lambda)$ must vanish at $\lambda' = +(n^2\pi^2/b^2 + \lambda/\kappa)^{1/2}$, $n = 1, 2, \cdots$, that is,

$$(13) \qquad \frac{(-)^n n\pi}{\lambda b(n^2\pi^2/b^2 + \lambda/\kappa)^{1/2}} + \int_0^b \sin\frac{n\pi y'}{b}\bar{w}_1(y')dy' = 0, \qquad n = 1, 2, \cdots.$$

The residue of $e^{\lambda' x}\bar{V}(y, \lambda', \lambda)$, *qua* function of λ' at the pole $\lambda' = -(n^2\pi^2/b^2+\lambda/\kappa)^{1/2}$ is

$$\frac{\exp\left[-x(n^2\pi^2/b^2+\lambda/\kappa)^{1/2}\right]}{b(n^2\pi^2/b^2+\lambda/\kappa)^{1/2}}\sin\frac{n\pi y}{b}\left\{\frac{(-)^n n\pi}{\lambda b(n^2\pi^2/b^2+\lambda/\kappa)^{1/2}} - \int_0^b \sin\frac{n\pi y'}{b}\,\bar{w}_1(y')dy'\right\}.$$

Using (13) in this gives for the result of the λ'-integration in (5)

$$\frac{\sinh y(\lambda/\kappa)^{1/2}}{\lambda \sinh b(\lambda/\kappa)^{1/2}} + \sum_{n=1}^{\infty}\frac{2\kappa n\pi(-)^n}{\lambda(\kappa n^2\pi^2+\lambda b^2)} \cdot \sin\frac{n\pi y}{b}\exp\left[-x\left(\frac{n^2\pi^2}{b^2}+\frac{\lambda}{\kappa}\right)^{1/2}\right],$$

where the first term comes from the pole $\lambda' = 0$.

For the λ-integration we require

$$\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{\exp\left[\lambda t - x(\lambda/\kappa+\alpha^2)^{1/2}\right]d\lambda}{\lambda(\lambda/\kappa+\alpha^2)} = \frac{1}{\alpha^2}e^{-\alpha x} - \frac{1}{\alpha^2}e^{-\kappa\alpha^2 t} + \frac{2}{\pi}\int_0^{\infty}e^{-\kappa t(\alpha^2+u^2)}\sin ux\,\frac{du}{u(u^2+\alpha^2)},$$

and

$$\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{e^{\lambda t}}{\lambda}\,\frac{\sinh y(\lambda/\kappa)^{1/2}}{\sinh b(\lambda/\kappa)^{1/2}}\,d\lambda = \frac{y}{b} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-)^n}{n}e^{-\kappa n^2\pi^2 t/b^2}\sin\frac{n\pi y}{b}.$$

Using these results we have finally

$$v(y, x, t) = \frac{y}{b} + \frac{2}{\pi}\sum_{n=1}^{\infty}e^{-n\pi x/b}\frac{(-)^n}{n}\sin\frac{n\pi y}{b} + \frac{4}{b^2}\sum_{n=1}^{\infty}n(-)^n\sin\frac{n\pi y}{b} \cdot \int_0^{\infty}e^{-\kappa t(u^2+n^2\pi^2/b^2)}\frac{\sin ux}{u(u^2+n^2\pi^2/b^2)}\,du. \tag{14}$$

Clearly problems in which the temperature of $y = b$ is a function of x or t may be dealt with in the same way provided the function satisfies fairly wide conditions.

5. **Heat conduction in the region $x > 0$, $0 < y < b$; the initial tem-**

perature unity; the surface maintained at zero for $t>0$. We have to solve

$$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}-\frac{1}{\kappa}\frac{\partial v}{\partial t}=0, \qquad t>0,\ x>0,\ 0<y<b,$$

with

$$\begin{aligned} u_0(y, x) &= 1, && x>0,\ 0<y<b,\\ w_0(y, t) &= 0, && t>0,\ 0<y<b,\\ w_1(y, t) &\ \text{unknown}, \end{aligned}$$

$v(y, x, t)=0$ when $y=0$ and $y=b$ for $x>0$ and $t>0$.

The subsidiary equation is

$$\frac{d^2\bar{V}}{dy^2}+q^2\bar{V}=\bar{w}_1(y)-\frac{1}{\kappa p'},$$

where $q^2=p'^2-p/\kappa$.

This is to be solved with $\bar{V}=0$, when $y=0$ and $y=b$. The solution is

$$\begin{aligned} \bar{V}(y, p', p) = &-\frac{\sin qb-\sin qy-\sin q(b-y)}{\kappa p' q^2 \sin qb}\\ &-\frac{\sin qy}{q\sin qb}\int_y^b \sin q(b-y')\bar{w}_1(y')dy'\\ &-\frac{\sin q(b-y)}{q\sin qb}\int_0^y \sin qy'\bar{w}_1(y')dy'. \end{aligned} \tag{15}$$

The zeros of $\sin b(\lambda'^2-\lambda/\kappa)^{1/2}$, *qua* function of λ', are at $\lambda' = \pm(n^2\pi^2/b+\lambda/\kappa)^{1/2}$, $n=0, 1, 2, \cdots$, and thus, in order that (6) may be satisfied, we must have

$$-\frac{b[1-(-)^n]}{n\pi\kappa(n^2\pi^2/b^2+\lambda/\kappa)^{1/2}}+\int_0^b \sin\frac{n\pi y'}{b}\bar{w}_1(y')dy'=0, \qquad n=1, 2, \cdots. \tag{16}$$

The residue of $e^{\lambda' x}\bar{V}(y, \lambda', \lambda)$ at the pole $\lambda'=-(n^2\pi^2/b^2+\lambda/\kappa)^{1/2}$ is

$$-\frac{1}{b}\left(\frac{n^2\pi^2}{b^2}+\frac{\lambda}{\kappa}\right)^{-1/2}\exp\left[-x\left(\frac{n^2\pi^2}{b^2}+\frac{\lambda}{\kappa}\right)\right]\sin\frac{n\pi y}{b}$$
$$\cdot\left\{\frac{b[1-(-)^n]}{\kappa n\pi(n^2\pi^2/b^2+\lambda/\kappa)^{1/2}}+\int_0^b \sin\frac{n\pi y'}{b}\bar{w}_1(y')dy'\right\}.$$

Using (16) in this the result of the λ'-integration in (5) is

$$
\begin{aligned}
&-\frac{4b^2}{\pi}\sum_{r=0}^{\infty}\frac{1}{(2r+1)\left[\kappa(2r+1)^2\pi^2+\lambda b^2\right]} \\
&\qquad\cdot\exp\left[-x\left(\frac{(2r+1)^2\pi^2}{b^2}+\frac{\lambda}{\kappa}\right)^{1/2}\right]\sin\frac{(2r+1)\pi y}{b} \\
&\qquad+\frac{\sinh b(\lambda/\kappa)^{1/2}-\sinh y(\lambda/\kappa)^{1/2}-\sinh(b-y)(\lambda/\kappa)^{1/2}}{\lambda\sinh b(\lambda/\kappa)^{1/2}},
\end{aligned}
$$

where the last term has been derived from the pole $\lambda'=0$; there are no poles at $\lambda'=\pm(\lambda/\kappa)^{1/2}$.

Carrying out the λ-integration in the usual way, we have finally

$$
\begin{aligned}
v(y,x,t)&=\frac{8}{\pi^2}\sum_{r=0}^{\infty}\frac{\sin(2r+1)\pi y/b}{(2r+1)} \\
&\quad\cdot\int_0^{\infty}\frac{\sin ux}{u}\exp\left[-\kappa t(u^2+(2r+1)^2\pi^2/b^2)\right]du.
\end{aligned}
\tag{17}
$$

The same method applies to cases in which the initial temperature $u_0(y,x)$ is a function of x and y under fairly general conditions on the function.

As an example let $u_0(y,x)=x$, for $x>0$, $0<y<b$.

The only change in $\bar{V}(y,p',p)$ is that the first term of (15) is replaced by

$$
\frac{-\sin bq+\sin qy+\sin q(b-y)}{\kappa p'^2q^2\sin bq}.
$$

Proceeding in the same way, we find that the only pole of $\bar{V}(y,\lambda',\lambda)$, *qua* function of λ', is $\lambda'=0$, where the residue is

$$
x\frac{\sinh b(\lambda/\kappa)^{1/2}-\sinh y(\lambda/\kappa)^{1/2}-\sinh(b-y)(\lambda/\kappa)^{1/2}}{\lambda\sinh b(\lambda/\kappa)^{1/2}}.
$$

Performing the λ-integration of (5), we obtain finally

$$
v(y,x,t)=\frac{4x}{\pi}\sum_{r=0}^{\infty}\frac{1}{2r+1}e^{-\kappa\pi^2(2r+1)^2t/b^2}\sin\frac{(2r+1)\pi y}{b}.
\tag{18}
$$

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THE MINIMAL NUMBERS OF BINARY FORMS[1]

RUFUS OLDENBURGER AND ARTHUR PORGES

1. **Introduction.** One of us proved that for certain fields K a form F of degree m can be written as a linear combination of mth powers of linear forms. Such a combination is termed a *representation* of F and the least possible number of terms in any such representation is called the *minimal* number of F with respect to K. The minimal number depends on both F and K. For fields K with characteristic greater than n, and binary forms F of degree n, it has been proved[2] that the minimal number ranges over at least $1, 2, \cdots, n$, and at most $1, 2, \cdots, n+1$, but the exact range was not determined. In the present paper the authors prove that the range is precisely $1, 2, \cdots, n$.

2. **Preliminary lemmas.** In what follows we use *identity* of polynomials in the usual sense, namely polynomials P and Q are identical if the coefficients of P equal the corresponding coefficients of Q.

Since the order of a field K is greater than m if the characteristic of K is greater than m, we have the following lemma.

LEMMA 1. *For a field K with characteristic greater than m a polynomial P of degree m is equal to a polynomial Q for all values of the variables if and only if P and Q are identical.*

An immediate consequence of Lemma 1 is the following lemma.

LEMMA 2. *For a field K with characteristic greater than m, a polynomial P of degree m not identically zero is different from zero for at least one set of values of the variables.*

LEMMA 3. *Let K be a field with characteristic greater than m. Let Δ be the determinant*

$$(1)\qquad \Delta = \begin{vmatrix} 1 & \cdots & 1 & b_1 \\ a_1 & \cdots & a_m & b_2 \\ \cdot & \cdots & \cdot & \cdot \\ a_1^m & \cdots & a_m^m & b_{m+1} \end{vmatrix}$$

of order $m+1$, $m \geqq 1$, with elements in K, and suppose that the b's are not all zero. The determinant Δ is not identically zero in the a's.

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[1] Presented to the Society, April 13, 1940.

[2] R. Oldenburger, *Polynomials in several variables*, Annals of Mathematics, (2), vol. 41 (1940), no. 3, pp. 694–710.

Lemma 3 is valid when $m=1$. For $m\geqq 2$ we have the following sequence of equalities:

$$\Delta = \begin{vmatrix} 1 & 0 & \cdots & 0 & b_1 \\ a_1 & (a_2 - a_1) & \cdots & (a_m - a_1) & b_2 \\ a_1^2 & (a_2^2 - a_1^2) & \cdots & (a_m^2 - a_1^2) & b_3 \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ a_1^m & (a_2^m - a_1^m) & \cdots & (a_m^m - a_1^m) & b_{m+1} \end{vmatrix}$$

$$= (a_2 - a_1) \cdots (a_m - a_1) \begin{vmatrix} 1 & 0 & \cdots 0 & b_1 \\ 0 & 1 & \cdots 1 & (b_2 - b_1 a_1) \\ 0 & a_2 & \cdots a_m & (b_3 - b_2 a_1) \\ \cdot & \cdot & \cdots\cdot & \cdot \\ 0 & a_2^{m-1} & \cdots a_m^{m-1} & (b_{m+1} - b_m a_1) \end{vmatrix}.$$

Thus

$$\Delta = (a_2 - a_1) \cdots (a_m - a_1)(M - a_1 N), \tag{2}$$

where

$$M = \begin{vmatrix} 1 & \cdots 1 & b_2 \\ a_2 & \cdots a_m & b_3 \\ a_2^2 & \cdots a_m^2 & b_4 \\ \cdot & \cdots\cdot & \cdot \\ a_2^{m-1} & \cdots a_m^{m-1} & b_{m+1} \end{vmatrix}, \qquad N = \begin{vmatrix} 1 & \cdots 1 & b_1 \\ a_2 & \cdots a_m & b_2 \\ a_2^2 & \cdots a_m^2 & b_3 \\ \cdot & \cdots\cdot & \cdot \\ a_2^{m-1} & \cdots a_m^{m-1} & b_m \end{vmatrix}.$$

The lemma now follows by induction.

It is to be remarked that if the characteristic of a field K is greater than m, each binary form of degree m with coefficients in K can be written as

$$b_1 x^m + m b_2 x^{m-1} y + \cdots + b_{m+1} y^m, \tag{3}$$

where the coefficient of $b_i x^{m-i+1} y^{i-1}$ is the binomial coefficient of $x^{m-i+1} y^{i-1}$ in the expansion of $(x+y)^m$.

3. **The range of the minimal number.** We proceed with the following theorem in which $I(m)$ denotes the integer

$$\tfrac{1}{2} m [m(m-1) + 2]. \tag{4}$$

THEOREM 1. *For a field K with characteristic greater than m, and order at least $I(m)$, the minimal number of a binary form F does not exceed m.*

We write F as in (4). We exclude the trivial case $F\equiv 0$. If $b_1\neq 0$ while $b_i=0$ for $i>1$, the form F is simply b_1x^m, whence the minimal number of F is 1; similarly, if $b_{m+1}\neq 0$ while $b_i=0$ for $i<m+1$. If $m\geqq 2$ while b_1 and b_{m+1} are not zero, and if further $b_i=0$ for $i=2, 3, \cdots, m$, the form F is nonsingular in the sense of Oldenburger,[3] and has minimal number 2.

If F is of the first degree, the minimal number of F is clearly 1, while if F is quadratic, the minimal number is identical[4] with the rank of F and does not exceed 2.

It remains to consider forms F of degree at least 3 for which $b_2, \cdots, b_m$ are not all zero. It is no restriction to assume that $b_{m+1}\neq 0$. For if $b_{m+1}=0$, the form F can be transformed nonsingularly into a form with this property, as is clear from the following argument. We write $F=F(x, y)$. Since $F\not\equiv 0$, there exist values a and b of x and y respectively such that $F(a, b)\neq 0$. We make the transformation

$$x = x' + ay', \qquad y = x' + by',$$

on F to obtain a form $F'(x', y')$. Evidently, $F'(0, 1)=F(a, b)$.

We consider the following equality:

$$b_1x^m + mb_2x^{m-1}y + \cdots + b_{m+1}y^m = \sum_{i=1}^{m} \lambda_i(x + a_iy)^m. \tag{5}$$

The equality (5) is identically satisfied if the λ's and a's are chosen so that the following system of linear equations is valid:

$$\begin{aligned} \lambda_1 + \lambda_2 + \cdots + \lambda_m &= b_1, \\ a_1\lambda_1 + a_2\lambda_2 + \cdots + a_m\lambda_m &= b_2, \\ a_1^2\lambda_1 + a_2^2\lambda_2 + \cdots + a_m^2\lambda_m &= b_3, \\ \cdots\cdots\cdots\cdots\cdots\cdots &, \\ a_1^m\lambda_1 + a_2^m\lambda_2 + \cdots + a_m^m\lambda_m &= b_{m+1}. \end{aligned} \tag{6}$$

We shall prove that the a's can be chosen so that the rank of the matrix of coefficients of the λ's in (6) is m, while the rank of the augmented matrix is also m. The determinant of the augmented matrix of (6) is Δ. We write Δ as in (2).

In the expansion of M, the terms of lowest degree α in the a's are those which have b_{m+1} as coefficient. The degree α is explicitly

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[3] R. Oldenburger, *Rational equivalence of a form to a sum of pth powers*, Transactions of this Society, vol. 44 (1938), pp. 219–249.

[4] M. Bôcher, *Introduction to Higher Algebra*, p. 135.

$\frac{1}{2}(m-1)(m-2)$. The terms of lowest degree, in the expansion of the polynomial a_iN for any i, are at least of degree $\alpha+1$. Thus the terms of degree α in the polynomial $M-a_iN$ [i in the range $2, 3, \cdots, m$] arise from M only, whence $M-a_iN$ is not identically zero in the a's. Thus the polynomial

$$(M - a_2N) \cdots (M - a_mN)N \tag{7}$$

of degree at most $I(m)-1$ is not identically zero in the a's. Evidently, $N\neq 0$ implies that $a_2, \cdots, a_m$ are distinct. Choose a set of values of $a_2, \cdots, a_m$ such that the polynomial (7) is different from zero. For this choice of the a's the polynomial $M-a_1N$ is linear in a_1, and $M-a_1N=0$ has a solution for a_1, distinct from $a_2, \cdots, a_m$. Thus there exist mutually distinct values of $a_1, a_2, \cdots, a_m$, such that $\Delta=0$. Since the a's are mutually distinct, the matrix of coefficients of the λ's in (6) has rank m. It follows that the augmented matrix has the same rank. It is well known that a system of linear equations has a solution if and only if the matrix of coefficients and the augmented matrix have the same rank.[5] Theorem 1 is now proved.

The following theorem was proved by Oldenburger.[6]

THEOREM 2. *For a field with characteristic greater than m, the minimal number of $x^{m-1}y$ is m.*

If the sum $\lambda_iL_i^m$ $(i=1, 2, \cdots, m)$ is chosen to be a minimal representation of $x^{m-1}y$, the sum

$$\lambda_1L_1^m + \cdots + \lambda_\rho L_\rho^m, \qquad \rho \leqq m,$$

is minimal. Thus for each ρ in the range $1, 2, \cdots, m$ there is a binary form of degree m with minimal number ρ. Applying Theorem 1 we have arrived at the following result.

THEOREM 3. *Let a field K be given as in Theorem 1. For the field K the range of the minimal numbers of binary forms of degree m is $1, 2, \cdots, m$.*

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[5] M. Bôcher, *Introduction to Higher Algebra*, p. 46.

[6] See first reference to Oldenburger above.

A REMARK CONCERNING SUFFICIENCY THEOREMS FOR THE PROBLEM OF BOLZA[1]

E. J. McSHANE

In recent years there have appeared several sufficiency theorems for minima in Bolza problems in which there is no assumption that the curve in question is normal. However, it is assumed in all these theorems that the multiplier rule, the transversality conditions and the strengthened conditions of Weierstrass, Clebsch and Jacobi hold for a set of multipliers with $\lambda_0 > 0$; whereas, among the necessary conditions the multiplier rule and the Weierstrass and Clebsch conditions have been shown[2] to hold for multipliers with $\lambda_0 \geqq 0$.

In the present note we close this slight gap by showing that the sufficiency theorems hold if we assume λ_0 non-negative instead of positive. The only case needing discussion is $\lambda_0 = 0$. In this case we show that the curve satisfying the hypothesis is isolated; no neighboring curve satisfies the side-equations and end-conditions. The curve is thus a minimizing curve in a trivial sense. Less trivially, we show[3] that the curve is a normal proper minimizing curve for a Mayer problem in which the end-conditions consist of a subset of the original end-conditions and the function to be minimized is a linear combination of the original end-functions.

The problem is that of minimizing a functional

$$(1) \qquad J[y] = g(x_1, y(x_1), x_2, y(x_2)) + \int_{x_1}^{x_2} f(x, y(x), y'(x))dx$$

in the class of curves $y^i = y^i(x)$, $x_1 \leqq x \leqq x_2$, $i = 1, \cdots, n$, which satisfy certain differential equations

$$(2) \qquad \phi_\alpha(x, y(x), y'(x)) = 0, \qquad \alpha = 1, \cdots, m < n,$$

and certain end-conditions

$$(3) \qquad \psi_\mu(x_1, y(x_1), x_2, y(x_2)) = 0, \qquad \mu = 1, \cdots, p \leqq 2n + 2.$$

We suppose the usual conditions[4] of continuity and differentiability satisfied, as well as the usual conditions on the ranks of the matrices of derivatives of ϕ_α and ψ_μ.

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[1] Presented to the Society, September 8, 1939.

[2] E. J. McShane, *On multipliers for Lagrange problems*, American Journal of Mathematics, vol. 61 (1939), pp. 809–819.

[3] At the suggestion of Professor M. R. Hestenes.

[4] See, e.g., G. A. Bliss, *The problem of Bolza in the calculus of variations*, Annals of Mathematics, (2), vol. 33 (1932), pp. 261–273.

Consider a curve $C: y_i = y_i(x)$, $x_1 \leqq x \leqq x_2$, which is of class C'' and satisfies equations (2) and (3). If λ_0 is a constant and $\lambda_\alpha(x)$ a set of functions, we define

$$F(x, y, y', \lambda) = \lambda_0 f(x, y, y') + \lambda_\alpha(x)\phi_\alpha(x, y, y').$$

The statements that C satisfies the multiplier rule, the strengthened Weierstrass condition II_N' and the strengthened Clebsch condition III′ shall have their usual meanings.[5] The curve C satisfies the transversality condition if there are numbers $e_1, \cdots, e_p$ such that

$$(4) \qquad (F - y_i' F_{y_i'})dx \Big|_1^2 + F_{y_i'} dy_i \Big|_1^2 + \lambda_0 dg + e_\mu d\psi_\mu = 0$$

is an identity in dx_1, dx_2, dy_{i1} and dy_{i2}.

For normal problems, in which we can suppose $\lambda_0 = 1$, the second variation of $J[y]$ due to the variations $[\xi_1, \xi_2, \eta(x)]$ is given by a functional[6]

$$(5) \qquad Q[\xi_1, \eta(x_1), \xi_2, \eta(x_2)] + \int_{x_1}^{x_2} 2\omega(x, \eta, \eta')dx.$$

Even for abnormal problems we shall say that the curve C satisfies condition IV′ if the functional (5) is positive for every variation $[\xi_1, \xi_2, \eta(x)]$ not identically zero which satisfies the equations of variation

$$(6) \qquad \phi_{\alpha y_i}\eta_i + \phi_{\alpha y_i'}\eta_i' \equiv 0, \qquad \alpha = 1, \cdots, m,$$

$$(7) \qquad \Psi_\mu(\xi_1, \eta(x_1), \xi_2, \eta(x_2)) = 0, \qquad \mu = 1, \cdots, p,$$

where as usual

$$(8) \qquad \begin{aligned} \Psi_\mu(\xi_1, \eta_1, \xi_2, \eta_2) = (\psi_{\mu x_1} &+ y_i'(x_1)\psi_{\mu y_{i1}})\xi_1 + \psi_{\mu y_{i1}}\eta_{i1} \\ &+ (\psi_{\mu x_2} + y_i'(x_2)\psi_{\mu y_{i2}})\xi_2 + \psi_{\mu y_{i1}}\eta_{i2}. \end{aligned}$$

We do not need the specific form of the quadratic function Q or of 2ω, but for a later paragraph it will be important to note that in (5) the integrand f enters only via the combination $F(x, y, y', \lambda)$, while in the coefficients of Q the functions g and ψ_μ enter only in the combination $\lambda_0 g + e_\mu\psi_\mu$ and its derivatives.

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[5] See, e.g., M. R. Hestenes, *Sufficient conditions for the problem of Bolza in the calculus of variations*, Transactions of this Society, vol. 36 (1934), pp. 795, 809, 813.

[6] For this computation, see, e.g., G. A. Bliss and M. R. Hestenes, *Sufficient conditions for a problem of Mayer in the calculus of variations*, Transactions of this Society, vol. 35 (1933); in particular, pp. 311, 312.

The sufficiency theorem referred to in the introduction is the following.[7]

THEOREM 1. *If the curve C is of class C'' and there are multipliers $\lambda_0 > 0$, $\lambda_\alpha(x)$ with which it satisfies the multiplier rule, the transversality condition and the conditions II_N', III' and IV', then there exists a neighborhood $\mathfrak{J}$ of C in $(x; y)$-space and a neighborhood N of $(x_1, y(x_1), x_2, y(x_2))$ in (x_1, y_1, x_2, y_2)-space with the following properties. If $\bar{C}$: $y = \bar{y}(x)$, $\bar{x}_1 \leqq x \leqq \bar{x}_2$, lies in $\mathfrak{J}$, and the point $(\bar{x}_1, \bar{y}(\bar{x}_1), \bar{x}_2, \bar{y}(\bar{x}_2))$ is in N, and $\bar{C}$ satisfies conditions* (2) *and* (3), *then either $\bar{C}$ is identical with C or $J[\bar{y}] > J[y]$.*

We now establish a slight generalization of this theorem.

THEOREM 2. *In Theorem* 1, *the hypothesis $\lambda_0 > 0$ can be weakened to $\lambda_0 \geqq 0$. Moreover, if the other hypotheses of Theorem* 1 *hold with multipliers $\lambda_0 = 0$, $\lambda_\alpha(x)$, the conclusion can be strengthened by replacing the last clause by the words "then $\bar{C}$ is identical with C."*

Observe that f, g, λ_0 never occur in the hypotheses save in the combinations $\lambda_0 f$, $\lambda_0 g$ and their derivatives. Suppose $\lambda_0 = 0$. All hypotheses continue to be satisfied if we change the problem by replacing λ_0 by 1 and replacing f and g by 0. For this new problem the hypotheses of Theorem 1 hold, and clearly the new functional $J[y]$ is identically 0. Therefore the conclusion of Theorem 1 holds, but the inequality $J[\bar{y}] > J[y]$ is impossible. This establishes Theorem 2.

Henceforth we assume that the hypotheses of Theorem 2 hold with $\lambda_0 = 0$. The curve C is then surely not normal; if

$$[\overset{j}{\xi}_1, \overset{j}{\xi}_2, \overset{j}{\eta}(x)], \qquad j = 1, \cdots, p, \tag{9}$$

is an aggregate of variations satisfying equation (6), the matrix

$$\| \Psi_\mu(\overset{j}{\xi}_1, \overset{j}{\eta}(x_1), \overset{j}{\xi}_2, \overset{j}{\eta}(x_2)) \|, \qquad \mu, j = 1, \cdots, p, \tag{10}$$

has rank less than p. Let $q < p$ be the greatest rank which the matrix (10) can have, and let the set (9) be so chosen that the matrix has rank q. By renumbering if necessary, we can bring it about that the upper left minor, corresponding to $\mu, j = 1, \cdots, q$, is nonsingular.

Consider now the Mayer problem of minimizing the function

$$J^*[y] = e_\mu \psi_\mu(x_1, y(x_1), x_2, y(x_2)) \tag{11}$$

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[7] For a proof, and for reference to other papers on the subject, see M. R. Hestenes, *A direct sufficiency proof for the problem of Bolza in the calculus of variations*, Transactions of this Society, vol. 42 (1937), pp. 141–154.

(where $e_1, \cdots, e_p$ are the constants entering in the transversality condition) in the class of arcs satisfying the differential equations (2) and the end-conditions

$$\psi_\kappa(x_1, y(x_1), x_2, y(x_2)) = 0, \qquad \kappa = 1, \cdots, q. \tag{12}$$

We shall prove that for this problem the curve C is normal and gives J^* a strong proper relative minimum. That is, the conclusion of Theorem 1 holds if we replace the reference (3) by (12) and change J to J^*.

The normality of C is evident; for this Mayer problem, the analogue of matrix (10) is the upper left q-square minor of (10), which is nonsingular.

Equation (11) is the analogue of (1) with

$$g^*(x_1, y_1, x_2, y_2) = e_\mu\psi_\mu(x_1, y_1, x_2, y_2), \qquad f^*(x, y, y') \equiv 0. \tag{13}$$

Define

$$\lambda_0^* = 1, \qquad \lambda_\alpha^*(x) = \lambda_\alpha(x). \tag{14}$$

Then

$$\begin{aligned} F^*(x, y, y', \lambda^*) &= \lambda_0^* f^* + \lambda_\alpha^*(x)\phi_\alpha(x, y, y') \\ &= \lambda_\alpha(x)\phi_\alpha(x, y, y') = F(x, y, y', \lambda). \end{aligned} \tag{15}$$

Since the multiplier rule and conditions II_N' and III′ involve only F and its derivatives, they are satisfied for our Mayer problem.

If we choose

$$e_\kappa^* = 0, \qquad \kappa = 1, \cdots, q, \tag{16}$$

we find, recalling (4) and (15),

$$\begin{aligned} (F^* - y_i' F^*_{y_i'})dx\,\Big|_1^2 + F^*_{y_i'}dy_i\,\Big|_1^2 + \lambda_0^* dg^* + e_\kappa^* d\psi_\kappa & \\ = (F - y_i F_{y_i'})dx\,\Big|_1^2 + F_{y_i'}dy_i\,\Big|_1^2 + \lambda_0 dg + e_\mu d\psi_\mu &= 0 \end{aligned}$$

identically in dx_1, dx_2, dy_{i1} and dy_{i2}. So the transversality condition is satisfied in our Mayer problem.

From (13), (14) and (16) we have

$$\lambda_0^* g^* + e_\kappa^* \psi_\kappa \equiv \lambda_0 g + e_\mu \psi_\mu. \tag{17}$$

Now in the light of (17) and (15) the sentence after (8) assures us that the sum (5) is identically equal to its analogue for our Mayer problem. Hence condition IV′ is satisfied. All the hypotheses of Theorem 1 have now been verified for our Mayer problem, and it follows that C minimizes $J^*[y]$, as stated.

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NEUTRAL ELEMENTS IN GENERAL LATTICES[1]

GARRETT BIRKHOFF

1. **Introduction.** O. Ore has defined "neutral" elements in modular lattices as elements a satisfying $a \cap (x \cup y) = (a \cap x) \cup (a \cap y)$ for all x, y and dually.[2] In the case of complemented modular lattices, the neutral elements compose the "center" in J. von Neumann's theories of continuous geometries and regular rings—that is, the set of elements having unique complements.[3]

The purpose of the present note is to extend the notion of neutral elements to general lattices. More precisely, call an element a of a lattice "neutral" if and only if every triple $\{a, x, y\}$ generates a distributive sublattice. It is proved that the neutral elements of any lattice L form a distributive sublattice, consisting of the elements carried into $[I, O]$ under isomorphisms of L with sublattices of direct products. Actually, this sublattice is the intersection of the maximal distributive sublattices of L.

Further, complements of neutral elements, when they exist, are unique and neutral. The sublattice of complemented neutral elements may be called the "center" of a lattice: it consists of those elements carried into $[I, O]$ under isomorphisms of L with direct products.

2. **Fundamental definition.** We define an element a of a lattice L to be "neutral" if and only if every triple $\{a, x, y\}$ generates a distributive sublattice of L.

LEMMA 1. *If a is "neutral," then the dual correspondences $x \to x \cap a$ and $x \to x \cup a$ are endomorphisms*[4] *of L.*

PROOF. By definition, $(x \cup y) \cap a = (x \cup a) \cap (y \cup a)$ and $(x \cap y) \cap a = (x \cap a) \cap (y \cap a)$, and dually. We note that this condition, which is sufficient to guarantee neutrality in the case of modular lattices, does not guarantee neutrality in general—see the graph below.

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[1] Presented to the Society, September 8, 1939.

[2] O. Ore, *On the foundations of abstract algebra* I, Annals of Mathematics, (2), vol. 36 (1935), pp. 406–437. For the definitions of lattices and modular lattices (called by Ore structures and Dedekind structures), as well as of sublattice, distributive lattice, O, I, and so on, compare the author's *Lattices and their applications*, this Bulletin, vol. 44 (1938), pp. 793–800—or the author's *Lattice Theory*, American Mathematical Society Colloquium Publications, vol. 25, 1940.

[3] J. von Neumann, *Lectures on Continuous Geometries*, Princeton, 1935–1936. Cf. also R. P. Dilworth, *Note on complemented modular lattices*, this Bulletin, vol. 45 (1939), pp. 74–76.

[4] We define an endomorphism as a homomorphism of L with itself.

LEMMA 2. *If a is neutral, then $x \cap a = y \cap a$ and $x \cup a = y \cup a$ imply $x = y$.*

PROOF. By direct computation, using the distributive law twice,

$$\begin{aligned} x &= x \cap (x \cup a) = x \cap (y \cup a) = (x \cap y) \cup (x \cap a) \\ &= (x \cap y) \cup (y \cap a) = y \cap (x \cup a) = y \cap (y \cup a) = y. \end{aligned}$$

Using x in the graph, we see that this condition by itself is also not sufficient. However, Lemmas 1–2 together are sufficient to guarantee neutrality.

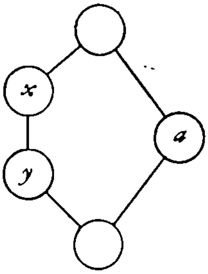

FIG. 1

Indeed, consider the correspondence $x \rightarrow [x \cap a, x \cup a]$ from L to the product[5] ST of the sublattice S of elements $s \leqq a$, with the sublattice T of elements $t \geqq a$. By Lemma 1, it is homomorphic onto a sublattice of ST; by Lemma 2, it is one-one; hence it is isomorphic. Moreover $x \rightarrow [a, a] = [I, O]$, since a is the I of S and the O of T.

But conversely, $[I, O]$ is obviously "neutral" in ST, since each component is. Hence it is neutral in every sublattice of ST, including L, and we conclude[6] that the following holds:

THEOREM 1. *An element of a lattice is neutral if and only if it is carried into $[I, O]$ under an isomorphism of the lattice with a sublattice of a direct product.*

3. **Neutral elements a sublattice.** Just as in the case of modular lattices, we have the following theorem:

THEOREM 2. *The neutral elements of any lattice form a distributive sublattice.*

PROOF. Let a and b be neutral. Then since the product of two endomorphisms of L is an endomorphism, it follows that the correspondence

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[5] By the "product" ST, is meant the system of couples $[s, t]$, $s \in S$, $t \in T$, where $[s, t] \cap [s', t'] = [s \cap s', t \cap t']$ and dually.

[6] N.B., we do *not* assume that L itself has an O or an I.

$$x \rightarrow [x \cap a \cap b, (x \cup a) \cap b, (x \cap a) \cup b, x \cup a \cup b]$$

defines a homomorphism of L onto a sublattice of a product $STUV$ of four sublattices of itself. Moreover by Lemma 2, $(x \cap a)$ and $(x \cup a)$ and therefore x, are determined uniquely by their images under the homomorphism. Hence the homomorphism is an isomorphism. But $a \cup b$ goes into $[I, I, I, O]$ under this—and hence into $[I, O]$ of $(STU)V$. Thus $a \cup b$ is neutral; dually, $a \cap b$ is neutral, which was to be proved.

4. **Intersection of maximal distributive sublattices.** The set of neutral elements of a lattice is also characterized in another way by the next theorem:

THEOREM 3. *The set of neutral elements of a lattice L is the intersection of its maximal distributive sublattices.*

PROOF. First, if a is not neutral, then some triple $\{a, x, y\}$ is not distributive. Hence *no* maximal distributive sublattice obtained by enlarging the distributive sublattice generated by $\{x, y\}$ can contain a. Consequently, the intersection of the maximal distributive sublattices of L contains no non-neutral elements.

Conversely, if a is neutral, and S is a distributive sublattice of L, consider the sublattice generated by $\{a, S\}$. The endomorphisms $x \rightarrow x \cap a$ and $x \rightarrow x \cup a$ carry it into sublattices generated by a distributive sublattice and I or O. But such sublattices are always distributive—hence so is the sublattice generated by $\{a, S\}$, since it is a sublattice of a product of distributive lattices. Thus every *maximal* distributive sublattice contains a, and the intersection of the maximal distributive sublattices contains every neutral element (as well as no non-neutral elements).

5. **Center of a lattice.** When one comes to complements of neutral elements, one finds that the following statement holds:

THEOREM 4. *Complements of neutral elements, when they exist, are unique and neutral.*

PROOF. Using Theorem 1, we see that $[I, O] \cap [x, y]$ is $[O, O]$ if and only if $x = O$, while $[I, O] \cup [x, y]$ is $[I, I]$ if and only if $y = I$. Hence $[I, O]$ has no complement except $[O, I]$ in the sublattice of ST isomorphic with L, proving uniqueness. Moreover $[O, I]$ is itself neutral, completing the proof.

COROLLARY 1. *The neutral elements of a complemented lattice form a Boolean algebra.*

We define the "center" of a lattice as the set of its complemented neutral elements.

THEOREM 5. *The center of any lattice L is a complemented distributive sublattice—and hence a Boolean algebra.*

PROOF. If a and b are neutral elements of L, with (neutral) complements a' and b', then

$$(a \cap b) \cap (a' \cup b') = (a \cap b \cap a') \cup (a \cap b \cap b') = O \cup O = O,$$
$$(a \cap b) \cup (a' \cup b') = (a \cup a' \cup b') \cap (b \cup a' \cup b') = I \cap I = I,$$

and so $a \cap b$ is complemented. Dually, $a \cup b$ is complemented, completing the proof.

We can now specialize Theorem 1 by proving the following:

THEOREM 6. *An element is in the center of a lattice L if and only if it is carried into $[I, O]$ under an isomorphism of L with a direct product.*

PROOF. By Theorem 1, such an element is neutral, and it has the complement $[O, I]$. Conversely, suppose a and a' are complementary neutral elements of L. Then for all x,

$$x = x \cap I = x \cap (a \cup a') = (x \cap a) \cup (x \cap a').$$

Hence the correspondence $x \rightarrow [x \cap a,\ x \cap a']$ is one-one between L and the couples $[u, v]$ with $u \leqq a$, $v \leqq a'$; the inverse correspondence is $[u, v] \rightarrow u \cup v$. But it obviously preserves inclusion; hence it is an *isomorphism*. Finally, it carries L into the product ST of the lattice S of elements $s \leqq a$, with the lattice T of elements $t \leqq a'$, while it carries a into $[a \cap a,\ a \cap a'] = [I, O]$ in ST.

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ON TOPOLOGICAL COMPLETENESS[1]

L. W. COHEN

Recently A. Weil[2] defined a uniform space as a set of points p such that for each α in a set A there is defined a set $U_\alpha(p) \subset S$, the class of sets $U_\alpha(p)$ satisfying the conditions:

I_A. $\prod_\alpha U_\alpha(p) = (p)$.

II_A. To each $\alpha, \beta \in A$ there is a $\gamma = \gamma(\alpha, \beta) \in A$ such that $U_\gamma(p) \subset U_\alpha(p) U_\beta(p)$.

III_A. To each $\alpha \in A$ there is a $\beta(\alpha) \in A$ such that if $p', p'' \in U_{\beta(\alpha)}(q)$, then $p'' \in U_\alpha(p')$.

For the uniform space S, Weil introduced the concept of Cauchy family $\{M_\beta\}$ of sets. Such a family is defined by the conditions that the intersection of any finite number of sets of the family is non-empty and that to each $\alpha \in A$ there is a $p_\alpha \in S$ and a $\beta(\alpha)$ such that $M_{\beta(\alpha)} \subset U_\alpha(p_\alpha)$. Weil gives a theory of completeness in these terms.

The writer has considered[3] a space S of points p and neighborhoods $U_\alpha(p)$ where α is an element of a set A such that:

I. $\prod_\alpha U_\alpha(p) = (p)$.

II. To each α and $\beta \in A$ and $p \in S$ there is a $\gamma = \gamma(\alpha, \beta; p)$ such that $U_\gamma(p) \subset U_\alpha(p) U_\beta(p)$.

III. To each $\alpha \in A$ and $p \in S$ there are $\lambda(\alpha)$, $\delta(p, \alpha) \in A$ such that, if $U_{\delta(p,\alpha)}(q) U_{\lambda(\alpha)}(p) \neq 0$, then $U_{\delta(p,\alpha)}(q) \subset U_\alpha(p)$.

The uniformity conditions here are lighter than those in II_A and III_A. A Cauchy sequence $p_n \in S$ was defined by the condition that for every $\alpha \in A$, n_α and $p_\alpha \in S$ exist such that $p_n \in U_\alpha(p_\alpha)$ for $n \geqq n_\alpha$. S is complete if every Cauchy sequence has a limit. It was shown that there is a complete space S^* which contains a homeomorphic image of S such that the image of a Cauchy sequence in S is a convergent sequence in S^*.

It is the object of this paper to show that Weil's space is a special case of the space $S_{\mathrm{I},\mathrm{II},\mathrm{III}}$ and that the notion of Cauchy family in this space leads to the same theory of completeness as that previously developed.

THEOREM 1. *If S satisfies III_A and $\beta^2(\alpha) = \beta(\beta(\alpha))$, then from $U_{\beta^2(\alpha)}(q) U_{\beta^2(\alpha)}(p) \neq 0$ follows $U_{\beta^2(\alpha)}(q) \subset U_\alpha(p)$.*

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[1] Presented to the Society, December 27, 1939.

[2] A. Weil, *Sur les Espaces à Structure Uniforme*, Paris, 1938.

[3] L. W. Cohen, *On imbedding a space in a complete space*, Duke Mathematical Journal, vol. 5 (1939), pp. 174–183. Also Duke Mathematical Journal, vol. 3 (1937), pp. 610–617, where the notion of topological completeness is introduced.

PROOF. Let $s \in U_{\beta^2(\alpha)}(q)U_{\beta^2(\alpha)}(p)$. Then from s, $q' \in U_{\beta^2(\alpha)}(q)$ we have $q' \in U_{\beta(\alpha)}(s)$. Therefore $U_{\beta^2(\alpha)}(q) \subset U_{\beta(\alpha)}(s)$. Similarly $U_{\beta^2(\alpha)}(p) \subset U_{\beta(\alpha)}(s)$. Now from $p \in U_{\beta(\alpha)}(s)$ and $U_{\beta^2(\alpha)}(q) \subset U_{\beta(\alpha)}(s)$ we have $U_{\beta^2(\alpha)}(q) \subset U_\alpha(p)$.

COROLLARY. *If S satisfies* III_A, *then S satisfies* III.

PROOF. For any $p \in S$ and $\alpha \in A$ we need only take $\lambda(\alpha) = \delta(p, \alpha) = \beta(\beta(\alpha))$. The result is stronger than III since $\delta(p, \alpha)$ is independent of p.

From now on a space S is one satisfying I, II, III. A family of sets $\{M_\beta\}$ is a Cauchy family if the intersection of any finite number of M_β is non-empty and if for any $\alpha \in A$ there is a $\beta(\alpha)$ such that $M_{\beta(\alpha)} \subset U_\alpha(p_\alpha)$ for some $p_\alpha \in S$. We will say that S is W-complete if, for every Cauchy family $\{M_\beta\}$, $\prod_\beta \overline{M}_\beta \neq 0$, where $\overline{M}_\beta$ is the closure of M_β. We shall always use the notations $\lambda(\alpha)$, $\delta(p, \alpha)$ in the sense of III.

THEOREM 2. *S is W-complete if and only if every Cauchy family $\{U_\alpha(p_\alpha)\}$ consisting of one $U_\alpha(p_\alpha)$ for each $\alpha \in A$ has the property that* $\prod_\alpha U_\alpha(p_{\lambda(\alpha)}) \neq 0$.

PROOF. Assume S is W-complete. If $\{U_\alpha(p_\alpha)\}$ is a Cauchy family, then $\prod_\alpha \overline{U}_{\lambda(\alpha)}(p_{\lambda(\alpha)}) \neq 0$. Since $\overline{U}_{\lambda(\alpha)}(p) \subset U_\alpha(p)$ follows from III, the condition $\prod_\alpha U_\alpha(p_{\lambda(\alpha)}) \neq 0$ is necessary. Assume now that the condition is satisfied and let $\{M_\beta\}$ be a Cauchy family in S. To each $\alpha \in A$ there are $p_\alpha \in S$ and $M_{\beta(\alpha)}$ such that $M_{\beta(\alpha)} \subset U_\alpha(p_\alpha)$. It is clear that $\{U_\alpha(p_\alpha)\}$ is a Cauchy family. Hence there is $p \in \prod_\alpha U_\alpha(p_{\lambda(\alpha)})$. For any $\gamma \in A$, consider the $\lambda(\gamma)$ and $\alpha = \delta(p, \gamma)$ of III. Since $p \in U_\alpha(p_{\lambda(\alpha)})U_{\lambda(\gamma)}(p)$, $U_\alpha(p_{\lambda(\alpha)}) \subset U_\gamma(p)$ and

$$M_{\beta(\lambda(\alpha))} \subset U_{\lambda(\alpha)}(p_{\lambda(\alpha)}) \subset U_\alpha(p_{\lambda(\alpha)}) \subset U_\gamma(p).$$

Hence

$$0 \neq M_\beta M_{\beta(\lambda(\alpha))} \subset M_\beta U_\gamma(p)$$

for all β, γ. Thus $p \in \prod_\beta \overline{M}_\beta$ and S is W-complete.

The space S^* referred to above is defined as follows. A family $\{U_\alpha(p_\alpha)\}$, one for each $\alpha \in A$, such that for any $(\alpha_1, \cdots, \alpha_n) \subset A$, $U_{\alpha_1}(p_{\alpha_1})U_{\alpha_2}(p_{\alpha_2}) \cdots U_{\alpha_n}(p_{\alpha_n}) \neq 0$ is denoted by Π. We write $\Pi' \sim \Pi''$ if for every $\alpha \in A$ there is a set $(\alpha_1, \cdots, \alpha_n) \subset A$ such that for some $p_\alpha \in S$

$$\prod_{i=1}^{n} U_{\alpha_i}(p'_{\alpha_i}) + \prod_{i=1}^{n} U_{\alpha_i}(p''_{\alpha_i}) \subset U_\alpha(p_\alpha).$$

The relation $\Pi'\sim\Pi''$ classifies all Π into mutually exclusive classes $C(\Pi)$ which are the points P of S^*. The neighborhoods $U^*_{\alpha_1\cdots\alpha_n}(P)$ are made up of all $Q=C(\Pi^Q)\ \varepsilon\ S^*$ where $\Pi^Q=\{U_\alpha(q_\alpha)\}$ is such that for some $\beta_i=\beta_i(\alpha_1,\cdots,\alpha_n;P)$, $i=1,\cdots,m=m(\alpha_1,\cdots,\alpha_n;P)$,

$$\prod_{i=1}^{m} U_{\beta_i}(q_{\beta_i})\subset\prod_{i=1}^{n} U_{\alpha_i}(p_{\alpha_i}).$$

The space S^* (for which A^* is the set of all finite subsets of A) satisfies I, II and

III*. For each $(\alpha_1,\cdots,\alpha_n)\subset A$ and for $P\ \varepsilon\ S^*$ there exist $\gamma_i=\gamma_i(\alpha_1,\cdots,\alpha_n;P)$, $i=1,\cdots,m=m(\alpha_1,\cdots,\alpha_n;P)$, such that if $U^*_{\gamma_1\cdots\gamma_m}(Q)\,U^*_{\gamma_1\cdots\gamma_m}(P)\neq 0$, then $U^*_{\gamma_1\cdots\gamma_m}(Q)\subset U^*_{\alpha_1\cdots\alpha_n}(P)$.

The mapping $f(p)=C(\Pi^p)$, where $\Pi^p=\{U_\alpha(p)\}$, is the homeomorphism on S to a subset of S^* referred to above.

THEOREM 3. *S^* is W-complete.*

PROOF. We first show that if $\{U^*_{\alpha_1\cdots\alpha_n}(P^{\alpha_1\cdots\alpha_n})\}$, where each $(\alpha_1,\cdots,\alpha_n)\subset A$ occurs just once, is a Cauchy family in S^*, then

$$\prod_{(\alpha_1\cdots\alpha_n)\subset A}\overline{U^*_{\alpha_1\cdots\alpha_n}(p^{\alpha_1\cdots\alpha_n})}\neq 0.$$

Consider the $U_\alpha^*(P^\alpha)$ for all $\alpha\ \varepsilon\ A$. Now $P^\alpha=C(\Pi^{P^\alpha})$, $\Pi^{P^\alpha}=\{U_\gamma(p^\alpha_\gamma)\}$, $U_\gamma(p^\alpha_\gamma)$ being a neighborhood in S for each $(\alpha,\gamma)\subset A$. Since, for each $(\alpha_1,\cdots,\alpha_n)\subset A$, $\prod_{i=1}^{n}U^*_{\alpha_i}(P^{\alpha_i})\neq 0$, we have

$$\prod_{i=1}^{n} U_{\alpha_i}(p^{\alpha_i}_{\alpha_i})\neq 0.$$

Hence the family $\Pi=\{U_\alpha(p^\alpha_\alpha)\}$ defines a $P=C(\Pi)\ \varepsilon\ S^*$. For any $(\gamma_1,\cdots,\gamma_m)\subset A$

$$\prod_{i=1}^{m} U^*_{\gamma_i}(P) = U^*_{\gamma_1\cdots\gamma_m}(P).$$

Both $U^*_{\gamma_i}(P)$ and $U^*_{\gamma_i}(P^{\gamma_i})$ are the set of all $Q=C(\Pi^Q)$, $\Pi^Q=\{U_\beta(q_\beta)\}$ such that for some $\beta_{ji}=\beta_j(\gamma_i)$, $j=1,\cdots,k_i$, $\prod_{j=1}^{k_i}U_{\beta_{ji}}(q_{\beta_{ji}})\subset U_{\gamma_i}(p^{\gamma_i}_{\gamma_i})$. Hence

$$U^*_{\gamma_i}(P) = U^*_{\gamma_i}(P^{\gamma_i}),\qquad U^*_{\gamma_1\cdots\gamma_m}(P) = \prod_{i=1}^{m} U^*_{\gamma_i}(P^{\gamma_i}).$$

Thus from the fact that $\{U^*_{\alpha_1\cdots\alpha_n}(P^{\alpha_1\cdots\alpha_n})\}$ is a Cauchy family we have for any sets $(\alpha_1,\cdots,\alpha_n)$, $(\gamma_1,\cdots,\gamma_m)\subset A$

$$U^*_{\alpha_1\cdots\alpha_n}(P^{\alpha_1\cdots\alpha_n})U^*_{\gamma_1\cdots\gamma_m}(P) = U^*_{\alpha_1\cdots\alpha_n}(P^{\alpha_1\cdots\alpha_n})\prod_{i=1}^{m} U^*_{\gamma_i}(P^{\gamma_i}) \neq 0.$$

It follows that

$$(1)\qquad P \,\varepsilon \prod_{(\alpha_1\cdots\alpha_n)\subset A} \overline{U^*_{\alpha_1\cdots\alpha_n}}(P^{\alpha_1\cdots\alpha_n}).$$

Now suppose that $\{M_\beta^*\}$ is a Cauchy family of sets in S^*. For every $(\gamma_1, \cdots, \gamma_n)\subset A$ there is a $\beta(\gamma_1, \cdots, \gamma_n)$ such that for some $P^{\gamma_1\cdots\gamma_n}\,\varepsilon\, S^*$, $M^*_{\beta(\gamma_1\cdots\gamma_n)}\subset U^*_{\gamma_1\cdots\gamma_n}(P^{\gamma_1\cdots\gamma_n})$. Thus

$$0 \neq \prod_{i=1}^{m} M^*_{\beta(\gamma_1^i,\cdots,\gamma_{n_i}^i)} \subset \prod_{i=1}^{m} U^*_{\gamma_1^i,\cdots,\gamma_{n_i}^i}(P^{\gamma_1^i,\cdots,\gamma_{n_i}^i})$$

for every finite set $(\gamma_1^i, \cdots, \gamma_{n_i}^i)\subset A$ and $\{U^*_{\gamma_1\cdots\gamma_n}(P^{\gamma_1\cdots\gamma_n})\}$ is a Cauchy family. Let $P \,\varepsilon\, S^*$ satisfy (1). For any $(\alpha_1, \cdots, \alpha_n)\subset A$ and $P\,\varepsilon\, S^*$ let $(\gamma_1, \cdots, \gamma_m)\subset A$ be the set of $\gamma_i=\gamma_i(\alpha_1, \cdots, \alpha_n; P)$ satisfying III*. From III* and (1)

$$U^*_{\gamma_1\cdots\gamma_m}(P^{\gamma_1\cdots\gamma_m})\subset U^*_{\alpha_1\cdots\alpha_n}(P).$$

Since $\{M_\beta^*\}$ is a Cauchy family, for any β and $(\alpha_1, \cdots, \alpha_n)\subset A$ we have

$$0 \neq M^*_\beta M^*_{\beta(\gamma_1,\cdots,\gamma_m)}\subset M^*_{\beta(\gamma_1,\cdots,\gamma_m)}\subset U^*_{\gamma_1\cdots\gamma_m}(P^{\gamma_1\cdots\gamma_m})\subset U^*_{\alpha_1\cdots\alpha_n}(P),$$

$$M^*_\beta U^*_{\alpha_1\cdots\alpha_n}(P)\neq 0.$$

Hence $P\,\varepsilon\prod_\beta \overline{M_\beta^*}$ and S^* is W-complete.

THEOREM 4. *If* $\{M_\beta\}$ *is a Cauchy family in* S *and* $f(S)\subset S^*$ *is the homeomorphism defined above, then there is a* $P\,\varepsilon\, S^*$ *such that*

1. $\prod_\beta \overline{f(M_\beta)}=(P)$,
2. *for any* $(\alpha_1, \cdots, \alpha_n)\subset A$ *there are* $\beta_1, \cdots, \beta_m$ *such that*

$$\prod_{i=1}^{m} f(M_{\beta_i})\subset U^*_{\alpha_1\cdots\alpha_n}(P).$$

PROOF. $f(M_\beta)$ is the class of $P=C(\Pi^p)$ for all $p\,\varepsilon\, M_\beta$ where for any $(\alpha_1, \cdots, \alpha_n)\subset A$ and $\lambda(\alpha_i)$ there are $\beta(\lambda(\alpha_i))$ and $p_{\lambda(\alpha_i)}$ such that

$$M_{\beta(\lambda(\alpha_i))}\subset U_{\lambda(\alpha_i)}(p_{\lambda(\alpha_i)})\subset U_{\alpha_i}(p_{\lambda(\alpha_i)}),$$

$$0\neq \prod_{i=1}^{n} M_{\beta(\lambda(\alpha_i))}\subset \prod_{i=1}^{n} U_{\alpha_i}(p_{\lambda(\alpha_i)}).$$

If $q\,\varepsilon\prod_{i=1}^{n} M_{\beta(\lambda(\alpha_i))}$, then for $\delta_i=\delta(p_{\lambda(\alpha_i)}, \alpha_i)$ we have

$$q\,\varepsilon\, U_{\delta_i}(q)U_{\lambda(\alpha_i)}(p_{\lambda(\alpha_i)}),\qquad U_{\delta_i}(q)\subset U_{\alpha_i}(p_{\lambda(\alpha_i)}),\qquad i=1, 2, \cdots, n.$$

Since

$$q \,\varepsilon\, \prod_{i=1}^{n} U_{\delta_i}(q) \subset \prod_{i=1}^{n} U_{\alpha_i}(p_{\lambda(\alpha_i)}),$$

the family $\{U_\alpha(p_{\lambda(\alpha)})\} = \Pi$ defines a $P = C(\Pi) \,\varepsilon\, S^*$ such that

$$f(q) \,\varepsilon\, U^*_{\delta_1\cdots\delta_n}(f(q)) \subset U^*_{\alpha_1\cdots\alpha_n}(P).$$

Hence for such $(\alpha_1, \cdots, \alpha_n) \subset A$ there are $P^{\alpha_1\cdots\alpha_n} = P$ and $\beta_i = \beta(\lambda(\alpha_i))$, $i = 1, \cdots, n$, such that

$$0 \neq \prod_{i=1}^{n} f(M_{\beta(\lambda(\alpha_i))}) \subset U^*_{\alpha_1\cdots\alpha_n}(P).$$

Thus the family of all finite products $\{f(M_{\beta_1}) \cdots f(M_{\beta_n})\}$ is a Cauchy family in S^* since $f(M_{\beta_1}) \cdots f(M_{\beta_n}) = f(M_{\beta_1} \cdots M_{\beta_n}) \neq 0$. We have

$$U^*_{\alpha_1\cdots\alpha_n}(P)f(M_\beta) \supset \prod_{i=1}^{n} f(M_{\beta(\lambda(\alpha_i))})f(M_\beta) \neq 0$$

for all β and all $(\alpha_1, \cdots, \alpha_n) \subset A$. In other words

$$P \,\varepsilon \prod_{\beta} \overline{f(M_\beta)}.$$

But the intersection of all $\overline{f(M_\beta)}$ is P, for if P' is any other point there are sets $(\alpha_1, \cdots, \alpha_n)$, $(\alpha_1', \cdots, \alpha_{n'}') \subset A$ such that

$$U^{*}_{\alpha_1\cdots\alpha_n}(P)\,U^{*}_{\alpha_1'\cdots\alpha_{n'}'}(P') = 0$$

and

$$P' \,\varepsilon\, S^* - \overline{U^*_{\alpha_1\cdots\alpha_n}}(P) \subset S^* - \prod_{i=1}^{n} \overline{f(M_{\beta(\lambda(\alpha_i))})} \subset S^* - \prod_{\beta} \overline{f(M_\beta)}.$$

We conclude with the remark that if S is W-complete, S is complete. Suppose p_n is a Cauchy sequence in S. Let $M_n = (p_n, p_{n+1}, \cdots)$. Then the intersection of any finite set of M_n is non-empty and for any $\alpha \,\varepsilon\, A$ there is an $n(\alpha)$ such that $M_{n(\alpha)} \subset U_\alpha(p_\alpha)$ for some $p_\alpha \,\varepsilon\, S$. Thus $\{M_n\}$ is a Cauchy family. S being W-complete, there is a $p \,\varepsilon \prod_n \overline{M}_n$. Now for p, any $\alpha \,\varepsilon\, A$, $\lambda(\alpha)$ and $\delta = \delta(p, \alpha)$, we have

$$M_{n(\delta)} \subset U_\delta(p_\delta), \qquad 0 \neq U_{\lambda(\alpha)}(p)M_{n(\delta)} \subset U_{\lambda(\alpha)}(p)U_\delta(p_\delta),$$

$$M_{n(\delta)} \subset U_\delta(p_\delta) \subset U_\alpha(p), \quad p_n \,\varepsilon\, U_\alpha(p), \quad n \geqq n(\delta) = n(\delta(p, \alpha)), \lim p_n = p.$$

UNIVERSITY OF KENTUCKY

REPORT OF THE WAR PREPAREDNESS COMMITTEE OF THE AMERICAN MATHEMATICAL SOCIETY AND MATHEMATICAL ASSOCIATION OF AMERICA AT THE HANOVER MEETING

In fulfilling its mission to prepare the Society and Association to serve the country in time of war the Committee recognizes three objectives

(1) The solution of mathematical problems essential for military or naval science, or rearmament.

(2) The preparation of mathematicians for research essential for objective (1).

(3) The strengthening of undergraduate mathematical education in our colleges to the point where it affords adequate preparation in mathematics for military and naval services of any nature. A study by a large group of mathematicians of the current routine military texts and sources wherever mathematics is involved in order that mathematicians may exert their proper influence on the teaching of military and naval science in time of war.

Organization. It is recommended that the present Subcommittee on Education be replaced by two new subcommittees corresponding to objectives (2) and (3) respectively and that the present Subcommittee on Research remain unchanged. So reorganized there would be three subcommittees designated as follows:

I. Committee on Research
II. Committee on Preparation for Research
III. Committee on Education for Service.

It is recognized that Committee III corresponds very closely to the special interests and aims of the Association. It is recommended that chairmen of Committees II and III be appointed at Hanover but that the completion of these committees be left until the new chairmen have had time to make recommendations as to the personnel of their committees.

Consultants. Having particular regard to the solution of problems, we recommend the appointment of Chief Consultants in each of the following fields: Ballistics, Aeronautics, Computation (numerical, mechanical, electrical), Cryptanalysis, Industry, Probability and Statistics. Other fields may well be added. The Chief Consultants would be responsible in research matters to the Research Committee without themselves taking on any of the executive duties of that committee. These consultants should also help the Committee on Preparation for Research when called upon. The three central committees should remain small so that they can act quickly. It is also recommended that there be *ordinary consultants* in each of the above fields. These ordinary consultants would be recommended for appointment so as to obtain some sort of correspondence with the subfields. Thus able young mathematicians so appointed might prepare themselves in advance to help in a given field. These consultants should also be appointed so as to meet the needs of particular sections of the country for help in its industries. It is suggested that any mathematician who feels competent in a field might ask to be made a consultant, and might be accepted as such if that seemed desirable. A flexible group is indicated.

DUTIES

1. *Committee on Research.* Problems would be received and assigned to consultants when available, otherwise assigned to members of the Research Committee. Recommendations as to educational needs would be made to the Committee on Education for Research. Suitable contacts would be made by this committee. The proper assignment and orientation of young consultants should be the duty of this committee.

2. *Education for Research.* This committee would continue work already done on bibliographies, texts, seminars, publications, etc. Professor A. W. Tucker, representing the Annals of Mathematics Studies, has stated that there will be no difficulty in publishing suitable texts concerning applications of mathematics in war or industry. New texts in practically all of the above fields are needed and competent mathematicians have already been invited to write such texts. A small loan fund may be necessary to pay the initial cost of such Studies since these texts are published on a cash basis without profit.

3. *Education for Routine Service.* The War Department intends in general to use sections of existing texts for instruction in the military schools. Those few texts which involve mathematics in an essential way should be carefully read by members of this committee, with two objects in view.

(1) *What elementary mathematics must be taught in order that these books be readily understood?*

(2) *In what way could the material in these books of a mathematical nature be better written to save time and achieve the desired results?*

The following three resolutions are recommended for adoption by the Council of the A.M.S. and Board of Governors of the M.A.A.:

1. That all competent students in the secondary schools take the maximum amount of mathematics available in their institutions. In the case of many schools, additions to the present curriculum will be necessary in order to furnish an adequate background for the military needs of the country.

2. That the colleges and universities at once make such revisions of their undergraduate courses in mathematics and add such courses to the curriculum as are necessary to prepare students in the elements of mechanics, probability, surveying, navigation and other essentials of military science.

3. That the graduate schools extend their courses in applied mathematics, such as dynamics, hydrodynamics, elasticity, aeronautics, ballistics, statistics, and that advanced students be urged to become highly qualified in one or more fields of applied mathematics.

These resolutions should be given immediate publicity in order that the changes recommended may be undertaken at once. In connection with resolutions 2 and 3 Committees II and III will be able to give a certain amount of advice at once and in a few months will undoubtedly be able to help considerably more.

We are appending two bibliographies corresponding respectively to the needs of those primarily interested in research and those primarily interested in education for service. These bibliographies are selective rather than exhaustive because it is felt that such bibliographies will best serve immediate needs. More extensive bibliographies are available and will be mailed to those interested upon request. The report of the War Preparedness Committee in extenso containing letters from various authorities and other important source material may be had upon writing to Secretary Richardson. Needless to say the Committee will help those who are working on war preparedness problems in any way in which it can.

Prepared for the Committee by

MARSTON MORSE
General Chairman

The above three resolutions were adopted by the Council of the American Mathematical Society and the Board of Governors of the Mathematical Association of America at a joint meeting at Hanover, N. H.

Professor M. H. Stone has been appointed Chairman of the Subcommittee on Preparation for Research and Professor W. L. Hart has been appointed Chairman of the Subcommittee on Education for Service. Chief Consultants have been appointed as follows:

Ballistics—Professor John von Neumann, Institute for Advanced Study
Aeronautics—Professor Harry Bateman, California Institute of Technology.
Computation—Professor Norbert Wiener, Massachusetts Institute of Technology.
Industry—Dr. T. C. Fry, Bell Telephone Laboratories
Probability and Statistics—Professor S. S. Wilks, Princeton University
Cryptanalysis—Professor H. T. Engstrom, Yale University.

SERVICE BIBLIOGRAPHY

Seacoast Artillery Gunnery, F. M. 4–10. U. S. Government Printing Office. Price 25 cents.

Seacoast Artillery, F. M. 4–15 and 4–10. U. S. Government Printing Office. To be published.

Gunnery, Fire Control and Position Finding for Anti-Aircraft Artillery. Special text No. 26, U. S. Government Printing Office. (*Restricted*)

Field Artillery, F. M. 6–40. 1939, U. S. Government Printing Office, Washington, D. C. Price 25 cents.

Field Artillery. Gunnery, 1936, Book 161. Printed: Field Artillery School, Fort Sill, Oklahoma.

Signal Communications, F. M. 24–5. U. S. Government Printing Office, Washington, D. C. Price 45 cents.

Practical Air Navigation. Special Publication No. 197, U.S. Department of Commerce, Coast and Geodetic Survey (temporarily out of print)

The Use of the Airway Radio Range and Other Radio Aids. Free publication of Civil Aeronautics Authority, Washington, D. C.

Aerodynamics. Bulletin No. 21, Civil Aeronautics Authority, Washington, D. C.

Anti-Aircraft Artillery. Gunner's Instruction, IX Expert Gunners, Coast Artillery Journal, 1115 17th Street, N.W., Washington, D. C.

Sound Ranging. F. M. 6–120, U. S. Government Printing Office, Washington, D. C.

Meteorology. Army Extension Courses, Special Text No. 193, U. S. Government Printing Office, Washington, D. C. Price 20 cents.

Coast Artillery Journal. $4 a year. 1115 17th St. N.W., Washington D. C.

Ordnance

T. J. Hayes. *Elements of Ordnance*, John Wiley and Sons, 1938. Chapters X, XI, XII, Exterior Ballistics, Probability of Hitting, Bombing from Airplanes, written by Kent, Dederick, Zornig, of Aberdeen Proving Ground.

Aerodynamics

B. Jones. *Elements of Practical Aerodynamics*, Wiley and Sons.

C. C. Carter. *Simple Aerodynamics*, Ronald Press.

Navigation

Dutton. *Navigation and Nautical Astronomy*, U. S. Naval Institute.

P. V. H. Weems. *Air-Navigation*, McGraw-Hill.

Optics

McKinley. *Applied Aerial Photography,* Wiley.
Reeves. *Aerial Photographs, Characteristics and Military Applications,* Ronald.
Hotine. *Surveying from Air Photographs,* Constable.
Gruber. *Photogrammetry,* Chapman.

Meteorology

McAdie. *Principles of Aerography,* Rand.
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Ballistics

T. J. Hayes. *Elements of Ordnance,* John Wiley and Sons. 1938. Chapters X, X-I, XII, Exterior Ballistics, Probability of Hitting, Bombing from Airplanes, written by Kent, Dederick, Zornig, of Aberdeen Proving Ground.
F. R. Moulton. *New Methods in Exterior Ballistics,* University of Chicago Press, 1926.
R. H. Fowler and others. *The Aerodynamics of a Spinning Shell,* Philosophical Transactions of the Royal Society of London, series A, vol. 221 (1921), pp. 295–387, and vol. 222 (1922), pp. 227–247.
A. E. H. Love and F. B. Pidduck. *Lagrange's Ballistic Problem,* Phil. Trans., vol. 222 (1922), pp. 167–226.

Computation

A. A. Bennett, W. E. Milne and H. Bateman. *Numerical Integration of Differential Equations,* Bulletin 92, National Research Council, Washington, 1933.

Probability and Statistics

H. L. Rietz, *Mathematical Statistics.* Chicago, Open Court, 1927.
G. Udny Yale and M. G. Kendall. *An Introduction to the Theory of Statistics,* London, Griffin, 1937.
R. A. Fisher. *Statistical Methods for Research Workers,* London, Oliver and Boyd, various editions.
J. Neyman. *Lectures and Conferences on Mathematical Statistics,* Washington, Graduate School of the U. S. Department of Agriculture, about 1938.
S. S. Wilks. *Statistical Inference,* Ann Arbor, Edwards Bros., 1937.
T. C. Fry. *Probability and Its Engineering Uses,* New York, Van Nostrand, 1928.
W. A. Shewhart. *Economic Control of Quality of Manufactured Products,* New York, Van Nostrand, 1931.

Cryptography

L. Sacco. *Manuale di Crittographia,* Rome, 1936.
L. S. Hill. *Concerning Certain Linear Transformation Apparatus of Cryptography,* American Mathematical Monthly, vol. 38 (1931), pp. 135–154.

References in other fields will be suggested later.

THE JUNE MEETING IN SEATTLE

The three hundred seventieth meeting of the American Mathematical Society was held at the University of Washington in Seattle, Washington, on Thursday, June 20, 1940, in conjunction with the meetings of the American Association for the Advancement of Science. The attendance was about fifty-two including the following thirty-nine members of the Society:

H. M. Bacon, C. R. Ballantine, J. P. Ballantine, E. F. Beckenbach, K. E. Benson, Z. W. Birnbaum, H. F. Blichfeldt, A. F. Carpenter, Sr. Maria Corona, C. M. Cramlet, C. H. Dowker, T. C. Doyle, K. S. Ghent, H. E. Goheen, F. L. Griffin, Israel Halperin, I. E. Highberg, Ralph Hull, C. F. Luther, L. H. McFarlan, W. E. Milne, C. W. Moran, R. E. Moritz, E. J. Moulton, A. F. Moursund, Hermance Mullemeister, L. I. Neikirk, John von Neumann. E. W. Pehrson, E. D. Pepper, T. S. Peterson, T. M. Putnam, J. H. Raymond, I. S. Sokolnikoff, P. M. Swingle, A. H. Taub, G. A. Whetstone, R. M. Winger, H. S. Zuckerman.

During the morning session Professor John von Neumann gave an address entitled *On operator rings and dimension.*

The titles and cross references to abstracts of papers read at the meeting are given below. Papers whose abstract numbers are followed by the letter *t* were read by title.

1. J. P. Ballantine: *Proof of Poincaré's geometric ring theorem.* (Abstract 46-7-375.)

2. E. F. Beckenbach: *An integral analogue of Laplace's equation.* (Abstract 46-7-376.)

3. P. M. Swingle: *Non-transfinite connected sets.* (Abstract 46-7-389.)

4. T. S. Peterson: *The Fredholm minors for Goursat's kernel.* (Abstract 46-7-386.)

5. Ralph Hull: *The structure of certain Fuchsian groups.* (Abstract 46-7-383.)

6. H. E. Goheen: *On the primitive groups of classes* $4p$ *and* $5p$. (Abstract 46-7-380.)

7. A. F. Moursund: *A note on Gibbs' phenomenon.* (Abstract 46-7-385.)

8. G. A. Whetstone and C. M. Cramlet: *On the invariance of the passivity conditions of systems of partial differential equations.* (Abstract 46-9-400.)

9. C. W. Vickery: *On cyclically invariant graduation.* II. (Abstract 46-7-390-*t*.)

10. H. S. Wall: *A continued fraction related to partition formulas.* (Abstract 46-7-392-*t*.)

11. C. W. Vickery: *On systems of Fourier coefficients.* (Abstract 46-7-391-*t*.)

12. R. S. Burington and J. M. Dobbie: *A new family of wing profiles.* (Abstract 46-7-377-*t*.)

13. P. M. Swingle: *Local connectedness and biconnected sets.* (Abstract 46-7-388-*t*.)

14. E. L. Dodd: *The probabilities for certain inequalities among the values of a moving average of chance variables.* (Abstract 46-7-379-*t*.)

15. Y. K. Wong: *On biorthogonal matrices.* (Abstract 46-7-393-*t*.)

16. L. B. Hedge: *Moment problem for a bounded region.* (Abstract 46-7-382-*t*.)

17. R. M. Robinson: *On the approximation of irrational numbers by fractions with odd or even terms.* (Abstract 46-7-387-*t*.)

T. M. PUTNAM,
Associate Secretary

BOOK REVIEWS

Einführung in die algebraische Geometrie. By B. L. van der Waerden. (Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen, vol. 51.) Berlin, Springer, 1939. 247 pp.

During the past 15 years the branch of mathematics known as algebraic geometry has experienced a considerable revival of interest. This has been due largely to the introduction into this field of methods and results of the modern developments in topology and algebra. Professor van der Waerden has been influential in both aspects of this revival, particularly on the algebraic side. In a series of articles, most of which appeared under the general title "Zur algebraischen Geometrie," he has developed a basic technique for the investigation of the algebraic properties of loci defined by polynomial equations. *Einführung in die algebraische Geometrie* is mainly a systematic presentation of some of the principles established in these articles, together with applications to classical problems of algebraic geometry.

The first three chapters of the book provide an introduction to the general theory which follows. There is first defined the important notion of a space (projective or affine) over a field K. The space is defined in the usual way by means of coordinates, these being elements of K. For the benefit of later developments, K is assumed to be algebraically closed and of characteristic zero, but it is otherwise arbitrary. The remainder of the first chapter is taken up with the derivation of the familiar properties of projective spaces; among the topics discussed are duality, projective transformations and their classification, Plücker coordinates for linear subspaces, correlations, null-systems and linear complexes, and some properties of hyperquadrics and of cubic curves in 3-space.

In the second chapter are introduced the basic algebraic concepts which are used throughout the rest of the book. An algebraic function of the variables $u_1, \cdots, u_n$ is defined as an element of an algebraic extension of the field $K(u_1, \cdots, u_n)$. The properties of these functions are then developed, first in general and then in the case when K is the complex field. The familiar fractional power series expansion of a function of one variable in the complex case is shown to have an exact analogue in a formal power series expansion in the general case. These formal power series play an important part in the discussion of plane algebraic curves, which is the topic of Chapter 3. An algebraic curve in the projective plane (over K) is defined in the customary manner, and the multiplicity of a point of intersection of two curves is determined by factoring a certain resultant connected with the curves; Bezout's theorem follows at once. The formal power series are used to introduce the notion of a branch of a curve; once this has been done the classification of singularities, their reduction by quadratic transformations and the derivation of Plücker's formulas follow quickly. A discussion of cubics and the point groups on them is also included.

Chapters 4 to 8 contain the basic theory of algebraic varieties. (We have translated "Mannigfaltigkeit" as "variety" rather than "manifold" for the latter word has a different meaning in topology.) An algebraic variety is the set of all points of a projective n-space whose coordinates satisfy a set (finite or infinite) of homogeneous polynomial equations. The most important varieties are the irreducible ones; those which cannot be expressed as the logical sum of two or more varieties. With each irreducible variety is associated a nonnegative integer d called its ***dimension***. At this point the author pauses to give a precise meaning to that much abused term "general point." A set of $n+1$ elements $(\xi_0, \cdots, \xi_n)$ of some extension of K is called a general point of a variety in n-space if a necessary and sufficient condition that $f(\xi_0, \cdots, \xi_n)=0$, f being a homogeneous polynomial with coefficients in K, is that $f(a_0, \cdots, a_n)=0$ for every point $(a_0, \cdots, a_n)$ of the variety. It is shown that a variety is irreducible if and only if it has a general point. The extended use of the word "point," implied by the above definition, to include $(n+1)$-tuples of an extension of K, is continued throughout the rest of the book. One such point η is said to be a ***specialization*** of another ξ if every homogeneous polynomial equation satisfied by ξ is also satisfied by η. The notions of general point and specialization are fundamental in the discussion of varieties. Chapter 4 closes with a constructive method of finding the irreducible components of a variety and a proof that every variety over the complex field is a topological complex.

Algebraic correspondences are considered in the fifth chapter. An algebraic correspondence between two varieties M and N is defined as a variety in the product space of M and N. If this variety is irreducible, it has a dimension q. It is shown that in this case to a general point of M (or N) corresponds an irreducible variety of N (M), of dimension b (d); furthermore, if M (N) is of dimension a (c), then $q=a+b=c+d$. The usefulness of this result is demonstrated by applications to the intersections of varieties with linear subspaces of the containing space, to the proof of the existence of 27 lines on a nonsingular cubic surface, and to the construction and investigation of a coordinate system for the d-dimensional varieties of an n-space. In

Chapter 6 the properties of correspondences are combined with the concept of specialization of points to answer the important question as to how the multiplicities of the solutions of an algebro-geometric problem are to be counted. The results are then applied to the investigation of tangent spaces of a variety and the intersection of a variety with hypersurfaces of its containing space.

Chapter 7 treats the general theory of linear systems of varieties of dimension $d-1$ on a variety of dimension d. The elementary properties of these systems, their relation to rational transformations, the behavior of a simple point of the variety under such a transformation, and the Bertini theorems are discussed, much use being made of the general points of the varieties involved. There is also a short discussion of divisor classes, their addition, and complete systems. In the next chapter these concepts are applied to the special case of linear series of point groups on a curve. The usual treatment is given here, commencing with Noether's theorem and leading up to the theorem of Riemann-Roch. The generalization of Noether's theorem for hypersurfaces is also given, together with some applications to space quartics. The last chapter is devoted to an account of Enriques' analysis of neighboring points on a branch of a plane curve, and an examination of their behavior under quadratic transformations.

The most striking difference between the classical introduction to algebraic geometry and the one outlined above is the latter's essential avoidance of all notions of continuity. It is this adherence to purely algebraic processes that enables one to replace the complex numbers of the classical theory by the elements of the arbitrary field K. In carrying over the classical proofs to the more general case it is necessary to have substitutes for the limiting processes which often occurred in these proofs; these substitutes are found in the use of formal power series and the specialization of general points. By applying these processes the known results can be formulated and proved in a way that shows their essentially algebraic nature and also avoids the loose reasoning that has characterized certain branches of algebraic geometry in the past.

In presenting this material van der Waerden has attempted to eliminate all superfluous algebra, in particular the use of ideal theory. The only non-elementary knowledge required of the reader is an acquaintance with the properties of algebraic and transcendental adjunctions of fields, the general elimination theory, and those properties of polynomials centering around Hilbert's basis theorem. By thus avoiding the more intricate algebraic ideas the author is able to keep the geometric background of his arguments in sight at all times, thus avoiding a tendency to over-abstraction which is apt to show itself in this type of work. The way the material is arranged also tends to emphasize the geometry—as each new principle is developed its use is illustrated by one or more applications to geometric problems, and in most cases some additional exercises are provided for the reader.

In his preface the author states that the book was written to provide the reader with the background necessary for the study of the deeper parts of algebraic geometry, especially the theory of surfaces. This requirement is perhaps a little vague, as the word "deep" may mean different things to different people, but by almost any criterion it may be said to have been well carried out. As noted above, the book contains a minimum of algebraic complications, and it certainly gives a good account of the basic notions of the subject. There are a few additional topics which the reviewer would have liked to see discussed, particularly the algebraic function field associated with an irreducible variety and its invariance under birational transformations. This is closely related to the notion of a general point and could easily have been introduced in Chapter 4. However, this is merely a detail; on the whole, we believe that the author has made a very good selection of the material at hand.

Technically the book lives up to the high standard we have been led to expect of its author and publisher. A few misprints and incorrect statements were noticed, but these are all of a trivial nature and can easily be detected and corrected by any conscientious reader. Probably the most serious defect is the lack of an index; this is particularly to be deplored in a book which is to serve as an introduction to a relatively unfamiliar branch of mathematics.

We recommend this book to the attention of every mathematician who is interested in either algebra or geometry, and particularly to those who believe that algebraic geometry is still a backward and unrigorous branch of mathematics. They will find here a clear, systematic exposition of an important new mathematical development, one which will undoubtedly have great influence in enlarging the interest in this fascinating field.

R. J. WALKER

The Decline of Mechanism in Modern Physics. By A. d'Abro. New York, Van Nostrand, 1939. 10+982 pp.

The author of this book is already known as a successful popular writer on science (cf. this Bulletin, vol. 34 (1928), p. 789, for review by T. C. Benton of *The Evolution of Scientific Thought from Newton to Einstein*, Boni and Liveright, 1927).

In brief, the book is good. The aim is "to give a semipopular presentation of the progressive growth of the new ideas, starting from the most elementary notions and giving due consideration to the mathematical development without which the new theories could never have been constructed." This aim is admirably fulfilled in three steps: Part I, "General Considerations" (114 pp.); Part II, "Physical Theories of the Classical Period" (312 pp.); Part III, "The Quantum Theory" (535 pp.). The three steps are progressively more technical.

The criticisms offered below are criticisms of detail. On the execution of his scheme, planned on such generous lines, the author is to be congratulated. It is to be hoped that potential readers will not be deterred by the formidable bulk of the book. Condensation is not desirable in a semipopular book, and there is advantage, rather than disadvantage, in a certain amount of repetition for emphasis.

The general scope of the book is indicated by the following samples of chapter headings (there are 41 chapters altogether): Assumptions in Science, The Significance of Theoretical Physics, Mechanistic Theories, Field Theories, Phenomenological Theories, Analytical Mechanics, Minimal Principles and Principles of Action, Thermodynamics, Planck's Original Quantum Theory, Bohr's Atom, De Broglie's Wave Mechanics, Heisenberg's Uncertainty Principle, Dirac's Theory of the Electron, The New Statistics.

The emphasis throughout is on theoretical physics rather than experimental. A novel feature is a popular presentation of pure mathematics, including functions of real variables, complex variables, elliptic functions, automorphic functions, groups of substitutions, differential equations, with a chapter on "The Controversies on the Nature of Mathematics."

A book of such length and of this character would be intolerable if dull or badly written. It is neither. It has a few dull patches but they are balanced by portions in which the interest is skilfully worked up. The style flows easily—perhaps a little too dignified—but science seldom smiles (in print).

Now for some criticisms. Before warming to his subject proper, the author had to (i) proclaim a new scientific era, (ii) kick the ancient Greeks. It is right and proper that the reader should know that this is an age of scientific revolution and that the Greeks were not the last word. But these things are so obvious that the information should be conveyed adroitly and, in the case of the Greeks, chivalrously, seeing that they are not in a position to answer back. The title of the book is not adroitly chosen. How many potential readers of a semipopular book know in what sense "mechanism" is to be understood?

Is it the insides of a watch? Or what? The author says what it is with a noble attempt to be clear and consistent, but flounders in the face of the impossible. One cannot tie words down like this. Half way through the book he seems to realize that all is not well: " . . . the meaning of the word mechanism needs revision, for today, since mechanics has been revised, there is no reason to associate mechanism with a form of mechanics which is recognized as a mere approximation." To save himself from intolerable confusion, the reviewer intends to continue to regard the insides of his watch as mechanism, and to think of the book under review as "Modern Physical Thought and the Quantum Theory."

As for the Greeks, we bare our teeth and rush to their defense on reading the following: "These qualities of courage, sincerity and modesty, which the Greeks seem to have lacked, prevented them from creating a science." We pass the lack of modesty (they were not alone), and even the courage and sincerity (though we have heard to the contrary), but on the point that they did not create a science, a stand must be made. Not only did the Greeks create *a* science, they created Science, in the sense in which it is understood in this book—namely deductive science. Any primitive people can collect data and record it: but that is not science. The dangerous and fascinating idea that nature is amenable to reason—that, for example, from a small number of axioms a geometry can be created which will not crack under application to astronomical observations or the surveying of fields—that courageous (if immodest) idea we owe to the Greeks. If not, to whom? Our forefathers in mediaeval Europe did not get it without the Greek contact.

Poor Aristotle comes in for special punishment, to quote: "We are told that a body dropped from the mast of a moving ship falls behind the mast. Obviously the experiment was not performed." Where does Mr. d'Abro think it would fall? It seems to the reviewer that a climb to the masthead would be an unnecessary exertion in order to verify such an obvious truth. Indeed Aristotle seems to have stuck to facts more than Galileo. He appreciated that terrestrial motions are conditioned by fluid resistance so that a terminal velocity (motus naturalis) cannot be exceeded, and the fact that Galileo succeeded where Aristotle failed might be ascribed in part to the fact that Galileo had the nerve to neglect experimental evidence and talk about motion in vacuo as if it existed physically. (See Nature, vol. 92 (1914), pp. 584, 606.)

In conclusion let it be repeated that the above are criticisms of detail. When he gets down to business the author proves himself a master of lucid exposition, and the book is to be warmly recommended to physicists and mathematicians and to those members of the public who are interested in getting a general view of modern theoretical physics. The book is well turned out, shows only a few trivial misprints (the omission of "h" from "psychological" in the table of contents is the most shocking), and has a much fuller index than those found in other books of this sort (if there are other books of this sort).

J. L. SYNGE

Colloque Consacré à la Théorie des Probabilités. Edited by M. Fréchet and E. Borel. (Actualités Scientifiques et Industrielles, nos. 734–740.) Paris, Hermann.

The proceedings of this colloquium are published in eight small volumes. They comprise an excellent collection of articles which would be an extremely valuable addition to the library of anyone interested in the theory of probability. Although very little of the material is of a purely expository nature, these volumes furnish a rather complete picture of the modern developments of this theory. The following is an outline of the contents of the various conferences.

Volume I. *Conférences d'Introduction et d'Initiation.* 1938

This volume contains two introductory addresses, 1. *Introduction*, by R. Wavre, 51 pages, and 2. *Allocution*, by M. Fréchet. These addresses are followed by:

3. *Les principaux courants dans l'évolution récente des recherches sur le calcul des probabilités*, by M. Fréchet. The author outlines the contributions, trends, and methods of the modern theory of probability. This paper constitutes only the first part of Fréchet's discussion. The remainder appears in Volume II.

4. *Promenade au hasard dans un réseau de rues*, by G. Pólya. The author considers certain probability problems leading to linear partial difference equations of the second order. Limiting cases of these problems admit of physical interpretations. In the limit the difference equations become differential equations. Moreover the solutions of these differential equations with suitable boundary conditions give asymptotic values for the solutions of the difference equations. The author discusses a promenade along a street of infinite length in which the direction of promenade is settled by the tossing of a coin at the end of each block. The corresponding physical problem is the diffusion of a salt solution in a tube. Furthermore the motion of rocks

in a river is related to the above promenade. An n-dimension promenade is also discussed. The reasoning is heuristic and the paper is suggestive of further research.

5. *Wahrscheinlichkeitsaussagen in der Quantentheorie der Wellenfelder*, by W. Heisenberg. The author states that a fundamental problem in modern physics is to ascertain at what points, in the description of natural phenomena, statistical considerations play a deciding role. As an example he considers the emission of electrons from a point source, the electrons being permitted to pass through two slits and strike a nearby plate. This may be regarded as a statistical phenomenon and hence we should expect that the probability of reaching a given point of the plate through one slit plus the probability of reaching this point through the other slit would be equal to the probability of reaching the point through one slit or the other. But this contradicts the fact that the problem can be regarded as a wave phenomenon and hence that there should be interference bands. The author mentions a number of other difficulties which have arisen in modern physics and suggests that the resolution of these difficulties may be accomplished by the introduction of new statistical considerations.

Volume II. *Les Fondements du Calcul des Probabilités*. 1938. 99pp.

This volume constitutes a fascinating debate on the foundations of the theory of probability. A variety of points of view (not only of the participants of the congress but also of those who were unable to attend) are presented and criticized. Frequently answers to criticisms appear. This volume furnishes excellent material for the study of the philosophical aspects of the theory of probability.

1. *Sur la définition des variables éventuelles*, by P. Cantelli. Since Cantelli was unable to attend the congress only an abstract of his paper appears in this series.

2. *Sur les axiomatiques du calcul des probabilités et leurs relations avec les expériences*, by W. Feller. The first part of this paper contains a general discussion of the foundations of the theory of probability. The second part is devoted to an exposition and critique of E. Tornier's theory of probability. The exposition is much clearer than that found in Tornier's book (*Wahrscheinlichkeitsrechnung und algemeine Integrationstheorie*). Feller criticizes Tornier's objections to the complete additivity in Kolmogoroff's system. He notes Tornier's failure to recognize that his system is a special case of that of Kolmogoroff and that his use of the continuity axiom is precisely that of Kolmogoroff. To these criticisms I should like to add the following.

In order to avoid the difficulties of the Regellosigkeitsprinzip of von Mises, Tornier constructs a theory which is oversimplified to such an extent that it presents a much less accurate picture of the physical universe. This sacrifice is unnecessary. See for example the paper by Wald or the modified systems described by Fréchet in this volume.

3. *Exposé et discussion de quelques recherches récentes sur les fondements du calcul des probabilités*, by M. Fréchet. The author starts with the classical concept of probability of Laplace and Poincaré. He recalls the familiar objections to this point of view and indicates how these objections might be overcome. Next he presents the theory of von Mises and the objections which have been raised against that theory. He also adds objections of his own. Then he describes the modifications of the von Mises theory by Popper, Reichenbach, Copeland, Wald, and Ville. He argues that although these modifications avoid the serious logical difficulties of the von Mises theory, they are open to a new objection, that is, that they are less natural and do not possess the strong intuitional appeal of the original theory. The author then turns to a point of view which he finds more acceptable and which has been described by Neyman as the modernized classical theory. This point of view is based on the postulates of Kolomogoroff and the modern theory of measure. Whether or not one agrees with the author he cannot fail to recognize that this article contains an excellent exposition of the various points of view and a clear presentation of the difficulties which attend them.

4. *Quelques remarques sur les fondements du calcul des probabilités*, by R. de Misès. The author gives a brief exposition of his own frequency theory of probability and points out the analogy between it and the theory of mechanics. He accepts the refinements contributed by Wald, Copeland, and other authors but states that these refinements do not alter essentially the fundamental character of his theory. As to Fréchet's contention that these refinements tend to rob his theory of its intuitive appeal; he points out that this is the case with any mathematical concept but that such refinements are necessary for a firmer foundation. He remarks that the modernized classical theory of probability is insufficient for the solution of applied problems and that without the frequency theory the modern methods such as chains of probabilities, convergence in probability, and arbitrary functions become merely problems in algebra and point set theory.

5. *Fréquence et probabilité*, by J. F. Steffensen. Steffensen upholds the frequency theory of probability as being more natural and more easily applicable in problems of statistics. He bases his discussion of

this theory on two hypotheses: first, that every possible sequence of successes and failures should be taken into consideration and second that if two fortuitous sequences are continued sufficiently far they will eventually differ. From these hypotheses he concludes that no sequence determined a priori by mathematical law can occur. He states that the probability approaches 1 that the relative frequency will differ from the probability by an arbitrarily small amount and gives a new proof of this fact. He then shows that the additive and multiplicative laws of probability can be similarly treated.

6. *Die Widerspruchsfreiheit des Kollektivbegriffes*, by A. Wald. A Kollektiv is an infinite sequence of elements taken from a given label space. Probabilities are associated with subsets of this label space and these probabilities are defined as limits of success ratios with respect to the Kollektiv. These limits must be invariant under the operation of selection. Thus far Wald agrees with von Mises. However instead of the absolute Kollektiv of von Mises, Wald's Kollektiv is relative to a given family of subsets of the label space and to a given system of selections. That is, probabilities are required to exist only for subsets belonging to the family, and invariance is required only with respect to selections of the given system. He shows that if the family of sets constitutes a denumerable field including the label space and if the system of selections is also denumerable, then there corresponds a continuum of Kollektivs. Thus the system can contain all selections which can be defined within a given symbolic logic. He shows further that if the denumerable field and the denumerable system are constructively defined, then there exist Kollektivs which can be constructively defined. The remainder of the paper is devoted to showing the relation of his system to those of Reichenbach, Popper, and Copeland and to answering objections raised against the frequency theory.

Volume III. *Les Sommes et les Fonctions de Variables Aléatoires*. 1938. 68 pp.

1. *Sur un nouveau théorème-limit de la théorie des probabilités*, by H. Cramér. The author considers the probability that $Z_1+Z_2+\cdots+Z_n\leqq x\cdot\sigma\cdot(n)^{1/2}$ where $Z_1, Z_2, \cdots$ are independent fortuitous variables having a common distribution function and such that the expected value of each is 0 and the expected value of the square of each is σ^2. It is well known that as n becomes infinite this probability tends to the probability integral if x is constant with respect to n. However the author is interested in this limit when x is allowed to become infinite with n. By means of a certain transformation using

characteristic (or moment generating) functions this problem can be studied with the aid of Liapounoff's theorem. He considers the case in which x is of order $n^{1/2}/\log n$ and the case in which x is of order $n^{1/2}$. He obtains theorems of Khintchine, Lévy, and Smirnoff as special cases of his results. In the latter part of the paper he applies his method to the study of homogeneous stochastic processes.

2. *L'arithmétique des lois de probabilités et les produits finis de lois de Poisson*, by P. Lévy. This paper deals with the field consisting of all probability laws. A probability law can be defined either by means of a distribution function or by means of the corresponding characteristic function. If two independent fortuitous variables have respectively the probability laws L and L', then the sum of the two variables has the law denoted by the product LL'. The characteristic function corresponding to the law LL' is the product of the characteristic functions of the laws L and L'. The set of all probability laws constitutes a field in which a unity is a law corresponding to a fortuitous variable which can take on only a single value. This memoir is a study of the decompositions of the elements of the field into factors.

3. *Généralisation des théorèmes de limite classiques*, by R. de Misès. The author studies the asymptotic distributions of certain functions of n fortuitous variables when n is allowed to become infinite. As an aid in this study he considers functionals of the corresponding distributions. For example the moments of a given distribution are linear functionals of the corresponding distribution function. These functionals are defined over convex sets of distribution functions and derivatives are defined at points of these sets in such a manner as to permit Taylor's developments of the functionals.

Volume IV. *Le Principe Ergodique et les Probabilités en Chaîne.* 1938. 66 pp.

1. *Statistische Probleme und Ergebnisse in der klassischen Mechanik*, by E. Hopf. The equations of motion of a mechanical system in general define a one-parameter group of measure preserving transformations T_t of the phase space Ω into itself (the parameter t being the time). A transformation is said to be metrically transitive provided any subset of Ω which is transformed into itself has either the measure 0 or else the same measure as Ω. The condition of metric transitivity is necessary and sufficient that the limit average time that a point spends in a subset A of Ω is equal to the measure of A divided by the measure of Ω (Birkhoff's ergodic theorem). If the system is metrically transitive but fails to possess a characteristic frequency,

then every subset of Ω becomes uniformly distributed over Ω as the time is indefinitely increased (mixture theorem). The rest of the paper is devoted to the discussion of the weaker condition, regional transitivity.

2. *Les fluctuations (changement aléatoires du nombre de points ou d'objets dans un compartiment)*, by B. Hostinský. The author considers three problems. The first concerns the distribution of N objects into n compartments. He computes the probability that x objects will be placed in a given compartment assuming that $1/n$ is the probability that a given object will be placed in that compartment. The limiting case in which N and n become infinite gives the Poisson law. The second problem concerns two urns with r balls in each. There are r white and r black balls, but otherwise the compositions of the urns are arbitrary. From each urn a ball is drawn at random and then placed in the urn from which the other was drawn. The author considers the limiting probability of a given composition as the number of such exchanges becomes infinite. This problem is analogous to the mixture of a colored liquid and a non-colored liquid in two containers which are connected. The third problem concerns two urns containing a total of three balls of the same color. Probabilities are given for the transference of a ball from one urn to the other and for failure to make such transference. These probabilities depend on the compositions of the urns. The limiting probabilities for various compositions are again computed. This problem is analogous to the exchange of compressible fluids or gases between two connected containers.

3. *Théorie générale des chaînes à liaisons complètes*, by O. Onicescu. The author considers a matrix composed of mutually exclusive fortuitous events (or states) $a_1, a_2, \cdots, a_m$ and the probabilities $x_1, x_2, \cdots, x_m$ corresponding to these events. Such a matrix can be transformed into a new matrix with new probabilities $y_i = f_{ik}(x_1, x_2, \cdots, x_m)$ for the given events, k being determined by the event a_k which does occur. Such a system of transformations constitutes a chain. If the matrix f_{ik} is independent of the probabilities, the transformation is completely fortuitous and the system constitutes a chain with complete liaisons. A chain is normal if the matrix f_{ik} possesses the ergodic property enjoyed by the Markoff chains. Such a matrix transforms the region $0 \leq x_i \leq 1$ into a closed region totally interior to this, and the limiting region as the number of transformations becomes infinite is a single point. The author considers also the continuous case in which the number of states is infinite. He cites as an example the equation of the flow of heat.

4. *Quelques problèmes nouveaux de la théorie des chaînes de Markoff*, by V. Romanovsky. The author considers a simple Markoff chain and a second chain such that there is a definite conditional probability of a given state on a given trial for every possible state of the corresponding trial of the first chain. The second chain is in general multiple and nonhomogeneous. A method is given for computing moments and the limiting correlation coefficient with respect to the chains. The author next studies cyclic chains. These chains may contain several cycles which occur fortuitously. A method of matrices is used to simplify the study of these chains. The latter part of the paper is devoted to methods of estimating by statistical means the law underlying a Markoff chain.

Volume V. *Les Fonctions Aléatoires*. 1938. 73 pp.

1. *Équations différentielles stochastiques*, by S. Bernstein. This paper is concerned with a difference equation of the form $\Delta y = A\Delta t + f(\Delta t)^{1/2}$ where A and f depend on y, t, Δt, and a fortuitous variable α_t and where the expected value of f is 0. Sufficient conditions are given for the existence of a limiting distribution function for y as the differences Δt approach 0. Bernstein shows that these conditions cannot be materially weakened. If $f \equiv 0$ and A is not a function of α_t, the problem reduces to the solution of the differential equation $dy/dt = A(y, t)$. Moreover if $(\Delta t)^{1/2}$ is replaced by Δt in the second term of the right-hand member of the difference equation and A is independent of α_t then y satisfies the equation $dy/dt = A(y, t)$ and the solution does not possess a fortuitous character. The method of this paper can be extended to systems of difference equations of the above form.

2. *Sur les fonctions aléatoires presque périodiques et sur la décomposition des fonctions aléatoires stationnaires en composantes*, by E. Slutsky. A fortuitous function y_t is assumed to be stationary, normalized, stochastically continuous, and subject to certain other minor conditions. The correlation $\rho(\tau)$ between y_t and $y_{t+\tau}$ can be decomposed into a constant plus a component with a discrete spectrum plus a component with a continuous spectrum. The portion with the discrete spectrum is an almost periodic function developable in a Fourier series. The function y_t admits of a decomposition into three non-correlated components the first of which is the mean value of y_t the second of which consists of a Fourier series closely related to the corresponding series for $\rho(\tau)$, and the third of which has a spectrum continuous in the mean. If $\rho(\tau)$ is almost periodic, then the probability is 1 that the Fourier series for y_t converges to y_t for almost every t.

3. *La théorie et les applications des fonctions indépendantes au sens stochastique*, by H. Steinhaus. If $f(t)$ is defined in the interval 0, 1, then the corresponding distribution function $F(x)$ is defined as the measure of the set of points on which $f(t) < x$ and is interpreted as the probability of this latter inequality. By suitable modifications this definition can be extended to any finite interval and also to the infinite interval. Independence is defined as independence in the probability sense. The functions $f(t)$, $g(t)$ defining the space filling curve of Peano furnish an example of independent functions. Such independent functions are an aid to calculation in many probability problems. The author applies this method to the probability limit theorem, the Maxwell distribution law, and to Brownian movements.

Volume VI. *Conceptions Diverses*. 1938. 57 pp.

1. *Sur la conception d'équivalence partielle*, by B. de Finetti. Finetti bases his theory of probability on the assumption that the laws of total and composite probabilities are valid and on a concept which he calls the equivalence of events. Events are said to be equivalent provided the probability that n of them will occur depends only on n and not on the particular set of n events chosen. He derives a formula (not the classical and highly dubious application of Bayes' principle) by means of which one can estimate the probability that one of the equivalent events will occur on the basis of an observed success ratio. Except for a very special case the formula shows that this probability must be close to the success ratio if the number of trials is large and that the events must be independent. The formula can be extended to groups of events where events in given groups are equivalent. This paper also contains a philosophical discussion concerning the relations between probability, causality, and induction.

2. *Sur la loi des grands nombres dans l'espace fonctionnel*, by V. Glivenko. Suppose that a fortuitous event has a continuous probability density defined in a closed interval. A histogram represents the observed frequency distribution in n trials of the event. The author shows that the probability is 1 that the maximum difference between the histogram and the probability density approaches 0 as the number of trials becomes infinite. This result is based on certain theorems which the author proves concerning moments of fortuitous functions. In particular he proves a generalization of the Bienaymé-Tchebycheff inequality and the strong law of large numbers in function space. The article though brief is complete and little previous knowledge of the theory of probability is assumed.

3. *Les fonctions aléatoires (lois de répartition dans un espace fonc-*

tionnel) et leurs applications, by A. Kolmogoroff. This paper was not completed at the time these volumes were published. It will appear in the collection *Exposés d'Analyse Générale* published under the direction of M. Fréchet.

4. *L'estimation statistique, traitée comme un problème classique de probabilité*, by J. Neyman. Let $X_1, X_2, \cdots, X_n$ be a system of fortuitous variables depending on the unknown parameters $\theta_1, \theta_2, \cdots, \theta_s$. An experiment determines the values of the fortuitous variables, that is, a point of the euclidean space R_n. Let a set function of points be defined such that to every point of R_n is assigned a set of points of the real axis. If this function is properly determined, then when $X_1, X_2, \cdots, X_n$ is substituted for the point of R_n there will be determined a probability that a given point θ_1 of the real axis will be included in such a set, θ_1 being one of the parameters. Let the function be further restricted so that this probability is a preassigned number α (the coefficient of confidence) for all possible values of the parameters. It follows that when an experiment determines a point of R_n the probability is α that the parameter θ_1 will be found in the corresponding set. Suppose that the number of parameters is 1 and that the above function is further restricted so that the probability α is maximized (maximum likelihood). The sets (functions of points of R_n) are then intervals (confidence intervals) and are uniquely determined.

Volume VII. *La Statistique Mathématique*. 1938. 57 pp.

1. *Certain coefficients of regression or trend associated with largest likelihood*, by E. Dodd. A pair of fortuitous variables x, y is assumed to be subject to a probability distribution which is known except for the values of certain parameters. A set of observations is made determining n values of each variable. The parameters are then determined by maximizing the likelihood of obtaining the values which were actually obtained. The author considers two examples and in each case the parameters are determined as certain means closely related to regression coefficients. The author then considers a general concept of mean and a corresponding theorem concerning maximum likelihood. However the conditions of this theorem are such as to permit an arbitrary assignment of value to the unknown parameter.

2. *Critique de la corrélation au point de vue des probabilités*, by C. Jordan. The author considers the correlation between two fortuitous variables u and v. A measure called the intensity of correlation is defined in terms of the discrepancies between certain conditional probabilities and corresponding unconditional probabilities. This

measure is equal to 0 if and only if the variables are independent and it attains its maximum 1 if and only if the variables are completely dependent. Although the intensity of correlation is never negative and hence cannot distinguish between positive and negative correlation, the author gives a simple independent method of accomplishing this distinction. The intensity of correlation is closely related to two measures given respectively by Pearson and Tschuprow. Jordan also discusses the conventional correlation coefficient, two additional measures due to Pearson, and Pearson's correlation ratios. The intensity of correlation is the only one of these measures which possesses all of the above mentioned desirable properties.

3. *Sur la loi de Poisson, la loi de Charlier et les équations linéaires aux différences finies du premier ordre à coefficients constants*, by N. Obrechkoff. The first portion of this paper concerns a system of functions consisting of the Poisson function $(a^x/x!)e^{-a}$ and its differences. Any function of the system is the product of the Poisson function with the appropriate Charlier polynomial. The author gives sufficient conditions for the development of a given function in a series formed from this system. The second part of the paper deals with a pair of dependent fortuitous variables such that each has a Poisson distribution when the other is fixed. This condition determines the formulas for the distribution of the two variables and the unrestricted distribution of each of the variables. The last part of the paper concerns a certain type of linear nonhomogeneous difference equation. The author gives a formula for the asymptotic behavior of the solution.

A. H. COPELAND

Tafeln und Aufgaben zur harmonischen Analyse und Periodogrammrechnung. By Karl Stumpff. Berlin, Springer, 1939. 172 pp.

This book is a sequel to the author's volume *Grundlagen und Methoden der Periodenforschung* (Berlin, Springer, 1937). It contains extensive tables for the carrying out of harmonic analysis of empirical curves based upon interpolation by means of trigonometric polynomials of proper degree through equally spaced ordinates. Tables are given for various numbers p of equal divisions. Tables are given for the proper sines and cosines by which to multiply the p ordinates in order to obtain the proper Fourier coefficients. These tables extend as far as $p=40$. They also include the best laborsaving arrangement for the ordinates. These tables are followed by the first thousand multiples of cosines and sines of various angles which occur for $p=8$, 12, 16, and 24, as well as the first hundred multiples of the cosines and sines of all angles of integral degrees in the first quadrant. These tables enable one to avoid a great deal of the multiplication of the ordinates by the proper cosines, which is inherent in the calculation of the coefficients. Further tables include the conversion from polar to rectangular coordinates (useful in passing from the two Fourier coefficients corresponding to a particular frequency to its amplitude and phase), Darwin's scheme for a large number of ordinates ($p=121$), tables of squares, square roots, $(\sin \alpha)/\alpha$, and other tables of interest in harmonic analysis. The last quarter of the book is devoted to problems and examples in harmonic analysis, illustrating the use of the various preceding tables; also to problems in the analysis of hidden periodicities.

The present volume should prove very useful to anyone engaged in frequent harmonic analysis.

H. PORITSKY

Tables of Partitions. By Hansraj Gupta. Madras, Indian Mathematical Society, 1939. 5+81 pp.

This work contains two tables. The first table gives the number $P(n)$ of unrestricted partitions of n for $n \leqq 600$. The second table is a table of double entry giving the number (n, m) of partitions of n the least element of which is m for $n \leqq 300$. As is explained in the introduction, the second table was calculated by the recursion formula $(n, m)=(n-m, m)+(n+1, m+1)$ together with certain special properties of the symbol (n, m); a specimen calculation is given. The first table is a result of the second since $P(n)=(n+1, 1)$. Adequate checks have been applied to insure accuracy. The introduction also contains summaries of certain papers by G. N. Watson and D. H. Lehmer on partitions.

The Indian Mathematical Society and the University of the Punjab are to be commended for bearing the cost of publication of this work. The table of unrestricted partitions has previously been published in two parts in the Proceedings of the London Mathematical Society. This table has been used by Chowla to disprove one case of a result conjectured by Ramanujan. This work should be very valuable to anyone interested in the subject of partitions.

T. A. PIERCE

Elementary Mathematics from an Advanced Standpoint. Geometry. By Felix Klein. Translated from the third German edition by E. R. Hedrick and C. A. Noble. New York, Macmillan, 1939. 9+214 pp., 141 figs.

Another volume of Klein's masterful lectures is available in English! This book—it should be found on the shelves of every mathematical library—will be read with pleasure and profit alike by the scholar, the student, and the teacher. It preserves, even in translation, all those qualities for which the lectures of Klein are so justly celebrated.

The first volume of Klein's *Elementarmathematik vom höheren Standpunkte aus* was translated also by Professors Hedrick and Noble (reviewed in this Bulletin, vol. 39 (1933), pp. 495–496), and this, the second, is similar in character and purpose: "In the foreground I shall place, let me say, the *encyclopedic ideal*—you will be offered a *survey of the entire field of geometry* into which you can arrange, as into a rigid frame, all the separate items of knowledge which you have acquired in the course of your study, in order to have them at hand when occasion to use them arises. Only afterward shall I emphasize that *interest in mathematical instruction* . . . " (see page 1). The emphasis is on *ideas*, as opposed to the details involved in their development, and on the *interrelations between different fields.*

Klein considers geometry from the point of view developed systematically in his *Erlanger Programm* of 1872. His book has a wide appeal in spite of the fact that the last statement might seem to indicate a theoretical treatment of interest only to the mathematical specialist. The explanation is found in the large number of applications to a wide variety of problems.

The book is divided into three parts: Part I, "The Simplest Geometric Manifolds"; Part II, "Geometric Transformations"; Part III, "Systematic Discussion of Geometry and Its Foundations." In Part I we find a discussion of all the classical geometric manifolds, that is, classes of objects which are investigated in geometry. Many of these manifolds (various types of vectors, tensors, and so on) are of interest to physicists. A detail is an explanation of the theory of Amsler's polar planimeter. We find in Part II a discussion of affine transformations, projective transformations, higher point transformations, and transformations with a change of space element. It is shown that the axonometric mapping of space on a plane used in descriptive geometry is an affine transformation with a vanishing determinant. Relief perspective and geographical maps are considered. There is an application of contact transformations to the theory of cog wheels. In the first section of Part III we find a classification of geometries according to their groups and a discussion of invariant theory. The second section of Part III contains the material of immediate interest to prospective teachers of high school geometry. Two methods of developing plane geometry, different from that followed by Euclid, are outlined. One emphasizes motions; the other takes distance, angle, and congruence as the fundamental notions. Finally, there is a critical discussion of Euclid's *Elements*.

A reading of this book makes one aware of recent progress in mathematics. To Klein, writing in 1908, geometry was coordinate geometry. He makes only bare references to point sets as objects of geometric study and to Hilbert space. He gives only a brief discussion of homeomorphisms and even then only in euclidean 3-space. Geometric manifolds with the complete generality of our modern abstract spaces were then unknown. Metric spaces (in the sense of Hausdorff) had been introduced only in 1906, and topological spaces were not yet clearly defined. The types of geometry that now occupy the center of the stage were only beginning to emerge. Nevertheless, they too fit into the framework erected by Klein.

G. B. PRICE

Advanced Calculus. By Ivan S. Sokolnikoff. New York, McGraw-Hill, 1939. 10+446 pp.

In most American colleges there is an abrupt discontinuity in mathematical education at the end of a first course in calculus. The drop in enrollment due to the completion of required mathematics by hordes of engineering students makes a sudden change in the needs and capabilities of the students. This discontinuity is perhaps more apparent to outsiders than to mathematicians themselves. It is the reason for the widely held popular belief that all useful mathematics ends with the calculus, that Newton made the last discoveries in mathematics, and that mathematics is dead except for a few frills indulged in by a handful of eccentrics. Even the name "Advanced Calculus" encourages this misconception—as though it were just calculus over again, with a few added tricks to be sure, but no new ideas.

It is in a second course in calculus that the number of students with primary interest in mathematics is for the first time an appreciable percentage of the total. The problem of the relative emphasis on method, that is, proofs, versus utility, to meet the respective needs of mathematics majors and students in the applied field, is as perplexing as any problem in the whole of mathematical education. Some sort of compromise is usually necessary. It would be very useful to the teacher if textbooks of good craftmanship were available with a variety of methods of effecting such a compromise. Unfortunately, a number of the existing books in this field are (or at least have the bulk of) treatises, and are not entirely suitable for textbook use.

The reviewer, therefore, welcomes the contribution of a book, entirely suitable for classroom use, from a man who has personal inclinations toward applied mathematics and who has already made a definite contribution in this field with a textbook leaning toward applications, but who now presents a treatment with considerable emphasis on real variable theory.

After an introductory chapter on the concepts of limit and continuity, there are chapters on the notion of derivative, functions of several variables, definite integral, multiple and line integrals, infinite series, power series, improper integrals, and Fourier series. A final chapter in implicit functions is more in the nature of an appendix; the treatment of partial differentiation of implicit functions in an early chapter is only claimed to be heuristic.

By limiting the scope of the book to the central problems of the calculus, the author is able to include a reasonable amount of formal material and at the same time give the student an introduction to "ϵ, δ" arguments. An attempt is made to reduce these arguments to terms understandable by the average student by such devices as prefacing the actual proofs with heuritistic treatments, and by the use of geometric counterexamples. Numerous exercises are also included.

The first chapter is evidently not intended to be taken too seriously; otherwise such a variety of new abstract ideas as Dedekind cuts, ϵ, δ definition of limit, Cauchy criterion for convergence, limit point, inferior and superior limits, uniform continuity, and so on, would overwhelm the average student just out of sophomore calculus. For such students the teacher might want to start with Chapter II and gradually refer back to the first chapter. The proofs in this opening chapter are incomplete, and many of the theorems are merely stated. For example, the proof of the Cauchy criterion for convergence is carried just to the point where one would ordinarily construct a cut; then the author says the remainder of the proof is too difficult and will be omitted. Such a procedure could have been avoided by merely postulating the existence of a least upper bound of any bounded set of real numbers.

Those who believe in being absolutely meticulous of statement in a book of this level will have a number of quarrels with the author. His definition of Riemann integral is inaccurate. Also, although he gives an ϵ, δ definition of limit of a function, he encourages the student to think in terms of the independent variable x approaching x_0 "by any conceivable sequence of steps," without ever making this idea precise. An extension of this vague idea to limits of functions of two independent variables has led the author to the erroneous conclusion that the existence of the simultaneous limit always insures the existence of the iterated limits. This conclusion is, in turn, involved in the proof of the theorem on the interchangeability of order of partial differentiation. (Fortunately, this does not invalidate the proof, but it makes desirable further elucidation of some of the steps.) Finally, it is stated, without proof, that the mere existence of f_{xy} and f_{yx} is sufficient to insure their equality; but counterexamples to this statement are known.

If it is desirable to use the intuitive idea of different modes of approach of the independent variable to its limit, the notion could easily be put on firm ground by giving the Heine sequence definition of a limit and stating its equivalence to the Cauchy definition. That this equivalence is not as simple as appears on the surface is well known, for the proof is impossible without the axiom of choice.

The chapters on infinite series, power series, and Fourier series appealed to the reviewer as very well done. The chapter on Fourier series contains among other topics a simple account of the Dirichlet conditions for convergence, Parseval's equation, the complex form of Fourier series, the asymptotic behavior of the coefficients, and the differentiation and integration of Fourier series.

P. W. KETCHUM

Geometria Descrittiva: *Lezioni Redatte per Uso Degli Studenti.* By Enea Bortolotti. Padua, Cedam, 1939. 715 pp., 500 figs. (Mimeographed.)

The term descriptive geometry has a wider meaning in Europe than it has with us; this is especially true in Italy where the study of geometry is an important part in the curriculum of all students of mathematics. Moreover, the school programs include work in determinants and matrices, projective and analytic geometry, and the elements of the calculus. Students of mathematics in the universities analogous to our undergraduates, devote all their time to the subject, hence are taking three courses simultaneously. By the time they reach descriptive geometry they will have had sufficient training to allow a teacher or an author to assume an acquaintance with many fundamental concepts.

Textbooks are ordinarily not employed at all; books written on a specific subject are for supplementary reading, usually voluntary, and not controlled. The students are thrown on their own at an early stage, and many of them show a decided precosity. The books are written for the better students who really want to learn about the subject treated.

The book under review is divided into four parts; most of the second can be read without the first, but the later parts make use of both. The first is devoted to descriptive geometry in the narrower sense and covers 243 pages. It starts with h, v orthogonal projections of figures bounded by straight lines and planes. Intuition is used freely, but each theorem is proved rigorously, including those on shades and shadows. Very few hints are given, but occasionally the reader is advised to go through the details of that particular construction. The second chapter of this part treats of perspectivity, both central and parallel, orthogonal and oblique. This is a helpful addition. So many books omit it in the elementary presentations, while more advanced treatments (sometimes in the same book) presuppose it. Now follows anaxometry. The discussion is so condensed that the reviewer frequently wondered whether a reader not having additional help could always follow the reasoning. The theorem of Pohlke is proved and frequently used. A commendable feature is the comparison of the respective advantages of the three methods developed in these three chapters. The closing chapter of this part gives a brief summary of a method used in topography, especially in mapping curves of equal level.

Thus far the subject matter has been almost entirely self-contained. All theorems used have been proved as needed. The second part, curves and surfaces, covers nearly half the volume; it constantly refers to proofs of theorems given in other Italian texts rather than spend time and space on them in the present work. It begins with plane curves, and provides an extensive treatment of plane differential geometry, including a discussion of singularities. It is carefully done, but the processes, applied to a complicated case would become so cumbersome as to be useless. This chapter is followed by a briefer one on space curves and developables. Thus far, no use has been made in this chapter of any of the earlier ones.

Now follows an extensive but condensed discussion of surfaces, from the standpoint of differential geometry; it includes normals, tangent planes, asymptotic lines, geodesics, lines of curvature, conjugate systems, and so on, with special emphasis on ruled surfaces and surfaces of revolution. Many names are used, as discoverers of theorems met, but there is no citation of sources. The mapping of one surface on another is discussed and applied in detail to various kinds of map drawing, in which one property or another is preserved, such as lengths, angles, areas, and so on. Here repeated use is made of the results of the preceding chapters.

A short chapter on graphical calculus and nomography is followed by one called "Complementi," in which details of the theory of linear transformations in one, two, and three dimensions, as well as correlations in two and three dimensions are developed, including the fixed points and a considerable number of particular cases. This is followed by a more detailed study of algebraic curves and surfaces. In the plane, polarity is introduced and used to derive Plücker's numbers, with application to cubic curves. The configuration of the points of inflexion, the constant cross ratio of the tangents from a point on the curve, nodal and cuspidal forms are provided for. Space cubic curves, and both kinds of space quartics are treated briefly.

Even with the omission of many proofs, the development is so rapid and condensed that a reader must be alert and patient to get all the points of the argument. On the other hand, one who has mastered this volume will be in possession of a large part of the knowledge which goes under the generic name of geometry.

VIRGIL SNYDER

Some Integrals, Differential Equations and Series Related to the Modified Bessel Function of the First Kind. By A. H. Heatley. (University of Toronto Studies, Mathematical Series, no. 7.) Toronto, University Press, 1939. 32 pp.

Let $I_n(x)$ denote Bessel's function, and let $T(m, n)$ denote the integral over $0<t<\infty$ of the function $I_n(2at)t^{m-n} \exp(-p^2t^2)$. Differential equations are given for $T(m, n)$, for $T(m, n) \exp(-a^2/p^2)$, and for $T(m, n) \exp(-a^2/2p^2)$; and these lead to explicit formulas for $T(2n+1, n)$ and for $T(n, n)$. Then recursion formulas lead to evaluation of $T(m, n)$ for other pairs of values of m and n.

Power series expansions of $T(m, n)$ and $T(m, n) \exp(-a^2/p^2)$ are obtained. These results of Part I (pp. 7–21) are used in Part II (pp. 21–30) to evaluate an integral used by the author (Physical Reviews, vol. 52 (1937), pp. 235–238) in the Langmuir collector theory.

Part III (pp. 30–32) deals briefly with integrals of $x^{n+m}e^{x}I_n(x)$ and $x^{n+m}e^{-x}I_n(x)$.

R. P. AGNEW

Theory of Probability. By Harold Jeffreys. Oxford, Clarendon Press, 1939. 7+380 pp.

This book of Jeffreys is an outstanding addition to the relatively few substantial treatises of probability in English. It is in line with the author's *Statistical Inference.* At the start, it resembles Keynes' *A Treatise on Probability* in its subjective or psychical approach to probability. But it carries the implications of this approach to a great variety of problems arising in the physical sciences, in biology and in economics. This subjective axiomatic approach to probability is exemplified also by a recent paper of B. O. Koopman, *The axioms and algebra of intuitive probability*, in the Annals of Mathematics, April, 1940.

Keynes, Jeffreys, and Koopman deal first with non-numerical probability. The probability of a proposition is an extension of logic by interpolation between the false or impossible proposition and the true or certain proposition. First, these subjective probabilities are assumed to be orderable. However, Keynes does not even require the order to be linear—he gives on page 39 a two-dimensional diagram showing several curves extending from the point of impossibility to the point of certainty. Points on different curves represent propositions with probabilities not comparable. But Jeffreys writes: "Axiom 1. Given $p, .q$ is either more or less probable than r, or both are equally probable; and no two of these alternatives can be true." After Theorem 2, Jeffreys sets up his "Convention 3" which associates certainty with the number 1. But he states that cases arise where ∞ is a more suitable choice for certainty. His Theorem 8 is: "Any probability can be expressed by a real number."

Jeffreys criticizes other conceptions or definitions of probability as follows: The "classical" definition of probability leads to nothing but combinatorial analysis. Probability is not introduced until human judgment assesses the elements to be equally probable. The extension of the classical definition by Neyman and Cramer to continuous probability, using the ratios of measures of sets, faces a similar difficulty. For with $f(x)$ monotonic, $f(x_2)-f(x_1)$ may be taken as the measure of the interval from x_1 to x_2 just as well as x_2-x_1. It takes a human judgment to select. In connection with probability as the ratio of measures of sets, Jeffreys might also have mentioned Lomnicki and Steinhaus, Fundamenta Mathematicae, 1923. As to probability as a limit, Jeffreys writes (p. 304): "The existence of the limit is taken as a postulate by Mises, whereas Venn hardly considered it as needing a postulate. Thus there is no saving of hypotheses in any case, and the necessary existence of the limit denies the possibility of complete randomness." In R. A. Fisher's "infinite population," there remains implicitly in the randomness the notion of "equally probable." In this respect, this infinite population resembles the Willard Gibbs "ensemble" (p. 300).

Jeffreys also criticizes certain applications of probability notions to particular problems. He considers (p. 309) that "Student" in his z-distribution found a direct probability instead of an inverse probability. He states (p. 323) "My main disagreement with Fisher concerns the hypothetical infinite population which is a superfluous postulate since it does not avoid the need to estimate the chance in some other way, and the properties of chance have still to be assumed since there is no way of proving them. Another is that, as in the fiducial argument, an inadequate notation enables him, like 'Student,' to pass over a number of really difficult steps without stating what hypotheses are involved in them."

Jeffreys writes Bayes' theorem as: Posterior probability is proportional to the product of prior probability and likelihood. This likelihood corresponds to productive probability. He discusses in considerable detail the troublesome prior or "a priori" probability. For the range 0 to $+\infty$ he takes frequently the prior probability to be proportional to dv/v, although he calls attention (p. 100) to the fact that the integral of dv/v becomes infinite at both ends of the range.

Among applications, Jeffreys is perhaps most interested in significance tests, treated in Chapters V and VI, 107 pages. But after a preliminary discussion of direct probabilities, he takes up in Chapters II to IV: multiple sampling and the multinomial law, the Poisson law, normal law of error, Pearson laws, negative binomial law, characteristic function, t and z distributions, the method of least squares, rectangular distribution, correlation, maximum likelihood, approximation and successive approximation, expectations, effects of grouping, smoothing, rank correlation, grades, contingency, and artificial randomness.

In Appendix I are given certain tables for chi-square and t^2. And in Appendix II on the factorial function, a short table is given of the diagramma function, the derivative of log $x!$; that is, of log $\Gamma(x+1)$. There is an index jointly for topics and authors.

EDWARD L. DODD

NOTES

A conference in topology was held at the University of Michigan on June 24–July 6, 1940. The following lectures were given: *Triangulated manifolds and differentiable manifolds,* S. S. Cairns, Queens College; *Topological functions,* E. W. Chittenden, University of Iowa; *Uniformity in topological space,* L. W. Cohen, University of Kentucky; *Extension and classification of continuous mappings,* Samuel Eilenberg, University of Michigan; *The role of local separating points in certain problems of continua structure,* O. G. Harrold, University of Virginia; *Abstract complexes,* Solomon Lefschetz, Princeton University; *Extensions of homeomorphisms in the plane,* Saunders MacLane, Harvard University; *Periodic and nearly periodic transformations,* P. A. Smith, Barnard College; *Regular cycles in compact spaces,* N. E. Steenrod, University of Chicago; *Topology of differentiable manifolds,* Hassler Whitney, Harvard University; *Uniform local connectedness,* R. L. Wilder, University of Michigan; *Topological transformation groups and foundations of geometry,* Leo Zippin, Queens College. It is planned to have the University of Michigan Press issue, during the coming year, a volume containing these lectures, as well as abstracts of a number of papers which were delivered during the conference.

Johns Hopkins University has conferred the doctorate of laws on Dr. Vannevar Bush, President of the Carnegie Institution of Washington.

Professor Emeritus J. L. Coolidge of Harvard University has been awarded an honorary doctorate of laws by Harvard University.

Duke University has conferred the doctorate of laws on Dean L. P. Eisenhart of Princeton University.

Iowa State College has conferred the doctorate of laws on Professor Maria M. Roberts in recognition of fifty years of service to that college.

After twenty-nine years of service as Johnson Professor of Mathematics at the College of Wooster, Wooster, Ohio, Professor B. F. Yanney has retired with the title Johnson Professor of Mathematics Emeritus.

Dr. H. T. Muhly of Johns Hopkins University and Dr. C. E. Shannon of the Massachusetts Institute of Technology have been awarded National Research Fellowships in mathematics.

The William Lowell Putnam scholarship for study at Harvard University during 1940 has been awarded to Mr. A. M. Gleason of Yale University.

Professor Joseph Allen of the College of the City of New York will retire September 1, 1940.

Assistant Professor Max Astrachan of Antioch College, Yellow Springs, Ohio, has been made chairman of the department of mathematics.

Assistant Professor Wealthy Babcock of the University of Kansas has been promoted to an associate professorship.

Dr. R. P. Bailey of Lafayette College has been promoted to an assistant professorship.

Dr. N. H. Ball of the U. S. Naval Academy has been promoted to an assistant professorship.

Dr. F. S. Beale of Lehigh University has been promoted to an assistant professorship.

Associate Professor A. H. Black of Lebanon Valley College, Annville, Pennsylvania, has been promoted to a professorship.

Assistant Professor Florence L. Black of the University of Kansas has been promoted to an associate professorship.

Dr. Augusto Bobonis of the University of Puerto Rico has been promoted to an assistant professorship.

Mr. J. M. Boswell of the University of Kentucky has been appointed professor of mathematics and physics at Cumberland College, Williamsburg, Kentucky.

Dr. A. T. Brauer of the Institute for Advanced Study will lecture at New York University during 1940–1941.

Dr. F. L. Brooks of Kent State University has been promoted to an assistant professorship.

Associate Professor Jewell H. Bushey of Hunter College has been elected chairman of the department of mathematics.

Dr. F. A. Butter of the University of Southern California has been promoted to an assistant professorship.

Mr. R. B. Carpenter of the Metropolitan Life Insurance Company of New York has retired.

Dr. J. A. Clarkson of the University of Pennsylvania has been promoted to an assistant professorship.

Sister Rose Margaret Cook of Loretto Heights College, Loretto, Colorado, has been promoted to a professorship.

Assistant Professor H. V. Craig of the University of Texas has been promoted to an associate professorship.

Associate Professor M. M. Culver of the University of Pittsburgh has been promoted to a professorship.

Dr. E. H. Cutler of Lehigh University has been promoted to an assistant professorship.

Dr. C. H. Denbow of Ohio University has been promoted to an assistant professorship.

Assistant Professor H. T. Engstrom of Yale University has been promoted to an associate professorship.

Assistant Professor O. J. Farrell of Union College, Schenectady, New York, has been promoted to an associate professorship.

Assistant Professor W. W. Flexner of Cornell University has been promoted to an associate professorship.

Assistant Professor H. L. Garabedian of Northwestern University has been promoted to an associate professorship.

Assistant Professor O. H. Hamilton of Oklahoma Agricultural and Mechanical College has been promoted to an associate professorship.

Dr. E. E. Haskins of Northeastern University, Boston, Massachusetts, has been promoted to an assistant professorship.

Dr. A. E. Heins of Purdue University has been promoted to an assistant professorship.

Dr. J. D. Hill of Michigan State College has been promoted to an assistant professorship.

Dr. Banesh Hoffmann of Queens College has been promoted to an assistant professorship.

Dr. J. T. Hurt of the Agricultural and Mechanical College of Texas has been promoted to an assistant professorship.

Assistant Professor B. W. Jones of Cornell University has been promoted to an associate professorship.

Assistant Professor Chosaburo Kato of Denison University, Granville, Ohio, has been promoted to an associate professorship.

Assistant Professor S. H. Kimball of the University of Maine has been promoted to an associate professorship.

Assistant Professor B. O. Koopman of Columbia University has been promoted to an associate professorship.

Dr. O. E. Lancaster of the University of Maryland has been promoted to an assistant professorship.

Assistant Professor V. S. Lawrence of Cornell University has been promoted to an associate professorship.

Associate Professor V. F. Lenzen of the University of California has been promoted to a professorship.

Assistant Professor Sophia H. Levy of the University of California has been promoted to an associate professorship.

Assistant Professor L. L. Lowenstein of Alfred University, Alfred, New York, has been promoted to an associate professorship and has been named chairman of the department of mathematics.

Dr. W. C. McDaniel of Southern Illinois State Normal University has been promoted to an assistant professorship.

Assistant Professor J. D. Mancill of the University of Alabama has been promoted to an associate professorship.

Dr. A. B. Mewborn of California Institute of Technology has been appointed to an assistant professorship at the University of Arizona.

Dr. T. W. Moore of the U. S. Naval Academy has been promoted to an assistant professorship.

Dr. Hermance Mullemeister of the University of Washington has been promoted to an assistant professorship.

Assistant Professor W. R. Murray of Franklin and Marshall College, Lancaster, Pennsylvania, has been promoted to an associate professorship.

Assistant Professor Sara L. Nelson of Georgia State College for Women has been promoted to an associate professorship.

Dr. Emma J. Olson of Kent State University has been promoted to an assistant professorship.

Assistant Professor F. W. Perkins of Dartmouth College has been promoted to a professorship.

Assistant Professor G. W. Petrie of South Dakota State School of Mines has been promoted to an associate professorship.

Associate Professor D. E. Richmond of Williams College has been promoted to a professorship.

Assistant Professor J. B. Rosser of Cornell University has been promoted to an associate professorship.

Brother Louis de La Salle Seiler of St. Mary's College, Winona, Minnesota, is now dean of studies.

Associate Professor L. P. Siceloff of Columbia University has been promoted to a professorship.

Professor W. G. Simon of Western Reserve University has been made vice president.

Assistant Professor Ruth G. Simond of Hampton Institute, Hampton, Virginia, has been promoted to an associate professorship.

Professor Lao G. Simons of Hunter College has retired.

Mr. R. E. Smith of the University of North Carolina has been appointed to a professorship at Atlantic Christian College, Wilson, North Carolina.

Associate Professor A. H. Sprague of Amherst College has been promoted to a professorship.

Dr. A. E. Taylor of the University of California at Los Angeles has been promoted to an assistant professorship.

Associate Professor M. S. Vallarta of Massachusetts Institute of Technology has been promoted to a professorship.

Assistant Professor O. E. Walder of South Dakota State College has been promoted to an associate professorship.

Associate Professor Helen M. Walker of Teachers College, Columbia University, has been promoted to a professorship.

Dr. M. E. Wescott of Northwestern University has been promoted to an assistant professorship.

Assistant Professor Hassler Whitney of Harvard University has been promoted to an associate professorship.

Dr. C. R. Wylie of Ohio State University has been promoted to an assistant professorship.

Assistant Professor G. A. Yanosik of New York University has been promoted to an associate professorship.

During 1940–1941, Assistant Professor G. M. Ewing of the University of Missouri will be on leave at Princeton University.

During 1940–1941, Associate Professor A. R. Turquette of Florida Southern College, Lakeland, Florida, will be on leave of absence to study at Cornell University.

Dr. Olaf Helmer and Dr. H. E. Vaughan of the University of Illinois have been appointed associates in mathematics.

Dr. W. K. Morrill of Johns Hopkins University has been appointed an associate in mathematics.

The following appointments to instructorships are announced: University of Alabama: Dr. R. H. Bruck, Dr. R. E. Gaskell; Amarillo College, Amarillo, Texas: Dr. G. A. Whetstone; Antioch College, Yellow Springs, Ohio: Dr. Leonard Tornheim; University of Arizona: Mr. F. E. Hohn; Brown University: Mr. G. E. Forsythe; Carnegie Institute of Technology: Dr. R. F. Clippinger, Dr. A. D. Hestenes; University of Cincinnati: Dr. Gabriel Horvay; College of the City of New York: Dr. S. F. Barber; Harvard University: Dr. A. L. Whiteman; Hunter College: Dr. Annita Tuller; University of Illinois: Dr. P. H. Anderson, Dr. M. M. Day, Dr. K. L. Nielsen; Indiana University: Dr. F. J. Weyl; University of Indiana Extension, East Chicago, Indiana: Mr. J. F. Paydon; University of Maryland: Dr. H. E. Newell; University of Michigan: Dr. W. D. Duthie, Dr. Samuel Eilenberg, Dr. A. V. Martin; Michigan State College: Dr. J. F. Heyda; University of Missouri: Dr. G. E. Schweigert; Newcomb College, Tulane University: Miss Mary E. Ladue; Northwestern University: Dr. O. G. Harrold, Dr. W. T. Scott; University of Notre Dame: Dr. J. P. Nash; University of Pennsylvania: Mr. H. N. Laden; Princeton University: Dr. Brockway McMillan, Dr. A. D. Wallace; Purdue University: Dr. H. L. Langhaar, Dr. A. W. McGaughey; Reed College, Portland, Oregon: Dr. L. Louise Johnson, Mr. R. A. Rosenbaum; University of Saskatchewan: Dr. D. C. Murdoch; Stanford University: Dr. T. C. Doyle; Suffolk University, Boston, Massachusetts: Mr. F. X. Sutton; U. S. Naval Academy: Dr. W. E. Bleick; University of Washington: Dr. R. A. Beaumont, Dr. R. S. Phillips; University of Western Ontario: Dr. R. H. Cole; University of Wisconsin: Dr. A. O. Lindstrum.

It is reported that Professor S. Ortu Carboni of Genoa died November 5, 1939, at the age of seventy-nine.

Dr. R. S. Martin of the University of Illinois died June 2, 1940.

ABSTRACTS OF PAPERS

SUBMITTED FOR PRESENTATION TO THE SOCIETY

The following papers have been submitted to the Secretary and the Associate Secretaries of the Society for presentation at meetings of the Society. They are numbered serially throughout this volume. Cross references to them in the reports of the meetings will give the number of this volume, the number of this issue, and the serial number of the abstract.

394. Leon Alaoglu, Garrett Birkhoff, and B. J. Pettis: *General ergodic theorems*. II. Preliminary report.

As a further application of the methods of the paper *General erogdic theorems*, Annals of Mathematics, (2), vol. 41 (1940), pp. 293–309, the following theorem is proved. The space E is a Banach space which is also a complete linear lattice; furthermore $\| \, |x| \, \| = \|x\|$, and the norm is a monotone non-decreasing function of positive elements. The abelian semi-group G consists of positive linear transformations on E to E; means of transformations of G are denoted by τ. Then, if $\limsup_\tau |\tau x| < +\infty$ for all x of E, the limit $\lim_\tau \tau x$ exists in order for all x for which it exists in norm, and the two are equal. These limits of means are defined on page 296 of the above paper. If E is reflexive, the hypothesis of lattice completeness can be dropped, and the limit in norm exists for all x. (Received June 19, 1940.)

395. M. M. Day: *Ordered sets and closures.*

A closure function on X is any function c defined on all subsets of X whose values are subsets of X; c is monotone if $Y \subset Y' \subset X$ implies $cY \subset cY'$. A set A is called effectively ordered if the order relation is transitive and if every element of A has a successor. A natural definition of convergence in a neighborhood space is given which reduces to the definition of the Moore-Smith limit if A is directed. Definitions of closure in terms of neighborhoods and of convergence of functions on effectively ordered sets are given and it is shown that the closures defined by either method are the monotone closures. Tukey's definition of order among directed sets is extended to effectively ordered sets and it is shown that much but not all of the theory that holds for directed sets carries over to effectively ordered sets. (Received June 8, 1940.)

396. M. M. Day: *Reflexive Banach spaces which cannot be made uniformly convex.*

The space $B = (B_1 \times B_2 \times \cdots)_p$ (S. Banach, *Théorie des Opérations Linéaires*, Warsaw, 1932, p. 243) is shown to be reflexive if all B_i are. If B_i is taken to be the i-dimensional space with $\|b\|_{B_i} = \sup |b_j|$ or $= \sum |b_j|$, $j \leqq i$, where $b = (b_1, \cdots, b_i)$, it is shown that B is not isomorphic to any uniformly convex space. By choosing subspaces isomorphic to these it can be shown, for example, that if $B_i = L^{q_i}$ or l^{q_i}, $1 < q_i < \infty$, and if there do not exist m and M with $1 < m \leqq q_i \leqq M < \infty$ for all i, then B is not isomorphic to a uniformly convex space. (Received June 10, 1940.)

397. H. L. Garabedian, H. S. Wall, and Einar Hille: *Hausdorff means and integral equations.*

If $\{a_n\}$, $\{b_n\}$ are two regular Hausdroff sequences, then $\{b_n\}$ is said to be a divisor of $\{a_n\}$ if there exists a regular Hausdorff sequence $\{c_n\}$ such that $a_n=b_nc_n$, $(n=0, 1, 2, \cdots)$. Let $f_a(x)=\sum a_n(-x)^n$, $f_b(x)=\sum b_n(-x)^n$. The authors show that $\{b_n\}$ is a divisor of $\{a_n\}$ if and only if there exists a regular mass function $\phi_c(u)$ such that $f_a(x)=\int_0^1 f_b(ux)d\phi_c(u)$, $|x|<1$. By means of the Stieltjes inversion formula they show that this integral equation can be transformed into the equation of R. Schmidt, namely: $\phi_a(u)=\int_0^1\phi_b(u/v)d\phi_c(v)$ (which holds for all except possibly a finite or countable set of values of u), and obtain in this way a theorem of Hille and Tamarkin (Proceedings of the National Academy of Sciences, vol. 19 (1933), pp. 573–577). Application is made to prove the inclusion relation $(H_m, \alpha)\supset(H, \alpha+m-1)$, $(\alpha>0, m=2, 3, 4, \cdots)$, where (H_m, α) denotes summability defined by the mth row in the difference matrix in which the base sequence is the sequence $\{(n+1)^{-\alpha}\}$ defining summability (H, α). (Received June 6, 1940.)

398. Edward Kasner and J. J. De Cicco: *Equilong and conformal transformations of period two.*

The main problem of this paper is to find all equilong transformations of period two. In equilong geometry the set of all equilong transformations of period two may be classified into three distinct types: (T_1) equilong involutions, (T_2) K symmetries, and (T_3) D inversions. This is in contrast with the conformal theory where Kasner has proved that the set of all conformal transformations of period two consists of two distinct types: $(\mathfrak{T}_1)$ conformal involutions; and $(\mathfrak{T}_2)$ conformal symmetries. There are thus five distinct types of transformations of period two; three equilong and two conformal. The equilong transformations of period two may be reduced equilongly to the canonical forms: (T_1) symmetry in the positive y_1-axis $Z=-z$ where z represents the dual variable $x+jy$ with $j^2=0$ and (x, y) the hessian or equilong coordinates of a line, (T_2) symmetry in the origin accompanied by reversal of orientation $Z=\bar{z}$, and (T_3) symmetry in the x_1-axis accompanied by reversal of orientation $Z=-\bar{z}$. It is recalled that the conformal transformations of period two may be changed conformally to $(\mathfrak{T}_1)$ symmetry in the origin $Z_1=-z_1$ where z_1 represents the complex variable x_1+iy_1 with $i^2=-1$ and (x_1, y_1) the cartesian coordinates of a point, and $(\mathfrak{T}_2)$ symmetry in the x_1-axis, $Z_1=\bar{z}_1$. (Received June 5, 1940.)

399. Otto Szász: *On the jump of almost periodic functions and Fourier integrals.*

In a previous paper (Duke Mathematical Journal, vol. 4 (1938)) the author proved a theorem expressing the generalized jump of a periodic function $f(x)$ with period 2π as the limit of $(\pi/\log 2)\{\bar{\sigma}_{2n}(x)-\bar{\sigma}_n(x)\}$, as $n\rightarrow\infty$, where $\bar{\sigma}_n$ is the nth arithmetic mean of the series conjugate to the Fourier series of $f(x)$. Here analogous results are established for functions almost periodic in the sense of Besicovitch, and for Fourier integrals of functions in L^p, $1\leqq p\leqq 2$. (Received June 8, 1940.)

400. G. A. Whetstone and C. M. Cramlet: *On the invariance of the passivity conditions of systems of partial differential equations.*

The system of r linear differential equations in one dependent and n independent variables $X_iu\equiv a_i\partial u/\partial x_\rho=0, i=1, \cdots, r; \rho=1, \cdots, n$, is said to be complete if the Poisson operator (X_iX_j) applied to u yields a linear combination of the X_iu. Under arbitrary analytic transformations on the independent variables the property of completeness is invariant. A well known extension of this result is the theorem: The property that a system be involutory is invariant. The authors have extended these theorems to systems of equations with more than one dependent variable. In terms of a canonical system C which is in orthonomic form and assumed to be passive the general theorem which includes the foregoing classical theorems is: When the system C is transformed by an arbitrary analytic transformation on the independent variables and the new system reduced to an orthonomic canonical form $\bar{C}$, this new system will be passive. (Received June 17, 1940.)

401. Stefan Bergman: *Boundary values of functions satisfying partial differential equations of elliptic type.*

Let $U(z, \bar{z})$, $z=x+iy$, $\bar{z}=x-iy$, be regular in $|z|<2$ and let it satisfy there the equation $L(U)=U_{z\bar{z}}+\mathrm{Re}\,(A U_z)+CU=0$, where A is a complex function of $z, \bar{z}$ regular in $|z|\leqq 2$, C a real function regular in the same region, $U_z=\partial U/\partial x-i\partial U/\partial y$. Under these circumstances $\sum_{n=1}^{\infty}|a_n|^2n^{-1}<\infty$ is a sufficient condition that U has boundary values almost everywhere on $|z|=2$ for radial approach. Here $a_n=\int_0^{2\pi}[\tilde{U}(\rho, \phi)K_1^{(n)}(\rho, \phi)+\tilde{U}_\rho(\rho, \phi)K_2^{(n)}(\rho, \phi)]d\phi$ where $\tilde{U}(\rho, \phi)=U(\rho e^{i\phi}, \rho e^{-i\phi})$, $\tilde{U}_\rho(\rho, \phi)=\partial\tilde{U}/\partial\rho$, $0<\rho<2$ arbitrary. $K_\nu^{(n)}(\rho, \phi)$ are certain functions depending only on L and connected in a simple way with the solution $E=V(z, \bar{z}, \zeta, \bar{\zeta})\lg|z-\zeta|+W(z, \bar{z}, \zeta, \bar{\zeta})$ of $M(E)=0, z\neq\zeta$, M denoting the equation adjoint to L. (See Sommerfeld, Enzyklopädie der mathematischen Wissenschaften, II A, 7 c, p. 515.) (Received July 27, 1940.)

402. Stefan Bergman and D. C. Spencer: *On distortion in pseudo-conformal mapping.*

In the theory of conformal mapping, theorems concerning the change of measure of various euclidean quantities play an important role. For example, the area of certain sub-domains, as the theorems of Golusin and Bermant show, cannot be too small (G. M. Golusin, Comptes Rendus de l'Académie des Sciences de l'URSS, 1937, pp. 617–619; A. Bermant, Comptes Rendus de l'Académie des Sciences de l'URSS, 1938, pp. 137–140). This paper contains analogous results for pseudo-conformal mappings. The concept of "B-area" (Stefan Bergman, Rendiconti dell' Accademia Nazionale dei Lincei, vol. 19 (1934), pp. 474–478) is used, as well as a new representation for four-dimensional volume. The main result is an analogue in two complex variables of the theorem of Golusin. (Received August 1, 1940.)

403. R. P. Boas: *A general moment problem.*

Let $\{\phi_n(x)\}$ be a set of normalized functions belonging to L^2 on $(0, 1)$. The object of this paper is to establish conditions on the ϕ_n under which any sequence $\{a_n\}$ of complex constants such that $\sum|a_n|^2<\infty$ can be represented in the form $a_n=\int_0^1f(x)\phi_n(x)dx$, where $f(x)\in L^2$. It is found that a necessary and sufficient condition is that the quadratic form $\sum\sum x_m\bar{x}_n\int_0^1\phi_m(x)\bar{\phi}_n(x)dx$ should have a positive lower bound. When the ϕ_n are orthogonal, the condition is trivially satisfied, and the theorem reduces to the Riesz-Fischer theorem. Generalizations of the Hausdorff-Young-Riesz theorems are also obtained, and the theory is applied to some specific sets of functions. (Received July 29, 1940.)

404. R. P. Boas and D. V. Widder: *Functions with positive differences.*

Notation: $\Delta_\delta^0 f(x)=f(x)$, $\Delta_\delta^n f(x)=\Delta_\delta^{n-1}f(x+\delta)-\Delta_\delta^{n-1}f(x)$ $(n=1, 2, \cdots)$. A simplified proof is given of a theorem of T. Popoviciu (Mathematica, Cluj, vol. 8 (1934), pp. 1–85) that if $f(x)$ is continuous in (a, b) and if, for some $n\geqq 2$, $\Delta_\delta^n f(x)\geqq 0$ for all x and $\delta(>0)$ such that x and $x+n\delta$ are in (a, b), then $f(x)$ is of class C^{n-2} in (a, b) and has continuous right- and left-hand $(n-1)$th derivatives. It can be deduced from results of S. Bernstein that if $\Delta_\delta^n f(x)\geqq 0$ for all integers n in a sequence $\{n_k\}$ such that n_{k+1}/n_k is bounded, $f(x)$ is analytic in (a, b); an independent proof is given of this theorem. It is shown that if n_{k+1}/n_k is unbounded a function $f(x)$ can be constructed, of class C^∞, such that $f^{(n_k)}(x)>0$ $(k=1, 2, \cdots)$, while $f(x)$ is not analytic. (Received July 31, 1940.)

405. A. T. Brauer: *On a property of k consecutive integers.*

Pillai has just proved the following theorem (Proceedings of the Indian Academy of Sciences, Section A, vol. 11 (1940), pp. 6–12): In every set of less than 17 consecutive integers there exists at least one integer which is relatively prime to all the others; there are sequences of k integers for $k=17, 18, \cdots, 430$, however, which have not this property. Pillai conjectures that the same is valid for every $k\geqq 17$. It is proved that this conjecture is true. (Received July 29, 1940.)

406. Herbert Busemann and Walther Mayer: *On the foundations of calculus of variations.*

The minimizing arcs of a two-dimensional variational problem with fixed end points need not be differentiable everywhere if the integrand $F(x, y; \dot{x}, \dot{y})$ is regular and continuous in all four variables. But if $F(x, y; \dot{x}, \dot{y})$ is regular and continuous and satisfies a Lipschitz condition with respect to x and y, all minimizing arcs are continuously differentiable. However, in general line elements will occur which are contained in no or in infinitely many minimizing arcs. (Received July 30, 1940.)

407. R. H. Cole: *The expansion problem associated with an ordinary linear differential equation and boundary conditions applying at a set of collinear points.*

The expansion problem associated with the differential equation $u^{[n]}+P_1u^{[n-1]}+\cdots+P_nu=0$ whose solutions are conditioned relative to an arbitrarily given finite set of points on a fundamental interval by n linear boundary relations is a familiar one. It was studied by Wilder (Transactions of this Society, vol. 18 (1917)) who restricted his investigation to the equation in which the parameter λ appears only in the coefficient function P_n in the form $P_n=P_{n,0}(x)+\lambda^n$, and to boundary conditions which are free from λ. The present paper deals with the question of the convergence of the expansion when the functions P_k are taken as polynomials in λ with coefficients which are functions of x, and when the coefficients of the boundary relations are arbitrary polynomials in λ. The reduction of the given system to matrix form permits the adoption of the formal expansion of an arbitrary vector (that is, a set of n arbitrary functions) developed by Langer (Transactions of this Society, vol. 46 (1939)) for problems of this type. His convergence theorem, however, is presented under regularity restrictions which are not fulfilled by the problem at hand. A convergence theorem is developed here which shows the uniform convergence of the simultaneous expansion of the n arbitrary functions under appropriate regularity conditions. (Received July 5, 1940.)

408. H. V. Craig: *On extensors.*

The first part of this paper is concerned with the transformation equations of certain quantities called "crossed extensors." The distinguishing feature of these equations is that the summations are not restricted to adjacent Greek and Latin indices. The second part contains certain applications of extensor methods to classical differential geometry. One of the results is such that it can be outlined briefly. Let ρ be the radius vector from a given origin to a point with coordinates x and let $x=x(t)$ be the equations of a parametrized arc. Obviously, ρ sup (M), the Mth derivative of ρ with respect to t, is a vector function of x sup $(\alpha)a$, $\alpha=0, \cdots, M$. The partial derivatives ρ sup (M) inf $(\alpha)a$ evidently transform cogrediently to an excovariant extensor. Consequently the scalar products of these vectors, denoted by g inf $\alpha a\beta b$, are the components of an extensor. Therefore the contraction of the metric tensor g sup bc with g inf αa Mc is a tensor-extensor which is represented by ρ sup b inf αa. The contraction of this quantity with V sup $a(\alpha)$, the derivatives of a contravariant vector V, over the reduced range $M-\beta$ to M, is the βth intrinsic derivative of V multiplied by a constant. From the extensor point of view, the tensor character of intrinsic differentiation is obvious without knowledge of the affine connection. (Received July 31, 1940.)

409. H. B. Curry: *A revision of the fundamental rules of combinatory logic.*

This paper is concerned with some general theorems concerning improvements in the formulation of the ultimate foundations of the theory of combinators. (Cf. the first part of the author's thesis, American Journal of Mathematics, vol. 52 (1930), pp. 509–525.) For the most part these improvements were suggested by the work of Rosser (cf. his thesis, Annals of Mathematics, (2), vol. 36 (1935), pp. 127–139) who formulated a weakened system of combinatory logic in which the rules had a simple character not possessed by the rules of the original system. This paper discusses ways of formulating the original system which are analogous to Rosser's formulation. A formulation is given based on the primitives of the original system, and also in terms of the Schönfinkel primitives, S and K. (Received July 27, 1940.)

410. H. B. Curry: *Consistency and completeness of the theory of combinators.*

Some theorems relating to the consistency and completeness of the theory of combinators were proved in the author's thesis (American Journal of Mathematics, vol. 52 (1930), pp. 509–536, 789–837). Later, analogous theorems were proved by Rosser (Annals of Mathematics, (2), vol. 36 (1935), pp. 127–150 and Duke Mathematical Journal, vol. 1 (1935), pp. 328–355); his theorems were, in some respects, more powerful, but they applied only to a weakened system. In this note it is shown that the strongest theorems can be extended to the full theory of combinators, with or without a normality condition similar to Rosser's. The methods involve some slight generalization of the methods used by Church and Rosser, including their theorem of conversion (Transactions of this Society, vol. 39 (1936), pp. 472–482). (Received July 30, 1940.)

411. John Dyer-Bennet: *A note on partitions of the set of positive integers.*

The partitions of the set of positive integers which are homomorphic with respect to addition are classified, and it is noted that they are also homomorphic with respect to multiplication. A necessary and sufficient condition that such a partition be homomorphic with respect to exponentiation is given. Elementary number-theoretic methods are used. (Received July 2, 1940.)

412. Benjamin Epstein: *A contribution to the theory of two complex variables in certain domains.*

In this paper the author investigates a.f. of 2 c.v. (analytic functions of two complex variables) in domains $\mathfrak{S}^4=\sum_{|\gamma|\leq 1}S^2[\theta_1(\gamma), \theta_2(\gamma), \infty]$, where $\mathfrak{S}^4$ is a domain of four-dimensional space which is the set sum of sectors $S^2[\theta_1(\gamma), \theta_2(\gamma), \infty]$ lying in the plane $z_2=\gamma$ and with boundary made up of two rays proceeding from the point $z_1=0$ and making angles $\theta_1(\gamma)$, $\theta_2(\gamma)$ respectively with the positive real axis. The following result is proved: Let $\mathfrak{g}^1$ be a point set of $|z_2|=1$ which has the property that the intersection of $\mathfrak{g}^1$ with every arc of $|z_2|=1$ of length Γ ($\Gamma\geqq\Gamma_0$, Γ_0 depending only on $\mathfrak{S}^4$) contains a set of measure equal to or greater than $\Gamma/p(p\leqq p_0<\infty)$; let $f(z_1, z_2)$ be a bounded a.f. of 2 c.v. regular in $\mathfrak{S}^4-\mathrm{E}[z_1=0, |z_2|\leqq 1]$; and let $\underline{\mathrm{Bd}}_{z_1\to 0}f(z_1, e^{i\phi_2})=0$, $e^{i\phi_2}\ \varepsilon\ \mathfrak{g}^1$, uniformly in ϕ_2. Then $\lim_{z_1\to 0}f(z_1, z_2)=0$, $|z_2|<1$. Other theorems of similar type can be proved. Investigations concerning a.f. of 2 c.v. in the domains $\mathfrak{S}^4$ have many applications, since it is often possible to reduce the study of a.f. of 2 c.v. in an arbitrary domain whose boundary hypersurface contains a segment of an analytic surface to the study of a.f. of 2 c.v. in a $\mathfrak{S}^4$-domain (cf. Bergmann, Journal für die reine und angewandte Mathematik, vol. 169 (1933), pp. 1–42). (Received July 30, 1940.)

413. G. C. Evans: *Surfaces of minimum capacity.*

The author considers the problem of finding for a given closed curve in space the surface cap which is of minimum electric capacity. For this it is assumed that the curve itself is of zero capacity, and that a sphere containing it can be mapped in a continuous one-one manner on itself in such a way that the curve is represented by a circle. A surface is found which satisfies the minimal condition by determining a certain two-valued harmonic function on a Riemannian manifold of two space sheets. The surface is a level surface of this function. It furnishes the minimum capacity among all surfaces which are made up of sufficiently smooth pieces and have the given curve as sole boundary. (Received July 29, 1940.)

414. H. T. Fleddermann: *The equality of measure functions.*

Besicovitch (Mathematische Annalen, vol. 98 (1927), p. 422) defines the density of a linearly measurable plane set at a point of it. In this definition he makes use of Carathéodory's measure. In this note, the definition is extended by replacing the Carathéodory measure by the Gross measure and by any general measure function. It is shown that if the density obtained by using the Gross measure is equal to 1, everywhere, then the density obtained with the Carathéodory measure is equal to 1, almost everywhere, and vice versa. It is further proved that under these conditions the Gross and Carathéodory measure of the set are equal. It follows from this and the fact that the Gross measure is a minimum measure that the density obtained by using any general measure function lying between the Gross and Carathéodory functions is also equal to 1 and the measure of the set is the same as the Gross or Carathéodory measure. As the Carathéodory measure is used as an upper bound, a class of measure functions is obtained for which it is a maximum. (Received July 30, 1940.)

415. J. S. Frame: *The double cosets of a finite group.*

Let H be a subgroup of order h of a finite group G of order $g=nh$. When the permutation group G_H of degree n on the right cosets is written as a group of matrices and completely reduced, let the irreducible representations Γ_i of degree n_i, $i=1, 2, \cdots, r'$, appear with multiplicity μ_i as components of G_H, and let $c_i=1, -1$, or 0 according as Γ_i has a symmetric, an alternating, or no bilinear invariant. Let G contain r double cosets $K_t=HS_tH/d_t$, $t=1, 2, \cdots, r$, consisting respectively of $k_t=h/d_t$ right cosets HS_t which are permuted transitively among themselves when multiplied on the right by elements of H. Let N_H of these be self-inverse. Then after some preliminary theorems about the distribution of elements within the double cosets of G, two principal theorems are proved. The first, which generalizes a result proved by the author in a previous paper (Proceedings of the National Academy of Sciences, vol. 26 (1940), pp. 132–139), states that $N_H=\sum_{i=1}^{r'} c_i\mu_i$. The second, which proves and generalizes a theorem conjectured by the author in another paper (Duke Mathematical Journal, vol. 3 (1937), pp. 8–17), states that $n^{r-2}\prod_{t=1}^{r} k_t=(\prod_{i=1}^{r'} n_i^{\nu}\overline{P}_1 P_1$, where $\nu=\mu_1^2$ and P_1 is an algebraic integer in the field of the characters of the components of G_H. (Received July 23, 1940.)

416. K. O. Friedrichs: *On hyperbolic differential operators.*

Uniqueness and existence of the solution of Cauchy's problem for linear hyperbolic differential equations can be based on the projection theorem in Hilbert space and the use of certain "mollifying" integral operators, which had been employed for elliptic differential operators in a previous paper of the author. "Hyperbolic" differential equations are here understood in a wider sense; they are written as a system of the first order and contain the classical wave equations as a special case. The differential operator is to be closed by extending it to a certain linear space of L^2-integrable functions. The values of the solution are prescribed on a spacelike surface; the conditions for these initial data are such that they persist in the process of propagation. (Received July 30, 1940.)

417. Orrin Frink: *Series expansions in linear vector space.*

If the set $P=\{p_n\}$ is closed and minimal in the real or complex Banach space B, it is shown that every element x of B has a uniquely determined series expansion (1) $\sum \xi_n p_n$ in terms of P, called its ***minimal expansion***, such that (I) the coefficient of p_n, in any sequence of linear combinations of P which converges to x, converges to ξ_n; (II) the coefficient ξ_n is a continuous additive function of x; (III) P is part of a biorthogonal system $\{p_n, f_n\}$ and conversely, if $\{p_n, f_n\}$ is biorthogonal, then $\{p_n\}$ is minimal; (IV) the partial sums of (1) are in one sense the best possible approximations to x. A condition is given which is sufficient that the series (1) converge absolutely to x. If P is normalized, it is shown that $\xi_n \to 0$. Semiregular summability methods are introduced, and it is shown that no such method sums (1) to the wrong limit, while at least one such method sums (1) to x if only a finite number of the coefficients ξ_n are zero. Applications are given to Schauder series and minimal polynomial series, including power series and Newton interpolation series, both real and complex. (Received July 27, 1940.)

418. Abe Gelbart: *On approximations of functions of two complex variables regular in multiply-connected four-dimensional domains.*

In functions of one complex variable it is shown that every analytic function $f(z)$ regular in a multiply-connected domain D^2 of n holes can be approximated uniformly by $(n+1)$ functions, the first n being rational with one singularity at a point in one of the holes, and the $(n+1)$st being a polynomial. With the aid of Bergman's integral formula for functions of two complex variables (Recueil Mathématique, vol. 43 (1936), pp. 851–862), it is shown that an analytic function of two complex variables, with n singularity surfaces in a four-dimensional region $\mathfrak{M}^4$, can be approximated uniformly in a multiply-connected closed region $\bar{\mathfrak{A}}^4$, interior to $\mathfrak{M}^4$ and excluding the n singularity surfaces, by $[n(n+1)+1]$ functions, the first $n(n+1)$ having just two singularity surfaces and the last being a polynomial. By imposing further conditions on the domain $\bar{\mathfrak{A}}^4$, it is shown that each of the $n(n+1)$ functions can be approximated uniformly in $\mathfrak{A}^4$ by rational functions that have the same two singularity surfaces. (Received July 30, 1940.)

419. Louis Green: *Twisted cubics associated with a space curve.* II.

This paper is an extension of results of the author (American Journal of Mathematics, vol. 62 (1940), pp. 285–306) on local projective properties of a space curve Γ. It is found that Γ possesses at each of its points a two-parameter family of five-point cubics which are intimately related to the osculating quadrics of Γ. By means of these cubics, new configurations related to Γ are obtained and new characterizations are given to some known ones. A number of the cubics are found to have special properties of interest. (Received July 31, 1940.)

420. D. W. Hall and W. T. Puckett: *Arc-preserving transformations of a certain class of spaces.*

It is shown that the class of cyclic strongly arcwise connected continua (W. T. Puckett, this Bulletin, abstract 46-5-316) consists exactly of all cyclic locally connected continua A such that every arc-preserving transformation $T(A)=B$, where B is not an arc, is topological. (Received July 29, 1940.)

421. Marshall Hall: *Projective planes.*

An investigation of projective planes, especially non-Desarguesian types. Consideration of the generation of planes, particularly "free" planes. A study is made of projections, collineations, and universal configurations. (Received July 22, 1940.)

422. P. R. Halmos and R. A. Leibler: *Square roots of measure preserving transformations.*

Let T be a one-to-one, measure preserving (not necessarily continuous) transformation of the m-dimensional unit cube into itself. The problem of the present paper is to find conditions necessary and sufficient for the existence of another such transformation S, called a square root of T, for which $S^2=T$. A complete solution is given in case T has a pure point spectrum: in other words, in case there exists a sequence of numbers α_n and a complete orthonormal set of functions $f_n(x)$ in L_2 such that $f_n(Tx)=\alpha_n f_n(x)$. The following two theorems (indicating, respectively, the nature of the necessary and of the sufficient conditions) are typical. I. If T has a square root, then the multiplicity of -1 with respect to the point spectrum of T is even (is

$0, 2, 4, \cdots, \infty$). II. If T is metrically transitive and has a pure point spectrum, and if -1 does not belong to the spectrum of T, then T has a square root. (Received July 30, 1940.)

423. J. G. Herriot: *Nörlund summability of double Fourier series.* Preliminary report.

Grünwald (Proceedings of the Cambridge Philosophical Society, vol. 35 (1939), pp. 343–350) has proved that the square partial sums s_{nn} of the double Fourier series of an integrable periodic function of two variables are summable $(C, 1)$ to the value of the function at any point of continuity. The author replaces the $(C, 1)$ method by a Nörlund method with nonnegative, nonincreasing coefficients and obtains necessary and sufficient conditions that the above result remains true. The necessity is proved by appealing to a theorem of Banach and Steinhaus on linear functionals. The sufficiency follows from Grünwald's result. In particular, it follows that the (C, α) method applied to square partial sums possesses the localization property if and only if $\alpha \geqq 1$. Again consider the Nörlund transforms t_{mn} of the partial sums s_{mn} obtained by a double Nörlund transformation $\{p_m q_n\}$, p_n and q_n being nonnegative and nonincreasing. Consider the limit of t_{mn} when $m, n \rightarrow \infty$ so that $m/n \leqq \lambda, n/m \leqq \lambda$ $(\lambda \geqq 1$, a fixed number). Necessary and sufficient conditions that this method of summability should possess the localization property are obtained. In particular, when $0 < \alpha$, $\beta \leqq 1$, (C, α, β) as above restricted possesses the localization property if and only if $\alpha = \beta = 1$. (Received July 25, 1940.)

424. Harold Hotelling: *Experimental determination of the maximum of an empirical function.*

In physical and economic experimentation to determine the maximum of an unknown function, for example, of a monopolist's profit as a function of price or of the magnetic permeability of an alloy as a function of its composition, the characteristic procedure is to perform experiments with chosen values of the argument x, each of which then yields an observation, subject to error, on the corresponding functional value $y = f(x)$. The values of x need, however, to be chosen on the basis of earlier experiments in order to make the determination efficient. The experimentation properly proceeds, therefore, in successive stages, with the values used at each stage determined with the help of the earlier work. The question, what distribution of x as a function of previous results should be used, is discussed in this paper on the basis of various hypotheses regarding the function, and further criteria. In particular, a conflict is shown to exist under some conditions between the criterion of minimum sampling variance and that calling for absence of bias. (Received August 1, 1940.)

425. Fritz John: *Discontinuous convex solutions of difference equations.*

This paper is a supplement to the author's paper (Acta Mathematica, vol. 71 (1939), pp. 175–189) which dealt only with continuous convex solutions. It includes as special cases results by A. E. Mayer (Acta Mathematica, vol. 70 (1938), pp. 57–62) and by H. P. Thielman (this Bulletin, vol. 46 (1940), p. 432). It is shown that a difference equation can have only a discontinuous convex solution if it has more than one continuous convex solution. A difference equation $\sum_k a_k f(x+k) = 0$ with constant a_k has a discontinuous convex solution when and only when the equation $\sum_k a_k x^k = 0$ has a positive real root. (Received July 25, 1940.)

426. Mark Kac: *On a question of Ramanujan.*

Let $d(n)$ denote the number of divisors of n. Ramanujan conjectured (*Collected Papers*, p. 333, Question 770) that the series $\sum_{k=1}^{\infty}(-1)^k[d(2k+1)]/(2k+1)$ is convergent. The present paper furnishes a short and simple proof of this statement. It turns out that the sum of the above series is $\pi^2/16$. (Received July 27, 1940.)

427. W. H. McEwen: *On the degree of convergence of a series associated with an integro-differential system.*

This paper deals with the expansion of a function $f(x)$ in terms of characteristic solutions of a given homogeneous integro-differential system involving a parameter. Denoting the partial sums of the series by $S_N(x)$ $(N=1, 2, \cdots)$, and letting m denote an arbitrary positive integer, hypotheses are obtained under which $f^{(k)}-S_N^{(k)} =0(1/N^{m-k})$ uniformly on the interval of x in question, where $k=0, 1, \cdots, m-1$. (Received July 31, 1940.)

428. H. E. Newell: *On the asymptotic forms of the solutions of an ordinary linear matric differential equation in the complex domain.* Preliminary report.

The matric differential equation $(d/dx)\,Y(x,\lambda)^{n\times n}=\{\lambda(\delta_{ij}r_j(x))^{n\times n}+(q_{ij}(x,\lambda))^{n\times n}\}\cdot Y(x, \lambda)^{n\times n}$ is discussed for the parameter λ large in absolute value, and for x in a finite simply connected region of the complex plane, within which the coefficient functions are analytic with the differences $r_i(x)-r_j(x)$, $(i\neq j)$, not zero, and on the boundary of which the coefficient functions may have poles, and the differences $r_i(x)-r_j(x)$, $(i\neq j)$, may have zeros. Under four distinct sets of conditions, three of which admit of meromorphic $q_{ij}(x, \lambda)$, regions of existence abutting a specified pole of an $r_j(x)$ or a specified zero of a difference $r_i(x)-r_j(x)$, $(i\neq j)$, are established for a solution of the form $P(x, \lambda)(\delta_{ij} \exp\{\lambda\int^x r_j(x)dx\})$, where $P(x, \lambda)$ reduces uniformly in x to the identity matrix when λ becomes infinite. Two of the sets of conditions are also sufficient that $P(x, \lambda)$ admit, for $|\lambda|$ large, of an asymptotic series expansion in λ^{-1}. The method employed is similar to that recently used by R. E. Langer in treating the case in which the coefficient functions are analytic with the differences $r_i(x)-r_j(x)$, $(i\neq j)$, bounded from zero. Finally, it is also shown that the results of both the present and Langer's papers apply to infinite as well as to finite x regions. (Received July 3, 1940.)

429. Rufus Oldenburger: *La teoria de los polinomios de orden superior.*

This paper will be published in Spanish in the scientific and engineering publication "Ingenieria" of the National University of Mexico, where it was given as a series of five lectures. In this paper, solutions are obtained for all of the open questions listed in the last paragraph of the author's article on polynomials (Annals of Mathematics, (2), vol. 41 (1940), p. 709), together with applications. (Received July 29, 1940.)

430. J. M. H. Olmsted: *Lebesgue theory on a Boolean algebra.* Preliminary report.

Any partially ordered linear space with an element 1 (cf. H. Freudenthal, Proceedings, Akademie van Wetenschappen, Amsterdam, vol. 39 (1936), p. 641) can be extended to a commutative ring, the multiplicative unit of which is the element 1. This extension is the space of all functions, from real numbers to elements of the underlying Boolean algebra, decreasing monotonely from the one element of the algebra to the zero element. Such functions are analyzed for an arbitrary Boolean algebra with countable superior and inferior. Furthermore, multiplication of functions and the existence of a measure on the Boolean algebra give rise to a theory of the L^p spaces. The Nikodym theorem for absolutely continuous functions is again available. (Received July 8, 1940.)

431. F. W. Perkins: *Polygonal and polyhedral means of functions.* Preliminary report.

Let R be a finite region of euclidean space of three dimensions. Let $G_P(a_P)$ be a closed subregion of R bounded by a regular polyhedron with center at P and radius a_P, and let $G_P(\rho)$ be the closed subregion bounded by a concentric similar polyhedron of radius $\rho \leqq a_P$, similarly oriented. Let $U(P)$ be a function of P, real and continuous in R. The author considers (as functions of P and ρ) various means of u associated with $G_P(\rho)$, especially the arithmetic means of u on the vertices, on the radial lines drawn to the vertices, on the triangular areas bounded by these radial lines and the edges, on the faces, and over the volume. Various relations involving these means are established, some of which are of particular interest in the theory of harmonic and subharmonic functions. Similar methods can be used in the plane; here some parts of the theory are related to topics studied by Frazer (*On the moduli of regular functions*, Proceedings of the London Mathematical Society, vol. 36 (1934), pp. 532–546). (Received July 25, 1940.)

432. G. Pólya: *Sur l'existence de fonctions entières satisfaisant à certaines conditions linéaires.*

The present paper deals with an interpolation problem for entire functions. Given two sequences of complex numbers $\{a_n\}$ and $\{A_n\}$ and a sequence of nonnegative integers $\{\alpha_n\}$. Does there exist an entire function $F(z)$ such that $F^{(\alpha_n)}(a_n)=A_n$ for all n? The author shows the existence of a solution when the sequence $\{a_n\}$ is periodic of period p, $\alpha_{n+p}=\alpha_n+p$, $\alpha_n \leqq \alpha_{n+1}$, $\lim |A_n/\alpha_n!|^{1/\alpha_n}=0$, and there is no polynomial of degree less than p satisfying the first p homogeneous equations. (Received July 20, 1940.)

433. E. J. Purcell: *Space Cremona transformations of order $m+n-1$.*

Consider a curve C_n of order n having $n-1$ points on each of two skew lines d and d', and a curve C_m' of order m having $m-1$ points on each of d and d' (m, n any integers). A generic point P determines a ray through it intersecting C_n once in α and d once in β. P also determines a ray through it intersecting C_m' once in γ and d' once in δ. Lines $\alpha\delta$ and $\beta\gamma$ intersect in P', the correspondent of P in a Cremona transformation of order $m+n-1$. Special positions of the defining curves give rise to involutions. The paper will appear soon in this Bulletin. (Received July 29, 1940.)

434. J. F. Ritt: *On a type of algebraic differential manifold.*

For manifolds of systems of differential equations involving one unknown function, operations of addition, multiplication and differentiation are defined and investigated. The paper will appear in an early number of the Transactions of this Society. (Received June 27, 1940.)

435. L. B. Robinson: *Singular solutions of nonlinear functional equations.*

Little is known about nonlinear functional equations. The author has computed some singular solutions and gives a simple specimen. Given the function $u(x)=x^{\lambda}/c$. This function satisfies an equation of the type $u'(x)=P(x)\{u(x^m)\}^2$ since it satisfies (I) $u'(x)=\lambda c x^{-2m\lambda+\lambda-1}\{u(x^m)\}^2$ and the transformation $x=1/\xi$ sends this into (II) $w'(\xi)=\lambda c\xi^{+2m\lambda-\lambda-1}\{w(\xi^m)\}^2$. Equation (II) admits a solution holomorphic at $\xi=0$ and depending on one parameter. It also has the singular solution $w(\xi)=\xi^{-\lambda}/c$. (Received July 25, 1940.)

436. R. M. Robinson: *Stencils for solving* $x^2\equiv a \pmod{m}$.

A set of 272 stencils for solving $x^2\equiv a \pmod{m}$ by the method of exclusion has been constructed on Hollerith cards. This provides a practical method for solving such congruences; the solution is particularly simple if $m<3000$. (Received July 29, 1940.)

437. E. Rosenthall: *On the diophantine equation* $x^3+y^3=z^3+w^3$.

The letters A, B, C, denote arbitrary algebraic integers of the quadratic number field $K(\rho)$, ρ an imaginary cube root of unity; all other letters represent rational integers. The conjugate of any integer Q is denoted by $\bar{Q}$. The following theorem is proved: All rational integer solutions of the equation $x^3+y^3=z^3+w^3$ are given by $x=-p(a+b)/q$, $y=p(a-2b)/q$, $z=-p(c+d)/q$, $w=p(c-2d)/q$ where $a+b\rho=(\overline{AB}\cdot B\bar{B}-AC\cdot C\bar{C})\bar{A}\bar{C}$, $c+d\rho=(\overline{AB}\cdot B\bar{B}-AC\cdot C\bar{C})AB$ and it suffices to take q equal to the G.C.D. of e, f, g, where $\overline{AB}\cdot B\bar{B}-AC\cdot C\bar{C}=e+f\rho$ and $\overline{AC}\cdot\overline{AB}-AC\cdot AB=g+2g\rho$. The proof is obtained by operating on a multiplicative equation in $K(\rho)$. Complete rational integer solutions are also obtained for each of the following equations: $x^3+y^3+z^3-3xyz=u^3+v^3+w^3-3uvw$ and $x^3+y^3+z^3-3xyz=u^3$. (Received June 29, 1940.)

438. A. R. Schweitzer: *Concerning general abstract relational spaces. II.*

In continuation of a paper reported in this Bulletin (abstract 46-3-190) the author constructs postulates for relational spaces $S_{n+1}(G, H)$ $(n=1, 2, 3, \cdots)$ when G is any substitution group on $n+1$ elements and H is an arbitrarily selected subgroup of G, including G. Distinction is made between "absolute" and "relative" invariance of the generating relation K_H effective between two $(n+1)$-ads of elements; the latter type of invariance is based on correspondence between the elements of the two $(n+1)$-ads. The symmetrical and transitive relation K_H is relatively invariant under G and absolutely invariant under H but not absolutely invariant under any substitution which is not in H. When G is the symmetric group the author's previous theory is obtained. When G is the alternating group and H coincides with G, instances include a system of postulates equivalent to the author's descriptive system ${}^3R_2$ (American Journal of Mathematics, vol. 31 (1909), pp. 387–388). When G is arbitrary the space $S_{n+1}(G, G)$ on $n+1$ elements consists of a definite set of $(n+1)$-ads corresponding to the group G on these elements. Other examples are given. (Received July 30, 1940.)

439. D. J. Struik and Alexander Wundheiler: *Affine representation of systems of forces.*

The well known geometrical representations of systems of forces (Study, Ball, and others) have an arbitrary character. This paper attempts a systematic treatment by introduction of a set of sliding affine figures (vectors, rotors, vector-rotors, rotor-vectors, and so on) obtained by combination of the properties of direction, orientation and normalization. Under the metric group certain figures merge together assuming traditional forms. Coordinates are assigned to these figures on the basis of their relations to projective figures. Merging conditions under the metric group with reflections are studied in particular. (Received August 1, 1940.)

440. H. S. Wall: *Real power series bounded in the unit circle.*

The author investigates the class T of power series $f(x)=\sum c_i(-x)^i$ with real coefficients, radius of convergence greater than or equal to 1, and with the property that $M(f)=\text{l.u.b.}_{|x|<1}|f(x)|\leqq 1$. The principal result is the theorem that to each function $f(x)$ of T there corresponds a function $F(x)$ of T which is representable in the form $F(x)=\int_0^1 x\,d\phi(u)/(1+xu)$ as a Stieltjes integral with respect to a bounded monotone mass function $\phi(u)$, such that $f(x)=[f(0)-x+(1-x)F(z)]/[1-xf(0)+(1-x)F(z)]$, $z=4x/(1-x)^2$. As $f(x)$ ranges over T, let $F(x)$ range over the subclass T_0 of T. Then a function is in T_0 if and only if it has a continued fraction representation of the form $(1+\lambda_0)(1-\lambda_1)x/4+(1+\lambda_1)(1-\lambda_2)x/1+(1+\lambda_2)(1-\lambda_3)x/4+\cdots$ (a 4 appears in every other partial denominator), where $-1\leqq\lambda_n\leqq+1$, $(n=0, 1, 2, \cdots)$. The sequence $\{\lambda_n\}$ is the sequence associated with $f(x)$ by the algorithm used by Schur. (Received July 5, 1940.)

441. J. H. C. Whitehead: *On incidence matrices, nuclei and homotopy types.*

W. Hurewicz (Proceedings, Akademie van Wetenschappen, Amsterdam, vol. 39 (1936), p. 125) has asked: "Are two closed manifolds homeomorphic if they are of the same homotopy type?" It appears, among other things, that two closed, polyhedral manifolds (using the word manifold in its narrowest sense), which are of the same homotopy type, need not be combinatorially equivalent, and need not even have the same nucleus (J. H. C. Whitehead, Proceedings of the London Mathematical Society, vol. 45 (1939)). In fact, two lens spaces of types (m, q) and (m, r) are of the same homotopy type provided they have the same intersection invariant, that is, provided $r\equiv\pm l^2q \pmod{m}$ for some l. This corrects an error in my note *On certain invariants introduced by Reidemeister* (Quarterly Journal of Mathematics (Oxford), vol. 10 (1939)) and it has not even been proved that these are topological invariants. For example, combinatorially inequivalent lens spaces of types (7, 1) and (7, 2) are of the same homotopy type but have different nuclei. Whether or no they are homeomorphic remains an open question. Reidemeister's invariants are the same for polyhedra with the same nucleus. (Received July 5, 1940.)

442. P. M. Whitman: *Splittings of a lattice.*

A "splitting" of a lattice L is defined as a partition of its elements into two "splinters": a principal ideal and a dual of a principal ideal. Necessary and sufficient conditions for the existence of splittings are given. The properties of the "edge" along which the splitting is made are investigated, and conditions given in terms of the nature of the edge and splinters that L be modular or distributive. Successive splittings form a lattice. The free lattice of n generators has a proper sublattice isomorphic to itself. (Received July 22, 1940.)

443. L. R. Wilcox: *A topology for semi-modular lattices.*

Let L be a semi-modular lattice (cf. Annals of Mathematics, (2), vol. 40 (1939), pp. 490–505) which is atomistic and satisfies the chain conditions. Let the set P of points (or atoms) of L be a metric space. It is shown under two mild assumptions of continuity that the topology of P can be extended to L so that L is a Hausdorff space. Indeed, if δ is the metric of P, and d is the dimension function on L, then neighborhoods of any element $a \in L$ are defined in the following manner. Let $\epsilon<0$, and let $p_1, \cdots, p_k$ be a finite system of independent points whose sum is a; then $U(a; \epsilon; p_1, \cdots, p_k)=[b; d(b)=d(a), \delta(p_i, b)<\epsilon]$, where $\delta(p_i, b)$ is the distance between the sets $[p_i]$ and $[p; p\leqq b]$. The purpose of the paper is to clarify and unify the foundations of projective, affine and euclidean differential geometry by lattice-theoretic methods. (Received August 1, 1940.)

444. Orla V. Wood: *On relations between solutions of the differential equation of the second order with four regular singular points.*

The differential equation of the second order with four regular singular points $(0, 1, a, \infty)$ has not been seriously studied, nor has a method been given of extending its integrals from one domain of existence to another. The author considers the difference equation arising from the recurrence relations satisfied by the coefficients in a series solution about the origin. The difference equation is homogeneous, of the second order, the coefficients being of degree zero, two, and four, respectively. The solution of this equation requires the determination of a form suitable to represent it at infinity. An asymptotic solution is found from consideration of a related difference equation which is normalized. Making use of this solution, a Barnes type contour integral is formed. The integral is shown to converge, and to be an analytic solution of the differential equation throughout the complex plane with a cut along the positive real axis. Application of Cauchy's residue theorem to the contour integral leads to an analytic continuation outside the unit circle of the function represented by the original series, and to a relation between solutions of the differential equation. (Received July 30, 1940.)

445. Alexander Wundheiler: *Some basic notions in meta-geometry.*

A figure is a set of points. A figure and a normal divisor of a group G determine a G-hyper-figure of the group G. Two hyper-figures are H-equivalent, H being a subgroup of G, if they can be associated in a way invariant under the group H. A class of H-equivalent hyper-figures determines an H-object under the group H. An H-object has definite coordinates in any coordinate system of the group H. Examples: figure—an attached vector; hyper-figure—a free vector; a free contravariant and a free covariant vector are equivalent under the metric group. (Received August 1, 1940.)

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Droste, H. W. Die Lösung angewandter Differentialgleichungen mittels Laplacescher Transformation. (Neuere Rechenverfahren der Technik, no. 1.) Berlin, Mittler, 1939. 35 pp.

Heffter, L. Grundlagen und analytischer Aufbau der projektiven, euklidischen und nichteuklidischen Geometrie. Leipzig and Berlin, Teubner, 1940. 8+199 pp.

Hogben, L. La Matematica nella Storia e nella Vita. 2 vols. Translated by F. Morra. Milan, Hoepli, 1939. 16+916 pp.

Kasner, E., and Newman, J. Mathematics and the Imagination. New York, Simon and Schuster, 1940. 14+380 pp.

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Moulton, E. J. See Curtiss, D. R.

Newman, J. See Kasner, E.

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BULLETIN

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In 1923 an Endowment Fund was collected for the support of the ever increasing number of important mathematical memoirs. Of this fund, which now amounts to some $59,000, a considerable proportion was contributed by members of the Society.

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The Bôcher Memorial Prize.

This prize was founded in memory of Professor Maxime Bôcher. It is awarded every five years for a notable research memoir in analysis which has appeared during the preceding five years in a recognized journal published in the United States or Canada; the recipient must be a member of the Society, and not more than fifty years old at the time of publication of his memoir.

First (Preliminary) Award, 1923: To G. D. Birkhoff, for his memoir *Dynamical systems with two degrees of freedom.*

Second Award, 1924: To E. T. Bell, for his memoir *Arithmetical paraphrases,* and to Solomon Lefschetz, for his memoir *On certain numerical invariants with applications to abelian varieties.*

Third Award, 1928: To J. W. Alexander, for his memoir *Combinatorial analysis situs.*

Fourth Award, 1933: To Marston Morse, for his memoir *The foundations of a theory of the calculus of variations in the large in m-space,* and to Norbert Wiener, for his memoir *Tauberian theorems.*

Fifth Award, 1938: To John von Neumann, for his memoir *Almost periodic functions and groups.*

The Frank Nelson Cole Prize in Algebra.
The Frank Nelson Cole Prize in the Theory of Numbers.

These prizes were founded in honor of Professor Frank Nelson Cole, on the occasion of his retirement as Secretary of the American Mathematical Society and editor of the Bulletin, after twenty-five years of service; the fund was later doubled by his son, Charles A. Cole. They are awarded at five-year intervals for contributions to algebra and the theory of numbers, respectively, under restrictions similar to those for the Bôcher prize.

First Award, 1928: To L. E. Dickson, for his book *Algebren und ihre Zahlentheorie,* Zurich, 1927.

Second Award, 1931: To H. S. Vandiver, for his several papers on Fermat's last theorem published in the Transactions of the American Mathematical Society and the Annals of Mathematics during the last five years, with special reference to a paper entitled *On Fermat's last theorem,* which appeared in volume 31 of the Transactions.

Third Award, 1939: To A. A. Albert, for his papers on the construction of Riemann matrices, published in volumes 35 and 36 of the Annals of Mathematics.

The Eliakim Hastings Moore Fund.

This fund was founded in 1922 in honor of Professor Eliakim Hastings Moore, on the occasion of the twenty-fifth anniversary meeting of the Chicago Section of the American Mathematical Society. The interest on the fund is to be used at the discretion of the Council of the Society, for the publication of important mathematical books or memoirs, or the award of prizes.

The Marion Reilly Fund.

Dean Marion Reilly willed to the American Mathematical Society a portion of her estate to be used for the advancement of research in pure mathematics. Thus far the Society has received approximately $16,800. The estate is not entirely settled, and the Society will receive additional funds within the next few years.

The Ernest William Brown Fund.

From the will of Professor E. W. Brown, a fund of $1000 is available, the interest on which, at the discretion of the Council, can be used for the furtherance of such mathematical interests as (a) the publication of important mathematical books, memoirs, and periodicals, and (b) lectures to be delivered on special occasions by invited guests of the Society.

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30	137.93	60	82.74
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45	115.13	75	47.56
50	105.24	80	37.44

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Sadowsky
Wilcox
Central Y.M.C.A. College
J. E. Davis
Gore
Chicago Teachers College
Mansfield
DePaul University
Corliss
Herzl Junior College
Gere
Loyola University
Gerst
O'Mara
Tordella
North Park College
C. G. Erickson
St. Francis Xavier College for Women
Garbe
Holland
E. C. Pixley
University of Chicago
A. A. Albert
Barnard
Bartky
Bliss
Bloom
Dickson
H. S. Everett
L. M. Graves
Hartung
Hazard
M. R. Hestenes
Householder
Karush
E. P. Lane
Leavens
Logsdon
Lunn
MacMillan
H. Meyer
H. L. Meyer
Perlis
Reid
Sanger
Schilling
Senior
Steenrod
Trimble
Y. K. Wong
J. W. A. Young
University of Illinois
Besancon
Woodrow Wilson Junior College
B. Friedman
Kinney
Rasmusen
Wright Junior College
Ballard
Erskine
Modesitt
Downers Grove
G. W. Whitehead
Elmhurst
Handel
Elsah
Hooper
Eureka
Newson
Evanston
E. R. Carter
Harris
Peck
Northwestern University
T. W. Anderson
Burcham
Burns
D. R. Curtiss
H. T. Davis
H. L. Garabedian
Griffiths
Harrold
Hellinger
Holgate
Kenney
E. J. Moulton
Scott
Simmons
Wall
Wescott
F. E. Wood
Galesburg
Sellew
R. C. Stephens
Godfrey
R. F. Jackson
Joliet
F. C. Smith
Kenilworth
W. W. Hart
Lake Forest
Curtis
Lisle
Shonka
Maywood
F. C. W. Olson
Monmouth
Beveridge
Naperville
I. F. Keeler
Northbrook
E. O. Schweitzer
Oak Park
Escott
R. E. Lane
Peoria
Comstock
Johanson
Rock Island
Cederberg
Springfield
N. W. Wells
Urbana
Blackwell
G. S. Ketchum
University of Illinois
P. H. Anderson
Armstrong
Baer
H. W. Bailey
Bourgin
O. K. Bower
Brahana
Bristow
Carmichael
P. W. Carruth
Chanler
Charnes
Coble
Crathorne
M. M. Day
Doob
Duffin
Emch
B. Epstein
R. H. Fox
R. W. Gibson
Goldman
Hansman
Hartley
Hazlett
Helmer
Hoersch
Judd
P. W. Ketchum
Leibler
H. Levy
Luke
Mendel
H. J. Miles
G. A. Miller
R. A. Miller
Nielsen
Niven
E. D. Pepper
J. W. Peters
Ramsey
Strobel
Topp
Townsend
Trjitzinsky
Turpin
Vaughan
Welker
W. Wilson
Yates City
Norval

INDIANA

Angola
V. R. Jones
Bloomington
Artin
Edmondson
Hennel
M. G. Moore
Rothrock
Stump
F. J. Weyl
Whaples
K. P. Williams
H. E. Wolfe
Collegeville
Schon
Crawfordsville
Polley
East Chicago
Godfrey
Paydon
Greencastle
Edington
Hanover
H. A. Meyer
Indianapolis
L. C. Cox
Gingery
Butler University
Getchell
Lafayette
Akeley
C. E. Clark
R. H. Downing
Eachus
A. E. Heins
Hodge
Lanczos
Langhaar

McGaughey
W. Marshall
C. D. Olds
Shreve
A. H. Smith
R. B. Stone
Webster
Westlund
Notre Dame
Gassensmith
Kelley
Kenna
Menger
Milgram
Nash
P. M. Pepper
C. V. Robinson
Paoli
Bentley
St. Mary-of-the Woods
Hayden
Terre Haute
Palmer
West Baden Springs
Lemmer

IOWA

Ames
E. S. Allen
Atanasoff
R. H. Cook
Gouwens
Herr
Higdon
Hinrichsen
Holl
Legvold
Lonseth
J. V. McKelvey
M. M. McKelvey
P. G. Robinson
E. R. Smith
H. F. Smith
Tintner
Cedar Falls
Kearney
Van Engen
Cedar Rapids
Coffin
Decorah
Jacobsen
Norlie
Des Moines
Hunter
P. C. Irwin
McCankie
MacKinnon
Nollen
Dubuque
Theobald
Grinnell
McClenon
Rusk
Hampton
Hilton
Iowa City
Chittenden
Conkwright
A. T. Craig
Gutzman
Kitchens
Knowler
R. O. Lane
C. E. Noble
Oberg
H. V. Price
Reilly
Rietz
Rigby
L. E. Ward
R. Woods
Mt. Vernon
Moots
Oskaloosa
Rothe
Sioux City
Graber
Gwinn

KANSAS

Atchison
Pretz
H. Sullivan
Emporia
O. J. Peterson
C. B. Tucker
Independence
L. E. Bell
Lawrence
W. Babcock
P. O. Bell
F. L. Black
H. E. Jordan
U. G. Mitchell
G. B. Price
G. W. Smith
Stouffer
J. J. Wheeler
Lindsborg
Marm
Manhattan
R. W. Babcock
Daugherty
Hyde
Janes
Mossman
Munro
Rawhouser
Remick
Sigley
Stratton
North Newton
Richert
Pittsburg
Shirk
R. G. Smith
St. Marys
Larguier
Salina
Arnoldy
Jensen
Topeka
R. H. Stark
Washburn College
Eberhart
W. A. Harshbarger
Wichita
Longenecker

KENTUCKY

Berea
Hutcherson
Bowling Green
Strayhorn
Yarbrough
Burlington
W. A. Barnett
Covington
Thuener
Lexington
Boyd
W. G. Clark
L. W. Cohen
H. H. Downing
F. John
Latimer
LeStourgeon
Pence
Louisville
Bullitt
Nazareth College
M. C. Fowler
University of Louisville
W. L. Moore
Stevenson
Murray
Carman
Richmond
Jenkins
R. S. Park
Williamsburg
Boswell
Winchester
Allison

LOUISIANA

Hammond
Cordrey
New Orleans
Derby
Riess
Loyola University
Fleddermann
Tulane University
H. E. Buchanan
G. F. Cramer
Duren
M. G. Humphreys
Kales
M. E. Ladue
Many
Neuhaus
M. C. Spencer
Weiss
Xavier University
J. Mayer
Ruston
Gatewood
Gentry
Kaltenborn
University
W. V. Parker
Rutt
Sanders
H. L. Smith

MAINE

Brunswick
Hammond
Holmes
Korgen
W. A. Moody
Gorham
Hadlock
Orono
Bryan
A. W. Jones
S. H. Kimball
Lamoreau
Waterville
Schoenberg

MARYLAND

Aberdeen
Dederick
Lorell
Annapolis
St. John's College
S. Buchanan
Comenetz
U. S. Naval Academy
H. C. Ayres
Ball
Bleick
Bramble
R. Church
Clements
Currier
Echols
Hawkins
Kells
Littauer
T. W. Moore
Rawlins

Root
J. B. Scarborough
L. T. Wilson
Baltimore
Dawkins
Parke
L. B. Robinson
F. D. Sutton
Goucher College
C. L. Bacon
F. P. Lewis
Torrey
Johns Hopkins University
Bourne
A. Cohen
I. S. Cohen
Collinson
Dowker
van Kampen
Kershner
A. Levine
Morrill
Murnaghan
Naim
L. J. Reed
Seidenberg
Sohl
J. Williamson
Wintner
Zariski
Cheltenham
Hartwell
Chestertown
W. J. Robinson
College Park
Dantzig
Lancaster
M. H. Martin
Newell
Titt
New Windsor
Terry
Westminster
Spicer
Woodstock
McCauley

MASSACHUSETTS

Amherst
B. L. Brown
Esty
Sprague
Belmont
C. W. Franklin
R. R. Struik
Boston
Benscoter
Brainard
T. J. Coolidge
Dysart
H. A. Garabedian
A. B. Gould
O'Connor
G. Putnam
Rines
Boston University
Bruce
Mode
Harvard University
T. H. Brown
E. B. Wilson
Northeastern University
E. E. Haskins
Spear
Suffolk University
F. X. Sutton
Brookline
A. L. Miller
Cambridge
V. A. Widder
E. W. Wilson
Harvard University
Alaoglu
Beatley
G. Birkhoff
G. D. Birkhoff
Caywood
H. E. Clifford
J. L. Coolidge
Eberlein
W. C. Graustein
Huntington
Loomis
Lyman
MacLane
von Mises
Nilson
Osgood
D. T. Perkins
B. J. Pettis
Pipes
Rabinow
Rulon
Shapley
H. M. Sheffer
M. H. Stone
J. H. Van Vleck
Walsh
H. H. Welch
Westergaard
Whiteman
P. M. Whitman
H. Whitney
D. V. Widder
Massachusetts Institute of Technology
R. H. Cameron
A. H. Clifford
Crout
P. Franklin
C. E. Fuller
Harvey
Hildebrand
N. Levinson
W. T. Martin
Moon
Nordling
H. B. Phillips
Reissner
Rule
Rutledge
D. C. Spencer
D. J. Struik
Vallarta
Wiener
F. S. Woods
Zeldin
Radcliffe College
R. B. Adams
Comegys
M. C. Graustein
Chestnut Hill
Marcou
O'Donnell
Chicopee
Madden
Hyannis
McMullen
Ipswich
Kinsman
Lynn
Bauer
Mattapan
McClay
Medford
J. L. Barnes
Ransom
Natick
R. Willis
Newton Highlands
Holley
Northampton
Munroe
Smith College
Benedict
N. H. McCoy
D. Montgomery
Rambo
Norton
C. A. Garabedian
Pittsfield
Washburne
Southbridge
Boeder
South Hadley
F. E. Baker
Litzinger
Springfield
W. B. Robinson
Swampscott
G. W. Evans
Tyngsboro
C. A. Richmond
Wellesley
L. P. Copeland
H. A. Merrill
Russell
C. E. Smith
M. E. Stark
M. M. Young
Weston
Burke
McCormick
Williamstown
Agard
J. G. Hardy
D. E. Richmond
Shepard
V. H. Wells
Winchester
M. S. Barnes
Worcester
A. H. Wheeler
Holy Cross College
Basch
O'Callahan
Worcester Polytechnic Institute
Gay
Morley
H. Rice

MICHIGAN

Ann Arbor
Herman E. Arnold
W. B. Ford
Grennan
Komm
University of Michigan
W. L. Ayres
Bartels
Bechenbach
Bradshaw
H. K. Brown
Churchill
Claytor
Coe
A. H. Copeland
C. C. Craig
Dushnik
Duthie
Dwyer
Eilenberg
Field
J. W. Glover

Goldstine
P. C. Hammer
Hay
T. H. Hildebrandt
Hopkins
S. Kaplan
W. Kaplan
Karpinski
Kazarinoff
Laporte
Love
A. V. Martin
E. W. Miller
S. B. Myers
Nesbitt
Nyswander
Odle
V. C. Poor
Rainich
Rainville
Rickart
A. Rosenthal
Running
R. L. Wilder

Detroit
Denton
M. J. Dunford
Fleiger
C. D. Jones
Weinberg

University of Detroit
J. F. Butler
Hausmann
McCarthy
Vezeau

Wayne University
Borgman
Coral
Folley
M. Friedman
J. H. Levin
D. C. Morrow
A. L. Nelson
Paxson
H. H. Pixley
M. C. Wolf

East Lansing
Beth
Dresher
V. G. Grove
Heyda
I. D. Hill
Hurd
Plant
Powell
Speeker
Van Schaack
C. P. Wells
Welmers
Zimmer

Highland Park
Tolonen

Hillsdale
Meighan

Howell
H. L. Olson

Ironwood
K. F. McLaughlin

Kalamazoo
Schneider
T. O. Walton

Marquette
Spooner

Owosso
S. W. Stewart

Royal Oak
P. Ladue

MINNESOTA

Coleraine
Kirchen

Duluth
Cothran

Minneapolis
Bearman
M. J. Norris
Thornton

University of Minnesota
H. A. Arnold
Brink
Bussey
Campaigne
S. E. Carlson
Dalaker
Gibbens
W. L. Hart
E. L. Hill
D. Jackson
F. Koehler
W. R. McEwen
Ness
Olmsted
Opatowski
Peebles
Scammon
Scherberg
Shumway
Turrittin
Underhill

Moorhead
M. R. Anderson

Northfield
Shover

Rochester
Hickman

St. Paul
Haltiner
M. P. Martin

College of St. Catherine
Wegner

College of St. Thomas
L. E. Bush
Thielman

Macalester College
E. J. Camp
Polansky

Winona
French
Seiler

MISSISSIPPI

Blue Mountain
Trott

Cleveland
J. A. Ward

Decatur
Dearman

Hattiesburg
Russ

Holly Springs
Maclin

Jackson
D. McCoy

Mathiston
Compton

State College
Ollivier
C. D. Smith
B. M. Walker
F. P. Welch

University
Hume

MISSOURI

Clayton
H. G. Brown
Rosskopf

Columbia
Haynes

Christian College
Spalding

University of Missouri
H. Betz
Blumenthal
Ewing
Michel
Schweigert
Shanks
Thurman
Wahlin
Wehausen
Westfall

Kansas City
Basye
Melcher

Rockhurst College
W. C. Doyle

St. Teresa's Junior College
Creane

University of Kansas City
Kibbey
Luby

Lexington
F. A. Miller

Parkville
Crull

St. Louis
Nagle
Rafferty
Short

Harris Teachers College
Osborn

Maryville College
Kernaghan

St. Louis College of Pharmacy
DeFoe

St. Louis University
Case
Regan

Washington University
Dunkel
Rider
Roever
Warschawski

Springfield
H'Doubler

Drury College
Finkel
W. L. Graves

Warrensburg
DeVore
J. H. Scarborough
Urban

MONTANA

Bozeman
Hurst
H. M. Schwartz

Helena
Canning
Topel

Missoula
E. F. A. Carey
Chatland
Lennes
A. S. Merrill

NEBRASKA

Lincoln
Huff

Union College
Ogden

University of Nebraska
Basoco
Brenke

C. C. Camp
Candy
Collins
Doole
Dribin
Gaba
T. A. Pierce
Rock
Runge
Omaha
Earl
Fitzpatrick
Wayne
J. W. Boyce
Hove

NEVADA

Reno
Beesley
Palm
Ross
Vance
F. Wood

NEW HAMPSHIRE

Concord
G. M. Conwell
Durham
D. C. Lewis
Slobin
Solt
Hanover
F. M. Morgan
Dartmouth College
B. H. Brown
Forsyth
C. N. Haskins
Mathewson
F. W. Perkins
R. Robinson
Silverman
C. E. Wilder

NEW JERSEY

Barber
Pratt
Belleplain
Durell
Hackettstown
LaMotte
Hightstown
Harrison
Huntley
Litterick
Hoboken
Haight
Stevens Institute of Technology
H. N. Davis
Deimel
Hazeltine
Korn
Olmstead
Jersey City
McGrath
Polachek
Montclair
Bull
Mountain Lakes
S. E. Hotelling
Newark
Charton
Conkling
H. C. Levinson
Rhodes
J. S. Thompson
Widmark
Newark College of Engineering
Fithian
Prudential Insurance Company
Bartleson
Mosesson
New Brunswick
Brasefield
Bunyan
H. S. Grant
Maker
Meder
R. Morris
C. A. Nelson
M. S. Robertson
Starke
Walter
C. R. Wilson
Paterson
Helly
Princeton
Institute for Advanced Study
J. W. Alexander
Ambrose
Bargmann
Bergmann
A. T. Brauer
Fubini
Gödel
Halmos
M. H. Heins
Maharam
W. Mayer
M. Morse
von Neumann
A. Schwartz
S. Sherman
Thrall
Veblen
H. Weyl
Princeton University
E. P. Adams
Bochner
Bohnenblust
A. A. F. Brown
Christie
A. Church
W. J. Dixon
L. P. Eisenhart
Elveback
Flood
Gillespie
Lefschetz
B. McMillan
Muhly
H. P. Robertson
Shannon
Singleton
Strodt
Tompkins
A. W. Tucker
Tuckerman
Tukey
Wallace
Wallman
Wedderburn
Wigner
Wilks
J. D. Williams
South Orange
L. M. Rauch
Stanwick
Trenton
Segal
Shuster
Upper Montclair
G. A. Campbell
New Jersey State Teachers College
D. R. Davis
Fehr
E. H. C. Hildebrandt
Mallory
Westmont
M. G. Epstein
West Orange
Edison

NEW MEXICO

Albuquerque
Barnhart
Newsom
Las Vegas
Rodgers
Portales
MacKay
Roswell
Alden
State College
Swingle

NEW YORK

Albany
Frankel
Hallett
Johannes
King
W. R. Thompson
New York State College for Teachers
Beaver
Birchenough
DoBell
Lester
E. C. Stokes
Alfred
Lowenstein
Aurora
M. P. Hollcroft
Wells College
Carroll-Rusk
T. R. Hollcroft
D. Manning
Beacon
H.-K. Hammer
Brooklyn
E. M. Blake
Deutsch
Douglas
Gilbarg
Groat
Holly
Lassar
L. L. Locke
Supnick
Wynne
Brooklyn College
Boeker
Borofsky
Boyer
Feld
Fialkow
Fleisher
Gorn
H. M. Griffin
Harkin
R. A. Johnson
Kennison
J. P. Kormes
A. W. Landers
MacNeish
A. J. Maria
D. H. Maria
L. T. Moore
Prenowitz
M. Richardson
J. E. Rosenthal
J. Singer
Sohn
J. Wolfe
Woodbridge
Long Island University
H. G. Lieber

L. R. Lieber
Polytechnic Institute of Brooklyn
W. J. Berry
Bond
Fenn
Greeley
Whitford
Pratt Institute
Cowles
J. E. Thompson
St. John's University
Cowan
St. Joseph's College for Women
Larkin
Buffalo
Coombs
O'Brian
Podmele
D'Youville College
M. M. Maloney
University of Buffalo
Gehman
Montague
E. R. Ott
Pound
Canton
Bates
Cedarhurst
R. F. Koch
Clinton
H. S. Brown
W. M. Carruth
Ferry
Crown Point
Henderson
Elmira
Suffa
Far Rockaway
L. Moore
Flushing
Hofmann
Queens College
R. G. Archibald
A. B. Brown
S. S. Cairns
Cope
N. A. Hall
Hartman
Hoffmann
Raudenbush
Sard
M. M. Sullivan
M. H. Turner
Zippin
Garden City
J. Bowden
Riblet
Geneva
Durfee
Hubbs
S. T. Parker
Hamilton
Aude
Munshower
Wardwell
Hempstead
Eaton
Stevens
Ithaca
E. A. Miller
Cornell University
Agnew
Bissinger
Carver
J. H. Curtiss
Ficken
Flexner
Givens
Golomb
Goodier
W. J. Harrington
Herzog
Hurwitz
B. W. Jones
Kac
Karapetoff
Lawrence
Malti
D. S. Miller
Monroe
Randolph
Rosser
Sharpe
V. Snyder
Stephany
G. B. Thomas
Trevor
G. L. Walker
R. J. Walker
Wray
Keuka Park
H. F. Archibald
Long Island City
Berman
Killian
New York
Alsberg
J. W. Arnold
Arrow
Barlow
M. F. Becker
Blanch
Boehm
M. A. Bowden
Brodie
L. Burgess
R. W. Burgess
Byerly
Capecelatro
R. V. Carpenter
Cosby
Crane
Dix
Downey
G. A. Ellis
F. H. Fowler
Frey
R. L. Gilman
Greitzer
Ingram
Isaacs
Jacobus
Joffe
Kagno
S. Y. Keyser
B. F. Kimball
Kiss
M. Kormes
Laderman
H. Levi
D. P. Ling
L. A. Lorch
Lydenberg
Mirick
Mittleman
Morrison
Neuman
J. R. Newman
Parsons
E. A. Porter
Preston
R. B. Robbins
Roos
Roosevelt
Scherk
Serbin
Shamos
R. C. Shook
L. G. Simon
L. G. Simons
Snook
Spitzbart
Steinman
Sternberg
Tanzola
Tepping
Traber
Vitale
W. Weaver
A. W. Whitney
E. C. Williams
York
American Mathematical Society
E. M. Hull
O. L. Wheeler
American Telephone and Telegraph Company
Belcher
Carson
Halbert
Bell Telephone Laboratories
Bode
Burton
Dietzold
R. M. Foster
Fry
Gray
Hebbert
Hoyt
P. C. Jones
MacColl
Mead
Molina
Riordan
Sandalls
Schelkunoff
Shewhart
Stibitz
Sunde
Vaage
Wise
Zobel
College of Mt. St. Vincent
O'Brien
Sheridan
College of the City of New York
Barber
Garrison
Gill
C. C. Grove
Hubert
Linehan
P. Newman
Philip
Post
F. G. Reynolds
S. L. Robinson
Shiffman
Wetzel
Wirth
Wright
Columbia University
A. C. Berry
Clingan
Crucet
Fiske
Fite
Gourin
Hawkes
H. Hotelling
Ingersoll
Jablon
Kasner
C. J. Keyser
Kolchin
Koopman
E. R. Lorch
McKean
Mullins

F. J. Murray
Northcott
Pfeiffer
C. L. Poor
Reeve
Rehberg
Ritt
Ruger
Schatten
Serghiesco
Siceloff
D. E. Smith
P. A. Smith
Sutherland
Upton
Wald
H. B. Williams
Cooper Union
Lehmann
MacDonald
F. H. Miller
Reddick
Sugar
Equitable Life Assurance Society
Murphy
Steinhaus
Fordham University
Kirby
Kubis
Murtaugh
Nardelli
Hunter College
R. L. Anderson
Aroian
Bradley
Jewell H. Bushey
J. H. Bushey
Cooper
Darkow
Guggenbühl
F. C. Hall
L. S. Hill
Infeld
M. K. Landers
M. Levin
I. C. McLaughlin
MacDuffee
M. S. Rees
Tuller
Weisner
A. M. Whelan
Manhattanville College of the Sacred Heart
Dever
Mira
Metropolitan Life Insurance Company
Alfieri
Bassford
Fisanick
J. A. Hamilton
Hohaus
Lotka
Penney
Mutual Life Insurance Company
Carner
Monsky
Strong
New York University
Adler
F. Bernstein
Cooley
Courant
Davids
Farnum
Flanders
Friedrichs
Graham
F. W. John
M. Kline
McKinsey
Mintzer
Payne
A. S. Peters
R. G. Putnam
I. F. Ritter
H. E. Robbins
S. G. Roth
Schlauch
Stoker
Tilley
Wahlert
Yanosik
Webb Institute of Naval Architecture
B. C. Keeler
Yeshiva College
Ginsburg
Lowan
Niagara University
Banks
Oneonta
K. A. Bush
A. V. Newton
Perry
G. W. Walker
Poughkeepsie
L. D. Cummings
Hopper
G. Smith
M. E. Wells
H. S. White
Wishard
Richmond
Wernick
Wolfowitz
Riverdale-on-Hudson
H. B. Mitchell
Rochester
W. Betz
Chesna
Herzberger
University of Rochester
Atkins
Galbraith
Gale
J. W. Green
Long
Seidel
Watkeys
Rockville Centre
A. M. C. Grant
St. Bonaventure
Scheier
Schenectady
General Electric Company
Alger
Poritsky
Union College
Burkett
Farrell
A. H. Fox
Garis
D. S. Morse
A. D. Snyder
Scotia
Leibing
Syracuse
Metzler
Roe
Syracuse University
A. D. Campbell
I. S. Carroll
Cowgill
Decker
J. E. Hart
Harwood
Troy
E. B. Allen
D. B. Ames
J. D. Campbell
Huston
Lengyel
Rempfer
Street
Warsaw
Schneckenburger
West Point
H. Jones
Willseyville
Seely

NORTH CAROLINA

Asheville
Frank
Chapel Hill
Blau
Browne
E. A. Cameron
Garner
Hobbs
Hoyle
Hurewicz
Jacobson
Lasley
Linker
Mackie
M. H. Williams
Clinton
Culbreth
Durham
Duke University
Boas
Carlitz
Civin
Dressel
Elliott
George
Gergen
J. A. Greenwood
Hickson
Jarnagin
Layne
E. R. C. Miles
W. L. Morris
J. H. Roberts
J. M. Thomas
L. I. Wade
North Carolina College for Negroes
A. Elder
Fallston
Lee
Greensboro
Pegram
Thaxton
Greenville
ReBarker
Raleigh
Efird
North Carolina State College
Bullock
Cell
J. M. Clarkson
H. A. Fisher
Gelbart
J. Levine
Mumford
Nahikian
H. V. Park
Winton

Salisbury
Dearborn
Patten
Wilson
R. E. Smith

NORTH DAKOTA

Fargo
I. W. Smith

OHIO

Ada
Fulton
Alliance
Hildner
Ashland
Stopher
Athens
Denbow
Marquis
D. D. Miller
F. W. Reed
Starcher
Bluffton
Hirschler
Bowling Green
Ogg
Overman
Cincinnati
Our Lady of Cincinnati College
Reingold
University of Cincinnati
I. A. Barnett
Brand
Farnau
Hancock
Horvay
Justice
Kersten
Lubin
Merriman
H. L. Miller
C. N. Moore
E. S. Smith
Szász
W. C. Taylor
Yowell
Western and Southern Life Insurance Company
Folz
Learson
Ryan
William J. Williams
Xavier University
Stechschulte
Cleveland
Burwell
Kraft
Case School of Applied Science
O. E. Brown
Burington
Dobbie
Focke
McCuskey
M. Morris
Rinehart
C. F. Thomas
C. C. Torrance
Fenn College
H. W. Alexander
W. A. Patterson
Tolar
Western Reserve University
M. G. Boyce
Musselman
Sauté
W. G. Simon
Cleveland Heights
E. M. Torrance
Columbus
G. E. Albert
Bamforth
Bareis
Blumberg
Hendrickson
Kuhn
LaPaz
Radó
Rasor
Reade
W. S. Snyder
Toops
J. H. Weaver
Wylie
Youngs
Dayton
Midgley
Delaware
Rowland
Gambier
Bumer
MacNeille
Granville
Kato
Ladner
Wiley
Green Camp
C. E. Leach
Hiram
Clarke
Kent
Brooks
F. Harshbarger
Manchester
E. J. Olson
Stelson
Marietta
T. Bennett
Sandt
Mt. St. Joseph
Corona
Oberlin
W. D. Cairns
M. M. Johnson
Sinclair
R. W. Wagner
Yeaton
Oxford
Miami University
W. E. Anderson
Pollard
Western College
Tappan
Painesville
R. M. Peters
South Euclid
Garvin
Springfield
Tripp
Toledo
Brandeberry
Wilberforce
R. Coleman
Davidson
Wilmington
Spinks
Wooster
C. O. Williamson
Yanney
Yellow Springs
Astrachan
Tornheim

OKLAHOMA

Edmond
L. W. Johnson
Langston
Forrest
Norman
LaFon
University of Oklahoma
Brixey
Court
Hassler
McFarland
Meacham
Randels
Reaves
Springer
Stillwater
J. Barnett
Diamond
O. H. Hamilton
P. E. Lewis
C. E. Marshall
Oboukhoff
H. W. Smith
Tulsa
Hodell
Ricker

OREGON

Corvallis
W. E. Milne
Scheffé
C. B. Smith
Sobczyk
Eugene
DeCou
Ghent
Kossack
Moursund
T. S. Peterson
Grants Pass
Feinler
Portland
Reed College
Goheen
F. L. Griffin
L. L. Johnson
R. A. Rosenbaum
University of Portland
Vassallo
Salem
C. F. Luther
Olshen

PENNSYLVANIA

Allentown
Deck
T. L. Koehler
Reichard
Altoona
Umberger
Annville
A. H. Black
Bethlehem
Beale
Cutler
Fort
Lamson
Latshaw
E. Park
Pitcher
Raynor
C. A. Shook
Smail
M. F. Smiley
Transue
Vanderslice
Bryn Mawr
Geiringer
Lehr
McKee
Oxtoby
A. P. Wheeler
California
Bernstorf

Carlisle
F. Ayres
Landis
Chambersburg
Davison
R. F. Johnson
Weeks
Clarion
C. E. Carey
Collegeville
Clawson
Dallas
Garrahan
Easton
R. P. Bailey
W. M. Smith
East Pittsburgh
Slepian
Emporium
Huntoon
Gettysburg
Arms
Livingood
Glenside
J. O. Patterson
Greensburg
McNeil
Harrisburg
Whited
Haverford
Allendoerfer
E. E. Betz
Gummere
Oakley
A. H. Wilson
Hazleton
Herpel
Immaculata
Anastasia Maria
Jenkintown
J. C. Durand
Kutztown
Knedler
Lancaster
W. R. Murray
LaPlume
Trapani
Latrobe
Yahner
Lewisburg
Drum
Gold
W. I. Miller
Lincoln University
Haviland
Meadville
Joseph
Narberth
W. M. Robertson
New Wilmington
H. L. Black
Philadelphia
Beamish
A. Blake
R. D. Brown
W. B. Campbell
Constable
Fudge
Hayward
McDonough
J. Sherman
Philadelphia College of Pharmacy and Science
Maddaus
Temple University
Gleason
Kratz
Lawton
C. N. Stokes
University of Pennsylvania
Beal
Caris
J. A. Clarkson
Erdös
H. B. Evans
J. R. Kline
Knobelauch
Laden
Lufkin
H. H. Mitchell
W. B. Morgan
Rademacher
Safford
Shohat
Shugert
Witmer
Pittsburgh
Briant
F. W. Brown
Cowley
Carnegie Institute of Technology
Clippinger
Dines
A. D. Hestenes
Hicks
R. P. Johnson
Landau
Morehead
Moskovitz
Neelley
E. G. Olds
Riggs
Rosenbach
Saibel
T. L. Smith
E. A. Whitman
Pennsylvania College for Women
Calkins
University of Pittsburgh
Culver
Foraker
Knipp
Speer
J. S. Taylor
Pottsville
Kendig
Kleinschmidt
Scranton
M. B. Walton
Selinsgrove
A. J. Smith
State College
H. B. Owens
Pennsylvania State College
T. Cohen
Curry
A. H. Frink
O. Frink
Gordon
C. H. Graves
Hagen
E. Johnson
Krall
E. I. Moody
F. W. Owens
Rupp
I. M. Sheffer
Stecker
C. C. Wagner
Swarthmore
Swann
Swarthmore College
Brinkmann
Dresden
Marriott
Uniontown
Sealander
Wallingford
J. A. Miller
Washington
Caton
Washington and Jefferson College
Atchison
Dorwart
Shaub

PHILIPPINE ISLANDS

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Melchor
Manila
Favila
Janairo
Jaramillo
Netzorg
Mapua Institute of Technology
Mijares
University of the Philippines
Gokhale
Perez
Virata

PUERTO RICO

Mayaguez
Morales
Sánchez-Díaz
Rio Piedras
Bobonis
Horne
Kuschke

RHODE ISLAND

Kingston
Pease
Providence
Brown University
C. R. Adams
R. C. Archibald
A. A. Bennett
Carlen
Coon
Demers
Feller
Forsythe
Frame
R. E. Gilman
D. W. Hall
Herriot
Lindsay
H. P. Manning
Neugebauer
R. G. D. Richardson
Schmidt
C. H. Smiley
Tamarkin
Providence College
McKenney
Précourt
West Warwick
Brant

SOUTH CAROLINA

Charleston
College of Charleston
Bushkovitch
The Citadel
Dye
Hedge
Columbia
Allen University
Peguese
University of South Carolina
J. B. Coleman
J. B. Jackson
Rock Hill
Royall
R. W. Stokes

Spartanburg
Ader
C. S. Pettis

SOUTH DAKOTA

Brookings
MacDougal
Walder
Wente
Rapid City
Petrie
Vermillion
Beaty

TENNESSEE

Knoxville
Eaves
Seward
Memphis
J. F. Locke
MacQueen
Nashville
J. K. Peterson
George Peabody College
Wren
Vanderbilt University
Dyer-Bennet
Miser
Sarratt
Ward-Belmont School
Clement
Falvey
Sewanee
Bruton

TEXAS

Abilene
Burnam
Mullings
Amarillo
Whetstone
Austin
Vickery
University of Texas
Batchelder
Coburn
W. B. Coleman
H. V. Craig
Decherd
Dodd
Ettlinger
R. E. Greenwood
F. B. Jones
Lubben
R. L. Moore
G. H. Porter
M. B. Porter
Sorgenfrey
Vandiver
College Station
Brewer
Chaney
Gross
Hurt
W. Jennings
Klipple
H. A. Luther
Corpus Christi
Seale
Crockett
Bright
Dallas
P. V. Montgomery
Southern Methodist University
Palmquist
P. K. Rees
Starr
Denton
E. H. Hanson
El Paso.
Leech
Fort Worth
Sherer
Houston
Rust
Slotnick
Rice Institute
Bray
A. C. Dean
Leighton
Lovett
Ulrich
H. A. Wilson
University of Houston
Aucoin
J. D. Hutchinson
Huntsville
Querry
Jacksonville
H. E. Clarkson
Kingsville
E. C. Kennedy
Lubbock
Gilbert
Ollmann
E. L. Thompson
Wakerling
Marlin
Falconer
Mercedes
W. F. Shaw
San Antonio
Incarnate Word College
Gough
Our Lady of the Lake College
M. L. Hill
St. Mary's University
F. A. Becker
San Antonio Junior College
Hurry
Tulloch
Waco
Hedberg
Walter J. Williams
Waxahachie
G. A. Newton

UTAH

Logan
Bird
Tingey
Salt Lake City
J. L. Gibson
J. J. Hayes
Horsfall
Pehrson
S. S. Smith
Stafford

VERMONT

Burlington
Bullard
Fraleigh
Swift
Winooski Park
Gleyzal

VIRGINIA

Blacksburg
Hatcher
J. E. Williams
Charlottesville
M. J. Cox
Hedlund
Linfield
McShane
R. C. Morrow
F. G. Myers
Oglesby
Sparrow
P. A. White
G. T. Whyburn
Dahlgren
Barker
L. T. E. Thompson
Ettrick
R. R. McDaniel
Fredericksburg
H. C. Carter
Hampton
H. A. Perkins
Simond
Hollins College
H. W. Allen
Hopewell
Fazel
Lexington
Virginia Military Institute
Byrne
Knox
Purdie
Washington and Lee University
Paxton
Lynchburg
Larew
Wiggin
Richmond
E. J. Willis
University of Richmond
Gaines
E. K. Ritter
C. H. Wheeler
Staunton
M. E. Taylor
Sweet Briar
N. Cole
Morenus
Williamsburg
Stetson

WASHINGTON

Lacey
Raymond
Pullman
L. G. Butler
Colpitts
Hacker
Knebelman
Seattle
J. P. Dean
Seattle Pacific College
Beegle
University of Washington
C. R. Ballantine
J. P. Ballantine
Beaumont
Benson
Birnbaum
A. F. Carpenter
Cramlet
Jerbert
McFarlan
Magnusson
Moritz
Mullemeister
Neikirk
R. S. Phillips
Taub
Winger

Zuckerman
Walla Walla
Bratton
Highberg
Woodinville
Anthony

WEST VIRGINIA

Bethany
Cramblet
Huntington
Hackney
Montgomery
W. F. Smith
Morgantown
Colwell
H. A. Davis
Eiesland
C. N. Reynolds
B. M. Turner
Philippi
B. D. Grant
Wheeling
Bagby

WISCONSIN

Beloit
Bisshopp
H. H. Conwell
Kenosha
A. E. May
Madison
Kelly
University of Wisconsin
F. E. Allen
J. H. Bell
Breit
Colvin
Cronvich
Crow
C. Eisenhart
W. S. Erickson
H. P. Evans
Hyers
Ingraham
S. B. Jackson
R. E. Johnson
Kiokemeister
Kleene
Langer
Leavitt
Lindquist
Lindstrum
March
W. L. G. Mitchell
Morkovin
Pell
Slichter
W. N. Smith
E. S. Sokolnikoff
I. S. Sokolnikoff
R. D. Specht
B. M. Stewart
Trump
Ulam
E. B. Van Vleck
Varino
Zilmer
Milwaukee
D. L. Bernstein
P. H. Evans
Marquette University
Pettit
Wilczewski
Milwaukee-Downer College
Beckwith
State Teachers College
Knight
R. E. Norris
University of Wisconsin
Bardell
Battig
Marden
Nordhaus
Parkinson
W. E. Roth
Vass
L. A. Wolf
Oshkosh
Beenken
I. Price
Platteville
Harrell
River Falls
Eide
Superior
C. W. Smith
West De Pere
DeCleene
Sromovsky
Wisconsin Rapids
M. B. McMillan

WYOMING

Lamont
Bellamy
Laramie
Rechard

CANADA

Alberta

Airdrie
Derry
Edmonton
J. W. Campbell
A. J. Cook
Sheldon

British Columbia

Vancouver
D. Buchanan
Gage
R. Hull
S. A. Jennings
Nowlan

Manitoba

Winnipeg
N. R. Wilson

New Brunswick

Sackville
W. H. McEwen

Nova Scotia

Halifax
Adshead
Walmsley
Wolfville
Jeffery
Macphail

Ontario

Hamilton
Findlay
Kingston
Halperin
Matheson
N. Miller
Watts
London
R. H. Cole
Kingston
Ottawa
G. V. Brady
Wyman
Toronto
Wolfenden
Ontario College of Education
A. A. Grant
University of Toronto
Beatty
R. Brauer
Burk
Coxeter
DeLury
Gilchrist
Krieger
Pounder
D. A. F. Robinson
G. de B. Robinson
Synge
Waddell
W. J. Webber
Victoria College
S. H. Gould

Quebec

Lennoxville
A. V. Richardson
Montreal
MacLean
Pall
A. N. Shaw
C. T. Sullivan
W. L. G. Williams
Quebec
Tremblay

Saskatchewan

Saskatoon
R. D. James
G. H. Ling
Murdoch

ARGENTINA

Buenos Aires
Baidaff
Barral-Souto
González-Dominguez
Rosario
B. Levi

AUSTRALIA

Adelaide
Wilton
Brisbane
Raybould
Hobart
J. C. Jaeger
Melbourne
Cherry
Perth
Weatherburn
Sydney
Carslaw
Thorne

BELGIUM

Ostend
Bouckaert

CANAL ZONE

Gatun
Eberman

CHINA

Peking
Hsü
Shanghai
Fan

COLOMBIA

Bogota
R. H. Cramer

CUBA

Havana
Corral-Aléman

DENMARK

Copenhagen
Royal Technical College
Jessen
University of Copenhagen
Bohr
Nörlund

ENGLAND

Birmingham
G. N. Watson
Bristol
Fraser
Cambridge
J. S. Turner
University of Cambridge
Collingwood
P. Hall
G. H. Hardy
Ingham
Larmor
H. W. Richmond
Smithies
Devon
Perryman
Hindhead Surrey
Western
Leeds
W. P. Milne
London
Brill
Dalal
University of London
Semple
H. Simpson
Taussky
Manchester
Mordell
Oxford
A. L. Dixon
Frecheville
J. H. C. Whitehead
Stocksfield-on-Tyne
Hardcastle
Suffolk
Curjel

FINLAND

Helsingfors
Ahlfors

FRANCE

Paris
Catholic Institute
Potron
University of Paris
Cartan
Fréchet
Rennes
Chevalley

GERMANY

Freiburg i. Br.
Bolza
Giessen
Geppert
Hannover
Prange
Leipzig
Ackermann-Teubner
Hopf

GREECE

Athens
Lanckton

HUNGARY

Budapest
Féjer
C. Jordan
de Kerékjárto
Szeged
Riesz

INDIA

Aligarh
Shah
Annamalainagar
Rao
Bangalore
Iyengar
Dacca
Vijayaraghavan
Delhi
Seth
Madras
Ananda-Rau
Poona
Paranjpye

IRAQ

Baghdad
Merrick

IRELAND

Dublin
A. J. McConnell
Rowe

ITALY

Milan
Ricci
Pisa
Tonelli
Rome
Levi-Civita
Volterra
Turin
Tricomi

JAPAN

Osaka
Nakayama
Sapporo
Kawaguchi
Sendai
Kubota
Takasu
Tokyo
Kuniyeda
Yoshida

LATVIA

Riga
Leimanis

MEXICO

Mexico
Salazar y Arce

NETHERLANDS

Delft
van Dantzig

NEW ZEALAND

Auckland
Forder
Dunedin
Martyn

NORTHERN IRELAND

Belfast
Gillis
McCrea
Povey

PALESTINE

Jerusalem
Levitzki
Nablus
Tukan
Petah Tikvah
Preiser

PERU

Lima
Catholic University of Peru
de Losada y Puga
University of San Marcos
García
Rosenblatt

SCOTLAND

Dundee
Copson
Edinburgh
Whittaker

SOUTH AFRICA

Cape Town
de Wet

STRAITS SETTLEMENTS

Singapore
Oppenheim

SWEDEN

Stockholm
Oseen

THAILAND

Bangkok
Vadhana

TURKEY

Istanbul
W. S. Kimball

U.S.S.R.

Kieff
Kryloff

WALES

Bangor
Berwick
Cardiff
Livens
Swansea
A. R. Richardson

BY-LAWS OF THE AMERICAN MATHEMATICAL SOCIETY

ARTICLE I

OFFICERS

Section 1. The officers of the Society shall be a President, three Vice Presidents, a Secretary, four Associate Secretaries, a Treasurer, a Librarian, and four Editorial Committees, one of four members for the *Bulletin*, one of three members for the *Transactions*, one of three members for the *Colloquium Publications*, and one consisting of three representatives of the Society on the Board of Editors of the *American Journal of Mathematics*.

Section 2. It shall be a duty of the President to deliver an address before the Society at the close of his term of office or within one year thereafter.

Section 3. The Librarian shall have charge of arrangements for the exchange of the Society's publications.

ARTICLE II

BOARD OF TRUSTEES

Section 1. There shall be a Board of Trustees consisting of five trustees elected by the Society, in accordance with Article IV.

Section 2. The function of the Board of Trustees shall be to receive and administer the funds of the Society, to have full legal control of its investments and properties, to make contracts and, in general, to conduct all business affairs of the Society.

Section 3. The Board of Trustees shall have the power to appoint a manager and such assistants and agents as may be necessary or convenient to facilitate the conduct of the affairs of the Society, and to fix the terms and conditions of their employment. The Board may delegate to the officers of the Society duties and powers normally inhering in their respective corporative offices, subject to supervision by the Board. The Board of Trustees may appoint committees to facilitate the conduct of the business of the Society and delegate to such committees such powers as may be necessary or convenient for the proper exercise of those powers. Agents appointed, or members of committees designated, by the Board of Trustees need not be members of the Board.

Nothing herein contained shall be construed to empower the Board of Trustees to divest itself of responsibility for, or legal control of, the investments, properties and contracts of the Society.

ARTICLE III

COUNCIL

Section 1. The Council shall consist of the officers specified in Article I, of ex-secretaries who have served as Secretary for ten years or more, of ex-presidents for a period of six years after the expiration of their respective presidential terms, and of fifteen additional elected members chosen in accordance with Article IV, Section 2.

Section 2. Its function shall be to formulate and administer the scientific policies of the Society and to act in an advisory capacity to the Board of Trustees.

Section 3. At meetings of the Council five members shall constitute a quorum for the transaction of business. Between meetings business may be transacted by a mail vote. This shall require at least fifteen favorable votes and also that the number of these votes be three-quarters of those transmitted to the Secretary before the time announced for the closing of the polls. In case three or more members of the Council request postponement, at time of voting, action on the matter at issue shall be postponed until the next meeting of the Council, unless at the discretion of the Secretary the question is made the subject of a second letter vote, in connection with which brief statements of reasons, pro and con, are circulated to members.

ARTICLE IV

ELECTION AND TERM OF OFFICE OF OFFICERS AND TRUSTEES

Section 1. The term of office shall be two years for the trustees, the President, the Vice Presidents, the Secretary, the Associate Secretaries, and the Treasurer; three years each for the Librarian, the Editorial Committees of the *Bulletin*, the *Transactions*, and the *Colloquium Publications*, and the fifteen elected members of the Council, five of the latter retiring annually; five years for the representatives of the Society on the Board of Editors of the *American Journal of Mathematics*. In every case, however, the trustees and officers shall continue until their successors shall have been duly elected and qualified. The President, the Vice Presidents, and the fifteen elected members of the Council shall not be eligible for immediate reelection to their respective offices.

Section 2. Election of trustees, officers, and members of the Council shall be by ballot at the Annual Meeting. An official ballot shall be sent to each member by the Secretary at least one month prior to the Annual Meeting, and such ballots, if returned to the Secretary in envelopes bearing the names of the voters and received prior to the closing of the polls, shall be counted at the Annual Meeting. Each ballot shall contain a name proposed by the Council for each office to be filled, with blank spaces in which the voter may substitute other names. A majority of all votes cast, whether in person or by mail, shall be necessary for election. In case of failure to secure a majority for any office, the members present at the Annual Meeting shall choose by ballot between the two having the highest number of votes.

Section 3. If the President of the Society die or resign, before the expiration of his term of office, the Council may, with the approval of the trustees, designate one of the Vice Presidents to serve as Acting President until the next Annual Meeting, when a President shall be elected by the Society. Such vacancies as may occur at any time among the other officers may be filled by the Council with the approval of the trustees.

Section 4. If any trustee die or resign during his tenure of office, the vacancy thus created shall be filled for his unexpired term by the Board of Trustees.

Section 5. If any elected member of the Council die or resign more than one year before the expiration of his term, the vacancy for the unexpired term shall be filled by the Society at the next Annual Meeting.

ARTICLE V

ELECTION OF MEMBERS

Section 1. Election of members shall be by vote of the Council.

Section 2. There shall be four classes of members, ordinary, contributing, institutional contributing, and sustaining.

Section 3. A firm, corporation, institution, association or individual interested in the support of mathematics, may be elected to sustaining membership. A sustaining member shall receive both the *Bulletin* and the *Transactions* (see Article X), and shall have the privilege of nominating one or more persons for election by the Council to ordinary membership in the Society, the number to be determined by the Council. Such nominations shall not be acted upon until at least thirty days after their presentation to the Council (at a meeting or by mail).

Section 4. A firm, corporation, institution, or association interested in the support of mathematics may be elected to institutional contributing membership. A contributing member shall receive the *Bulletin* or have the privilege, for each twenty-five dollars contributed, of nominating one person to ordinary membership in the Society. Such nominations shall not be acted upon until at least thirty days after their presentation to the Council (at a meeting or by mail).

Section 5. Except in the case of nominees of sustaining members and of institutional contributing members, application for admission to ordinary membership shall be made by the applicant, on a blank provided by the Secretary, and shall be approved by two members of the Society. Such applications shall not be acted upon until at least thirty days after their presentation to the Council (at a meeting or by mail), except in the case of members of other societies entering under special action of the Council.

Section 6. An ordinary member may become a contributing member by paying the dues for such membership.

ARTICLE VI

DUES

Section 1. Persons elected to ordinary membership in the Society by the Council, under the provisions of Article V, Section 5, shall be admitted to membership upon the payment, within sixty days of the date of their election, of an initiation fee of five dollars.

Section 2. The annual dues of persons elected by the Council to ordinary membership under the provisions of Article V, Section 5, shall be eight dollars, with the following exceptions: (1) during the first three years of membership, the annual dues shall be six dollars; a person shall be considered to have completed his first year of membership on January 1 following his election; (2) the amount of dues may be altered by reciprocity agreements with other societies; (3) the Council may make special rulings in exceptional cases, with the approval of the Board of Trustees; four dollars and fifty cents of the dues of each member shall be for a year's subscription to the *Bulletin*. Each new member shall pay in proportion to the unexpired fraction of the year at the time of his election.

Section 3. The minimum dues for a contributing member shall be fifteen dollars per year. Members may, upon their own initiative, pay larger dues.

Section 4. The dues for an institutional contributing member for any year shall not be less than twenty-five dollars. Institutions may pay larger dues.

Section 5. The dues of a sustaining member for any year shall not be less than one hundred dollars. A sustaining member who contributes annually at least five hundred dollars shall be designated as a Patron of the Society.

Section 6. Persons elected to ordinary membership as nominees of sustaining members or institutional contributing members under the provisions of Article V, Sections 3-4, shall not be required to pay an initiation fee. They shall not be required to pay dues, so long as it is agreed that they are designated by the sustaining member or institutional contributing member under these provisions. If a nominee of a sustaining member or an institutional contributing member later becomes a dues-paying member, he shall pay dues at the rate of six dollars a year for the remainder (if any) of his first three years of membership, and eight dollars a year thereafter.

Section 7. If the annual dues of any member remain unpaid beyond a reasonable time, the Board of Trustees shall remove his name from the list of members, after due notice.

Section 8. Any member not in arrears of dues may become a life member on the payment of a sum to be determined in accordance with actuarial principles.*

Section 9. Any member who has been retired from active service on account of age and has been a member of the Society for twenty years or more may, upon notification to the Secretary of such retirement, have his dues remitted, on the understanding that he will thereafter receive the programs of the meetings, but not the *Bulletin*.

ARTICLE VII

MEETINGS

Section 1. The Annual Meeting of the Society shall be held between the fifteenth of December and the fifteenth of January next following. Notice of the time and place of this meeting shall be mailed by the Secretary or an Associate Secretary to the last known post office address of each member of the Society. The times and places of the Annual and other meetings of the Society shall be designated by the Council.

Section 2. The President may call a meeting of the Council whenever the affairs of the Society make it desirable; he shall call a meeting of the Council on written request of five of its members.

Section 3. The Board of Trustees shall have its regular Annual Meeting on the first day of the Annual Meeting of the Society. Special meetings of the Board of Trustees may be called by the Chairman of the Board upon three days' notice of such meeting mailed to the last known post office address of each trustee. He shall call a meeting upon the receipt of a written request of two of the trustees. Meetings for the transaction of business may also be held by common consent of all the trustees.

ARTICLE VIII

SECTIONS

Whenever it shall appear to the Council that a sufficient number of members of the Society are desirous of conducting in any locality periodic meetings for the reading and discussion of mathematical papers, the Council may authorize the formation of a Section to be composed at each sectional meeting of such members of the Society as may be present; and the Council shall have the right to withdraw such authorization.

ARTICLE IX

PROGRAMS

Section 1. Papers intended for presentation at any meeting or sectional meeting of the Society shall be passed upon in advance by a program committee ap-

* See table on page 6.

pointed by or under the authority of the Council; and only such papers shall be presented as shall have been approved by such committee. Papers in form unsuitable for publication, if accepted for presentation, shall be referred to on the program as preliminary communications or reports.

Section 2. No matter of general business shall be considered at any meeting of the Society except the Annual Meeting without the recommendation of the Council.

ARTICLE X

PUBLICATIONS

Section 1. The Society shall publish an official organ called the *Bulletin of the American Mathematical Society* which shall be sent to each member of the Society; four dollars and fifty cents of the dues paid by each member shall be for a year's subscription to this publication. It shall publish a series of volumes called *Colloquium Publications,* which shall embody in book form new mathematical developments. It shall publish a journal, the object of which shall be to make known important researches presented at meetings of the Society; this journal shall be called the *Transactions of the American Mathematical Society.* It shall also coöperate in the conduct of the *American Journal of Mathematics.*

Section 2. The editorial management of the *Bulletin, Transactions,* and *Colloquium Publications* and the participation of the Society in the editorial management of the *American Journal of Mathematics* shall be in charge of committees as specified in Article I.

ARTICLE XI

AMENDMENTS

These By-Laws may be amended or suspended at any meeting of the Society on recommendation of the Council and by a two-thirds vote of the members present, provided notice of such proposed action and of its general nature shall have been given in the call for such meeting.

REPORT OF THE TREASURER FOR THE PERIOD DECEMBER 1, 1937 TO NOVEMBER 30, 1938

BALANCE SHEET

November 30, 1938

Assets

CASH IN BANKS:			
Corn Exchange Bank Trust Company	$ 9,531.78		
Bank for Savings	2,782.29		
Union Dime Savings Bank	5,986.55	$ 18,300.62	
INVESTMENTS*		124,318.99	
Total Assets			$142,619.61

Reserves and Surplus

RESERVES:			
Endowment Fund	$70,481.07		
Bôcher Fund	2,182.46		
Cole Fund	3,323.82		
Moore Fund	4,319.16		
Reilly Fund	17,733.35		
Life Memberships	5,763.88		
Sinking Fund	9,980.77		
Investment Losses	665.44		
International Congress	3,884.63		
Bulletin Reprinting and Index Fund	498.15		
Transactions Reprinting and Index Fund	220.43		
Transactions Life Subscription	116.60		
Colloquium	10,813.91	$129,983.67	
SURPLUS AT NOVEMBER 30, 1938		12,635.94	
Total Reserves and Surplus			$142,619.61

Statement of General Receipts and General Disbursements

GENERAL RECEIPTS:		
Dues from Ordinary Memberships	$13,406.16	
Dues from Contributing Memberships	1,124.96	
Dues from Institutional Memberships	6,455.00	
Initiation Fees	629.86	
Interest Income from Endowment Fund and General Investments	3,762.48	
Miscellaneous	566.25	
Total General Receipts		$25,944.71

* Value of investments, exclusive of Brennan, Conlon, Sullivan and Davenport bonds and mortgages, based on market quotations on November 30, 1938, is $88,666.80.

GENERAL DISBURSEMENTS:		
Secretaries (clerical)	$ 4,894.84	
Treasurer	615.65	
Officers' Traveling	476.40	
Office Furniture and Fixtures	169.22	
Library	436.24	
Membership Committee	28.93	
Gibbs Lecture	111.48	
American Journal Subvention	2,500.00	
Emergency	61.60	
Total General Disbursements		9,294.36
EXCESS OF GENERAL RECEIPTS OVER GENERAL DISBURSEMENTS		$16,650.35

SURPLUS AT DECEMBER 1, 1937		$18,142.71
Additions:		
Adjustment—Amortization of Investments	$ 70.79	
Adjustment—Life Membership Reserve	225.38	
Adjustment—Transactions Subscription Reserve	1.19	
Excess of General Receipts over General Disbursements (see above)	16,650.35	16,947.71
Total		$35,090.42
Deductions:		
Appropriation to:		
Investment Account for Bonds in Default	$ 3,600.00	
Reserve for Investment Losses	665.44	
Sinking Fund	4,000.00	
Bulletin Account	9,189.52	
Transactions Account	3,249.52	
Transactions Reprinting Account	500.00	
Colloquium Account	1,250.00	
SURPLUS AT NOVEMBER 30, 1938		$12,635.94

BULLETIN

Receipts:		
Subscription Sales	$ 1,694.64	
Reprint Sales	73.80	$ 1,768.44
Appropriation from General Receipts		9,189.52
Total		$10,957.96
Disbursements:		
Printing (2 numbers for 1937, 10 for 1938)	$ 6,757.79	
List of Members	620.13	
Salaries, Annuity and Pension	3,093.49	
Advertising	20.16	
Editors' Traveling	5.00	
Miscellaneous—Office Furniture, Supplies, etc.	461.39	10,957.96

Bulletin Reprinting and Index

BALANCE AT DECEMBER 1, 1937		$ 108.55
Sales of Back Volumes		389.60
BALANCE AT NOVEMBER 30, 1938		$ 498.15

Transactions

Receipts:		
Subscription Sales	$ 4,425.30	
Back Volume Sales	684.30	
Reprint Sales	208.22	
Contributions	17.00	$ 5,334.82
Appropriation from General Receipts		3,249.52
Total		$ 8,584.34
Disbursements:		
Printing (1 number for 1937, 5 for 1938)	$ 6,499.92	
Salaries, Annuity and Pension	1,755.45	
Advertising	20.16	
Miscellaneous—Postage, Supplies, etc	308.81	8,584.34

Transactions Reprinting and Index

BALANCE AT DECEMBER 1, 1937		$ 38.59
Appropriation from General Receipts		500.00
Total		$ 538.59
Reprinting		318.16
BALANCE AT NOVEMBER 30, 1938		$ 220.43

Colloquium

BALANCE AT DECEMBER 1, 1937		$ 9,806.32
Receipts:		
Sales of Colloquium Volumes	$ 1,976.47	
Sales of Colloquium Volume 16 (Bliss) for National Research Council	133.39	
Appropriation from General Receipts	1,250.00	3,359.86
Total		$13,166.18
Disbursements:		
Printing (Volume 22)	$ 970.58	
Salaries	969.49	
Miscellaneous—Shipping, Supplies, etc. (including cost of shipping Volume 16 for National Research Council)	220.18	
Circulars and Advertising	62.13	
Payment to National Research Council for Volume 16 (Bliss)	114.89	
Colloquium Lecture Expenses	15.00	2,352.27
BALANCE AT NOVEMBER 30, 1938		$10,813.91

INTERNATIONAL CONGRESS

Contributions	$ 4,500.00
Disbursements	615.37
BALANCE AT NOVEMBER 30, 1938	$ 3,884.63

SEMICENTENNIAL

BALANCE AT DECEMBER 1, 1937		$ 2,000.00
Receipts:		
Registration Fees, Ticket Sales, etc.	$ 3,307.28	
Publications	1,019.31	4,326.59
Appropriation from Sinking Fund		2,019.23
Total		$ 8,345.82
Disbursements:		
Meeting Expenses	$ 3,650.57	
Publications	4,295.25	
Enlarged List of Members	400.00	8,345.82

SPECIAL FUNDS

	Total	Principal	Income
ENDOWMENT FUND			
Balance at December 1, 1937	$70,456.07	$70,456.07	
Additions:			
Income and Gifts	2,444.39	25.00	$2,419.39
Adjustment for Amortization of Investments	45.52		45.52
Total	$72,945.98	$70,481.07	$2,464.91
Deduction:			
Interest Distributed to General Income	2,464.91		2,464.91
Balance at November 30, 1938	$70,481.07	$70,481.07	
BÔCHER FUND			
Balance at December 1, 1937	$ 2,107.49	$ 1,188.00	$ 919.49
Additions:			
Income	73.59		73.59
Adjustment for Amortization of Investments	1.38		1.38
Balance at November 30, 1938	$ 2,182.46	$ 1,188.00	$ 994.46
COLE FUND			
Balance at December 1, 1937	$ 3,211.35	$ 2,093.13	$1,118.22
Additions:			
Income	110.39		110.39
Adjustment for Amortization of Investments	2.08		2.08
Balance at November 30, 1938	$ 3,323.82	$ 2,093.13	$1,230.69

MOORE FUND

Balance at December 1, 1937	$ 4,173.89	$ 2,100.62	$2,073.27
Additions:			
Income	142.59		142.59
Adjustment for Amortization of Investments	2.68		2.68
Balance at November 30, 1938	$ 4,319.16	$ 2,100.62	$2,218.54

REILLY FUND

Balance at December 1, 1937	$ 6,729.84	$ 6,531.00	$ 198.84
Additions:			
Income	510.56		510.56
Adjustment for Amortization of Investments	9.61		9.61
Received from Estate of Marion Reilly	10,483.34	10,483.34	
Balance at November 30, 1938	$17,733.35	$17,014.34	$ 719.01

Bennington P. Gill
Treasurer

New York, N.Y.
November 30, 1938

REPORT OF THE AUDITING COMMITTEE
1938

We, the undersigned Auditing Committee, appointed by the American Mathematical Society, have this day audited the accounts of the Treasurer and in our opinion the following statement is correct:

CASH BALANCE AT DECEMBER 1, 1937:		
Corn Exchange Bank Trust Company	$3,579.88	
Bank for Savings	2,727.35	
Union Dime Savings Bank	5,868.30	$12,175.53
Cash Receipts—December 1, 1937 to November 30, 1938		46,612.49
Interest on Savings Accounts in Bank for Savings and Union Dime Savings Bank		173.19
Total		$58,961.21
Cash Disbursements—December 1, 1937 to November 30, 1938		40,660.59
CASH BALANCE AT NOVEMBER 30, 1938:		
Corn Exchange Bank Trust Company	$9,531.78	
Bank for Savings	2,782.29	
Union Dime Savings Bank	5,986.55	
		$18,300.62

Ralph G. Archibald
Albert E. Meder, Jr.

New York, N.Y.
December 20, 1938

REPORT OF THE TREASURER FOR THE PERIOD DECEMBER 1, 1938 TO NOVEMBER 30, 1939

BALANCE SHEET

November 30, 1939

ASSETS

CASH IN BANKS:			
Corn Exchange Bank Trust Company	$ 3,037.95		
City Bank Farmers Trust Company	5,554.34		
Bank for Savings	2,838.35		
Emigrant Industrial Savings Bank	7,508.88		
Providence Institution for Savings	3,500.00		
Union Dime Savings Bank	6,107.17	$ 28,546.69	
INVESTMENTS*		174,519.86	
Total Assets			$203,066.55

RESERVES AND SURPLUS

RESERVES:			
Endowment Fund	$70,506.07		
Bôcher Fund	2,171.62		
Brown Fund	1,012.52		
Cole Fund	3,466.50		
Moore Fund	4,504.31		
Reilly Fund	18,494.11		
Life Memberships	5,720.15		
Sinking Fund	10,258.72		
Investment Losses	1,655.49		
International Congress	5,716.62		
Bulletin Reprinting and Index Fund	311.76		
Transactions Reprinting and Index Fund	220.43		
Transactions Life Subscription	115.37		
Colloquium	8,075.38		
Mathematical Reviews	61,431.89	$193,660.94	
SURPLUS AT NOVEMBER 30, 1939		9,405.61	
Total Reserves and Surplus			$203,066.55

STATEMENT OF GENERAL RECEIPTS AND GENERAL DISBURSEMENTS

GENERAL RECEIPTS:		
Dues from Ordinary Memberships	$13,541.71	
Dues from Contributing Memberships	1,163.92	
Dues from Institutional Memberships	6,530.00	
Initiation Fees	299.64	
Interest Income from Endowment Fund and General Investments	4,481.26	
Miscellaneous	456.24	
Total General Receipts		$26,472.77

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* Value of investments, exclusive of Brennan, Sullivan and Conlon bonds and mortgages, based on market quotations on November 30, 1939, is $146,375.13.

GENERAL DISBURSEMENTS:		
Secretaries (clerical)	$ 5,500.00	
Treasurer	644.18	
Officers' Traveling	700.00	
Library	1,146.13	
Membership Committee	200.00	
American Journal Subvention	2,500.00	
Emergency	333.36	
Accrued Interest Purchased	191.45	
Total General Disbursements		11,215.12
EXCESS OF GENERAL RECEIPTS OVER GENERAL DISBURSEMENTS		$15,257.65

SURPLUS AT DECEMBER 1, 1938		$12,635.94
Additions:		
Adjustment—Amortization of Investments	$ 47.65	
Adjustment—Life Membership Reserve	273.74	
Adjustment—Transactions Subscription Reserve	1.23	
Excess of General Receipts over General Disbursements (see above)	15,257.65	15,580.27
Total		$28,216.21
Deductions:		
Appropriation to:		
Investment Account for Bonds in Default	$ 3,600.00	
Reserve for Investment Losses	990.05	
Bulletin Account	9,632.72	
Transactions Account	3,337.83	
Colloquium Account	750.00	
Mathematical Reviews Account	500.00	18,810.60
SURPLUS AT NOVEMBER 30, 1939		$ 9,405.61

BULLETIN

Receipts:		
Subscription Sales	$ 1,785.09	
Reprint Sales	81.39	$ 1,866.48
Appropriation from General Receipts		9,632.72
Total		$11,499.20
Disbursements:		
Printing (2 numbers for 1938, 10 for 1939)	$ 7,419.04	
Salaries, Annuity and Pension	3,223.84	
Advertising	37.82	
Editors' Traveling	210.50	
Miscellaneous—Office Furniture, Supplies, etc.	608.00	11,499.20

BULLETIN REPRINTING AND INDEX

BALANCE AT DECEMBER 1, 1938	$ 498.15
Sales of Back Volumes	409.12
Total	$ 907.27
Reprinting	595.51
BALANCE AT NOVEMBER 30, 1939	$ 311.76

TRANSACTIONS

Receipts:		
Subscription Sales	$ 4,846.73	
Back Volume Sales	714.28	
Reprint Sales	186.64	
Contributions	17.00	$ 5,764.65
Appropriation from General Receipts		3,337.83
Total		$ 9,102.48
Disbursements:		
Printing (1 number for 1938, 5 for 1939)	$ 6,792.53	
Salaries, Annuity and Pension	1,710.00	
Advertising	37.82	
Editors' Traveling	260.80	
Miscellaneous—Postage, Supplies, etc.	301.33	9,102.48

TRANSACTIONS REPRINTING AND INDEX

BALANCE AT DECEMBER 1, 1938	$ 220.43
BALANCE AT NOVEMBER 30, 1939	$ 220.43

COLLOQUIUM

BALANCE AT DECEMBER 1, 1938		$10,813.91
Receipts:		
Sales of Colloquium Volumes	$ 2,404.51	
Sales of Colloquium Volume 16 (Bliss) for National Research Council	105.22	2,509.73
Appropriation from General Receipts		750.00
Total		$14,073.64
Disbursements:		
Printing (Volumes 23 and 24)	$ 4,559.28	
Salaries	1,006.25	
Miscellaneous—Shipping, Supplies, etc. (including cost of shipping Volume 16 for National Research Council)	240.00	
Circulars and Advertising	61.52	
Payment to National Research Council for Volume 16 (Bliss)	81.21	
Honorarium	50.00	5,998.26
BALANCE AT NOVEMBER 30, 1939		$ 8,075.38

Mathematical Reviews

Receipts:		
Contributions	$65,000.00	
Subscription Sales	328.53	
Income	379.85	$65,708.38
Appropriation from General Receipts		500.00
Total		$66,208.38
Disbursements:		
Salaries	$ 2,421.00	
Periodicals, Books, etc.	175.38	
Circulars and Advertising	411.42	
Traveling	362.86	
Office Furniture	824.23	
Miscellaneous	581.60	4,776.49
BALANCE AT NOVEMBER 30, 1939		$61,431.89

International Congress

BALANCE AT DECEMBER 1, 1938		$ 3,884.63
Receipts:		
Contributions and Fees		3,805.00
Total		$ 7,689.63
Disbursements:		
Clerical	$ 1,251.23	
Stationery and Printing	436.93	
Traveling	61.15	
Miscellaneous—Supplies, Telegrams, Furniture, etc.	223.70	1,973.01
BALANCE AT NOVEMBER 30, 1939		$ 5,716.62

Special Funds

	Total	Principal	Income
ENDOWMENT FUND			
Balance at December 1, 1938	$70,481.07	$70,481.07	
Additions:			
Income and Gifts	3,017.78	25.00	$2,992.78
Adjustment for Amortization of Investments	31.82		31.82
Total	$73,530.67	$70,506.07	$3,024.60
Deduction:			
Interest Distributed to General Income	3,024.60		3,024.60
Balance at November 30, 1939	$70,506.07	$70,506.07	

BÔCHER FUND

Balance at December 1, 1938	$ 2,182.46	$ 1,188.00	$ 994.46
Additions:			
Income	88.22		88.22
Adjustment for Amortization of Investments	.94		.94
Total	$ 2,271.62	$ 1,188.00	$1,083.62
Deduction:			
Prize	100.00		100.00
Balance at November 30, 1939	$ 2,171.62	$ 1,188.00	$ 983.62

BROWN FUND

Received from Estate of E. W. Brown	$ 1,000.00	$ 1,000.00	
Income	12.37		$ 12.37
Adjustment for Amortization of Investments	.15		.15
Balance at November 30, 1939	$ 1,012.52	$ 1,000.00	$ 12.52

COLE FUND

Balance at December 1, 1938	$ 3,323.82	$ 2,093.13	$1,230.69
Additions:			
Income	141.18		141.18
Adjustment for Amortization of Investments	1.50		1.50
Balance at November 30, 1939	$ 3,466.50	$ 2,093.13	$1,373.37

MOORE FUND

Balance at December 1, 1938	$ 4,319.16	$ 2,100.62	$2,218.54
Additions:			
Income	183.20		183.20
Adjustment for Amortization of Investments	1.95		1.95
Balance at November 30, 1939	$ 4,504.31	$ 2,100.62	$2,403.69

REILLY FUND

Balance at December 1, 1938	$17,733.35	$17,014.34	$ 719.01
Additions:			
Income	752.76		752.76
Adjustment for Amortization of Investments	8.00		8.00
Balance at November 30, 1939	$18,494.11	$17,014.34	$1,479.77

BENNINGTON P. GILL
Treasurer

New York, N.Y.
November 30, 1939

REPORT OF THE AUDITING COMMITTEE

1939

We, the undersigned Auditing Committee, appointed by the American Mathematical Society, have this day audited the accounts of the Treasurer and in our opinion the following statement is correct:

CASH BALANCE AT DECEMBER 1, 1938:		
Corn Exchange Bank Trust Company	$9,531.78	
Bank for Savings	2,782.29	
Union Dime Savings Bank	5,986.55	$ 18,300.62
Cash Receipts—December 1, 1938 to November 30, 1939		116,562.39
Interest on Savings Accounts in Union Dime Savings Bank, Bank for Savings and Emigrant Industrial Savings Bank		185.56
Total		$135,048.57
Cash Disbursements—December 1, 1938 to November 30, 1939		106,501.88
CASH BALANCE AT NOVEMBER 30, 1939:		
Corn Exchange Bank Trust Company	$3,037.95	
Bank for Savings	2,838.35	
City Bank Farmers Trust Company	5,554.34	
Emigrant Industrial Savings Bank	7,508.88	
Providence Institution for Savings	3,500.00	
Union Dime Savings Bank	6,107.17	
		$ 28,546.69

RALPH G. ARCHIBALD
ALBERT E. MEDER, JR.

New York, N.Y.
December 18, 1939

PUBLICATIONS

Journals

Bulletin of the American Mathematical Society.
Transactions of the American Mathematical Society.
Mathematical Reviews.

Books

F. Klein, **The Evanston Colloquium Lectures on Mathematics**, 1894, x, 109 pp.; reprinted, 1911, xii, 109 pp.	$1.00
Mathematical Papers Read at the International Mathematical Congress, 1896, xvi, 411 pp.	4.70
Semicentennial Publications.	
1. R. C. Archibald, **A Semicentennial History of the American Mathematical Society, 1888-1938, with Biographies and Bibliographies of the Past Presidents**, 1938, xii, 261 pp., 30 plates.	3.00
2. **Semicentennial Addresses**, 1938, vi, 315 pp.	4.50

Colloquium Series

1.	H. S. White, **Linear Systems of Curves on Algebraic Surfaces**; F. S. Woods, **Forms of Non-Euclidean Space**; E. B. Van Vleck, **Selected Topics in the Theory of Divergent Series and of Continued Fractions**; 1905, xii, 187 pp.	3.00
2.*	E. H. Moore, **Introduction to a Form of General Analysis**; M. Mason, **Selected Topics in the Theory of Boundary Value Problems of Differential Equations**; E. J. Wilczynski, **Projective Differential Geometry**; 1910, x, 222 pp.	
3_1.	G. A. Bliss, **Fundamental Existence Theorems**, 1913, x, 107 pp.; reprinted, 1934, iv, 107 pp.	2.00
3_2.	E. Kasner, **Differential-Geometric Aspects of Dynamics**, 1913, iv, 117 pp.; reprinted, 1934, iv, 117 pp.	2.00
4.*	L. E. Dickson, **On Invariants and the Theory of Numbers**; W. F. Osgood, **Topics in the Theory of Functions of Several Complex Variables**; 1914, xviii, 230 pp.	
5_1.*	G. C. Evans, **Functionals and their Applications. Selected Topics, Including Integral Equations**, 1918, xii, 136 pp.	
5_2.*	O. Veblen, **Analysis Situs**, 1922, viii, 150 pp.; second ed., 1931, x, 194 pp.	
6.	G. C. Evans, **The Logarithmic Potential. Discontinuous Dirichlet and Neumann Problems**, 1927, viii, 150 pp.	2.00
7.	E. T. Bell, **Algebraic Arithmetic**, 1927, iv, 180 pp.	2.70
8.	L. P. Eisenhart, **Non-Riemannian Geometry**, 1927, viii, 184 pp.; reprinted, 1934, viii, 184 pp.	2.70
9.	G. D. Birkhoff, **Dynamical Systems**, 1927, viii, 295 pp.	3.35
10.	A. B. Coble, **Algebraic Geometry and Theta Functions**, 1929, viii, 282 pp.	3.35
11.*	D. Jackson, **The Theory of Approximation**, 1930, viii, 178 pp.	
12.*	S. Lefschetz, **Topology**, 1930, x, 410 pp.	

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* Out of print.

13.	R. L. Moore, **Foundations of Point Set Theory**, 1932, viii, 486 pp.	6.00
14.	J. F. Ritt, **Differential Equations from the Algebraic Standpoint**, 1932, x, 172 pp.	2.70
15.*	M. H. Stone, **Linear Transformations in Hilbert Space and their Applications to Analysis**, 1932, viii, 622 pp.	
16.	G. A. Bliss, **Algebraic Functions**, 1933, x, 218 pp.	3.35
17.	J. H. M. Wedderburn, **Lectures on Matrices**, 1934, viii, 200 pp.	3.35
18.	M. Morse, **The Calculus of Variations in the Large**, 1934, x, 368 pp.	5.35
19.	R. E. A. C. Paley and N. Wiener, **Fourier Transforms in the Complex Domain**, 1934, viii, 184 pp. + portrait plate.	3.35
20.	J. L. Walsh, **Interpolation and Approximation by Rational Functions in the Complex Domain**, 1935, x, 382 pp.	6.00
21.	J. M. Thomas, **Differential Systems**, 1937, x, 118 pp.	2.00
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THE BASES OF PROBABILITY[1]

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The subject for consideration today forms an aspect of a somewhat venerable branch of mathematical theory; but in essence it is part of a far older department of thought—the ancient science of logic. For it is concerned with a category of propositions of a nature marked by features neither physical nor mathematical, but by their rôle under the aspect of the reason. Their essential characteristic is their involvement of that species of relation between the knower and the known evoked by such terms as probability, likelihood, degree of certainty, as used in the parlance of intuitive thought. It is our threefold task to transcribe this concept into symbols, to formulate its principles, and to study its properties in their inner order and outward application.

As prelude to this undertaking it is necessary to set forth certain conventions of logic. Propositions are the elements of symbolic logic, but they may play the rôle of *contemplated propositions* (statements in quotation marks) or of *asserted propositions* (statements regarded as true throughout a given manipulation or deduction); and it is necessary to take account of this in the notation for the logical constants. We shall employ the symbols for negation ($\sim$), conjunction or logical product ($\cdot$), and disjunction or logical sum ($\vee$), and regard them as having no assertive power: they combine contemplated propositions into contemplated propositions and asserted propositions into asserted propositions *of the same logical type*. Quite other shall be our convention regarding implication ($\subset$) and equivalence ($=$): they combine contemplated propositions into asserted propositions, and shall not be used to combine asserted propositions in our present study. If a and b stand for contemplated propositions, the assertion that a is false (true) shall be written $a=0$ ($a=1$), and the assertion that a implies b, $a\subset b$ or $a\sim b=0$; it is thus quite different from the contemplated proposition $\sim(a\sim b)$. Finally it is universally asserted that $a\sim a=0$, $a\vee\sim a=1$, and in fact all the laws of Boolean algebra are regarded as assertions. We shall assume their elements to be familiar, and shall accept without question the intuitive logical background implied in their manipulation and interpretation.[2]

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[1] An address delivered before the New York meeting of the Society on February 24, 1940, by invitation of the Program Committee.

For the details of the theory here expounded, see the two publications of the present author. *The axioms and algebra of intuitive probability*, Annals of Mathematics, (2), vol. 41 (1940), pp. 269–292 (herein to be abbreviated as AAP) and *Intuitive probability and sequences* (forthcoming in the Annals of Mathematics) (abbreviation PS).

The technical logician will observe that there is here involved a meta-mathematical question. But space forbids us to introduce and explain here the modern terminology of this subject, and compels us to throw a perhaps undue burden on the terms "contemplated" and "asserted proposition."

The next step towards our goal consists in defining a category of propositions which are to form the substratum upon which the structure of our theory is to be erected. We will designate by ***experimental propositions*** such statements of the outcome of a particular physical or biological event as may in principle be verified by the performance of a single crucial experiment. Thus "it will rain on this roof tomorrow at this hour" or "Mr. X made a mistake in his accounts last Monday" are experimental propositions, whereas "Newtonian mechanics is correct" is not: the motion of bodies can always be accounted for by assuming sufficiently complicated laws of force in Newton's equations, so its truth, while experimental in meaning, is not determined by a crucial experiment but rather by its ability to harmonize the results of many such experiments in an acceptably simple manner. Experimental propositions shall be denoted by lower case Latin letters; and inasmuch as a finite set of crucial experiments may always be regarded as constituting a single crucial experiment, finite combinations of letters by means of $(\sim \cdot \vee)$ also denote experimental propositions. Finally, experimental propositions shall be regarded as contemplated propositions.[3]

At this point the category of propositions may be introduced which forms the subject of the theory of probability. We will take as the conceptual germ from which the whole theory springs the ordering of two events in the relation "not more probable than," a relation to be denoted by the partial ordering symbol $(\prec)$—which like $(\subset, =)$ shall have assertive force. One could of course develop a theory of such assertions as $a \prec b$; but it would prove insufficient for the purposes of probability, as one needs for example to compare the probability of a assuming h true with its probability assuming h false. The definitive form of propositions sought is the following, in which a, b, h, k are experimental propositions and $h \neq 0$, $k \neq 0$:

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[2] For references see E. V. Huntington, Transactions of this Society, vol. 35 (1933), pp. 274–304.

[3] In its occurrence in a/h the experimental proposition h is in a sense "temporarily asserted," i.e., a is viewed on the assumption that h is true. But we are applying the term asserted proposition only to those held as true on *both* sides of $a/h \prec b/k$.

a on the presumption h is no more
probable than b on the presumption k

and this is symbolized as $a/h \prec b/k$ (or equivalently, $b/k \succ a/h$). Thus from the four contemplated experimental propositions and the logical symbol $(/ \prec /)$ an asserted proposition $a/h \prec b/k$ (to be called a $(\prec)$ proposition) is formed which is to constitute the building stone of the whole theory of probability.

Before proceeding further, two difficulties must be surmounted. Firstly it may be objected that $a/h \prec b/k$ is a proposition of such vague and subjective order that it may not merely be held by one person and rejected by another, but that one and the same individual may sometimes assent to it and at other times and in a different mood reject it. If this is so, how in the nature of things can such propositions form the subject matter of a precise mathematical science? Secondly it may be objected that the probability of a proposition a depends on a body of knowledge going far beyond the fact that h is true: It will involve propositions of higher logical types such as the laws of logic—and of probability itself; and perhaps even matters of subconscious moods, associations, artistic taste, and the like.

We believe that these two objections are answered at one stroke by adhering to the following convention, or rather, clarification of the use and laws of $(\prec)$ propositions. *A given individual at a given moment* may be regarded as assenting to a certain set of $(\prec)$ propositions; ignoring what he may hold at any other moment or what others may believe, that set of $(\prec)$ propositions which he holds at that given moment must have certain relations with one another which may be called *relations of consistency*. It is to their formulation and study that we conceive the present science to be devoted. So viewed, the analogue with strict logic is clear: many may disagree with me when I assert $a=0$; but every one whose mind is constructed on normal lines will agree that if $a=0$ then $(\sim\sim a)=0$. Along with the first objection, this convention answers the second, for by positing a given individual at a given moment in the consideration of any set of $(\prec)$ propositions, the body of knowledge becomes fixed throughout, and so does not require explicit symbolization.

A third objection which might be voiced is against the restriction of the application of probability to experimental propositions. Why can we not compare the probability of two physical theories, for example? In answer to this we can say only that with the present restriction many grave logical difficulties are avoided and a theory is obtained which covers all the classical cases of mathematical probability and many others as well; and our limitation of the scope is a matter of practical strategy rather than of principle. It would undoubtedly be of interest to extend the present ideas to propositions of higher logical types, classes of propositions, logical systems and the like.

There is evidently no difference in intuitive meaning between a/h and ah/h (or, dually, $a \vee \sim h/h$), and we will make the notational convention that any one of these symbols be replaceable by any other. This has an interesting algebraic counterpart. Let $\mathfrak{A}$ be the Boolean ring determined by all the experimental propositions considered in a given discussion.[4] To make the presumption that $h \varepsilon \mathfrak{A}$ is true (equivalently, that $\sim h$ is false) is to make the presumption that any two propositions a and b of $\mathfrak{A}$ for which $a \sim b \subset \sim h$ and $b \sim a \subset \sim h$ are equivalent, i.e., both true or false simultaneously. But this "identification" of all so-related pairs a, b is precisely the formation of the quotient ring $\mathfrak{A}/(\sim h)$ whose elements $a/(\sim h)$ are the remainder classes with respect to the principle ideal $(\sim h)$. It is purely for convenience that we write a/h in lieu of $a/(\sim h)$. Thus if $\mathcal{A}$ is the class of all remainder classes in $\mathfrak{A}$ with respect to all its principal ideals, the $(\prec)$ symbol introduces a partial ordering of the elements of $\mathcal{A}$.[5]

We are now ready to undertake our second task and lay down the axioms which govern any aggregate of $(\prec)$ propositions. It will be noted that they all have $(\subset, =)$ or $(\prec)$ propositions as hypothesis and as conclusion, and that in each case where the conclusion is a non-trivial $(\prec)$ proposition, this is true of the hypothesis as well. Finally, a tacit assumption is always made: *no denominator* $=0$.

THE AXIOMS

V. AXIOM OF VERIFICATION. *If* $k \subset b$, *then* $a/h \prec b/k$.

I. AXIOM OF IMPLICATION. *If* $a/h \prec b/k$ *and* $h \subset a$, *then* $k \subset b$.

R. AXIOM OF REFLEXIVITY. *If* $h = k$ *and* $ah = bk$, *then* $a/h \prec b/k$.

T. AXIOM OF TRANSITIVITY. *If* $a/h \prec b/k$ *and* $b/k \prec c/l$, *then* $a/h \prec c/l$.

A. AXIOM OF ANTISYMMETRY. *If* $a/h \prec b/k$, *then* $\sim a/h \succ \sim b/k$.

C. AXIOMS OF COMPOSITION. *Let* $0 \neq a_1 \subset b_1 \subset c_1$ *and* $0 \neq a_2 \subset b_2 \subset c_2$.

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[4] For Boolean rings and their ideals, see M. H. Stone, Transactions of this Society, vol. 40 (1936), pp. 37–111.

[5] One should guard against the notion that this ordering has any simple relation with the ordering of the remainder classes with respect to class inclusion.

C_1. *If* $a_1/b_1 \prec a_2/b_2$ *and* $b_1/c_1 \prec b_2/c_2$, *then* $a_1/c_1 \prec a_2/c_2$.
C_2. *If* $a_1/b_1 \prec b_2/c_2$ *and* $b_1/c_1 \prec a_2/b_2$, *then* $a_1/c_1 \prec a_2/c_2$.

D. AXIOMS OF DECOMPOSITION (QUASI-CONVERSES OF C). *Let* $0 \neq a_1 \subset b_1 \subset c_1$, $0 \neq a_2 \subset b_2 \subset c_2$, *and* $a_1/c_1 \prec a_2/c_2$. *Then if either symbol in* (i): $(a_1/b_1, b_1/c_1)$ *has the* $(\succ)$ *relation with either in* (ii): $(a_2/b_2, b_2/c_2)$, *then the remaining symbol in* (i) *has the* $(\prec)$ *relation with that in* (ii). (*Thus* D *contains four axioms.*)

P. AXIOM OF ALTERNATIVE PRESUMPTION. *Let* $a/hb \prec r/s$ *and* $a/h{\sim}b \prec r/s$; *then* $a/h \prec r/s$.

S. AXIOM OF SUBDIVISION. *For each positive integer* n *the following axiom is posited*:

S_n. *If* $a_1 \vee \cdots \vee a_n = a \neq 0$, $b_1 \vee \cdots \vee b_n = b \neq 0$, $a_i a_j = b_i b_j = 0$ *for all* $i \neq j$, *and lastly if*

$$a_1/a \prec a_2/a \prec \cdots \prec a_n/a,$$

$$b_1/b \succ b_2/b \succ \cdots \succ b_n/b,$$

then $a_1/a \prec b_1/b$.

The Axioms V, R, T, A are simply the transcription into the present language of facts so familiar as scarcely to require comment. The partial ordering property of $(\prec)$ expresses itself by R and T; it leads to the definition of equiprobability $a/h \approx b/k$, inferior probability $a/h < b/k$ and incomparability $a/h \| b/k$ in the usual manner. The question of whether one can go further and assume that the entities a/h form the elements of a lattice with respect to $(\prec)$ will naturally be raised; until now we have been unable to make any use of this idea, and if our experience is borne out we shall be in the presence of the first non-trivial example of a partially ordered set which is not a lattice.

Axiom I is in sharp contrast with the familiar circumstance that the numerical probability of an event may be unity (i.e., the same as a certain event) without that event's being certain. This is because numerical probability gives but a blurred rendering of the ultimate logical relations between probability and certainty.

As for Axiom C and its converse D, the following verbal rendering of C_1 may be given: If a_1 depends for its possibility on b_1, and likewise a_2 on b_2, and if c_1 is less likely to lead to b_1 than is c_2 to b_2, and if finally b_1 is in turn less likely to lead to a_1 than is b_2 to a_2, then c_1 is less likely to lead to a_1 than c_2 to a_2. So stated, it exhibits a sort of inner transitivity. All the other cases have a corresponding phraseology.

One might be disposed to regard Axiom P as a theorem which could be proved by arguing that since the hypothesis $a/hb \prec r/s$, $a/h{\sim}b \prec r/s$ tells us that a on the presumption h is not more likely than r/s both when b is true and when b is false, it must be so in all cases, i.e., $a/h \prec r/s$. Carrying this species of reasoning a little further, we could prove that such a relation as $a/h \approx {\sim}a/h$ is impossible: Since the assertion $(a \vee {\sim}a) = 1$ is always made we would conclude that the only possibilities are $a = 1$ or ${\sim}a = 1$ (i.e., $a = 0$); in either case $a/h \approx b/k$ is impossible—hence it is never possible. In essence this is the old objection of the elementary student who voices it by saying that since an event will either happen or not happen, it is absurd to say that its probability of happening could ever be $\frac{1}{2}$. We all know how to answer him by general reference to the dependence of probability on a body of knowledge;[6] but we are in a position here to give the answer in a precise logical form: The fallacy lies in confusing the assertion $(a \vee {\sim}a) = 1$ with the assertion "$a = 1$ *or* ${\sim}a = 1$" [which might be written $(a = 1) \vee ({\sim}a = 1)$]. *The distinction between an asserted disjunction and a disjoined assertion is fundamental:* $(u \vee v) = 1$ *must never be confused with* $(u = 1) \vee (v = 1)$. The disregard of this distinction has led to more difficulties in the foundations of probability than is often imagined. It is now clear that the above proof of Axiom P is fallacious since it confuses $(b \vee {\sim}b) = 1$ with $(b = 1) \vee ({\sim}b = 1)$. As a matter of fact the same proof would have provided an infinite extension of Axiom P, an extension which leads to paradoxes.[7]

Axiom S is epitomized in the idea that if a first event is less likely of occurrence than its opposite and if a second is more likely than its opposite, then the first is less likely than the second. While we have not succeeded in simplifying the general case (beyond restricting S_n to prime values of n), we still feel confident that those more skilful than ourselves may have better success.

As a purely formal matter it may be remarked that in a system *completely* ordered by $(\prec)$, Axioms P and S are logical consequences of the rest.

Our second task being complete, we pass to the third, the deduction of all the useful theorems of probability from the axioms. But before proceeding it may be remarked that we have traversed the path of all mathematical disciplines: One proposes to study a subject of which one is made aware through the intuition, the senses, and such non-mathematical modes of perception. In undertaking it one introduces symbols and statements of axioms and laws in terms of these. But from this point on everything (except the interpretation) becomes purely mathematical: The symbolic abstractions become the ultimate objects of study, the axioms and laws become the postulates. Thenceforth we may discard the intuitionalistic introduction of our symbols and axioms and regard the latter as pure conventions (postulates) pertaining to the former, which are taken as "undefinables."

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[6] The notion that the uncertainty resides in the events themselves rather than in the mind of the individual contemplating them, the appeal to the "principle of uncertainty," etc., betrays merely a misconception both of probability and of quantum mechanics.

[7] Cf. PS, §2.

The further developments fall into three groups of theorems: the theorems on comparison, the theorems on numerical probability, and the theorems on statistical weight or frequency in a sequence.

In the first group we shall confine ourselves to citing the following typical ones. No comments appear necessary.

THEOREM. *If* $ah\neq 0$ *or* h, *then* $0/1<a/h<1/1$.

THEOREM. *If* $a_1/h_1 \prec a_2/h_2$, $b_1/h_1 \prec b_2/h_2$, *and* $a_1b_1h_1 = a_2b_2h_2 = 0$, *then* $a_1 \vee b_1/h_1 \prec a_2 \vee b_2/h_2$.

THEOREM. *If* $a/hc_i \prec r/s$ $(i=1, \cdots, n)$ *and* $hc_ic_j=0$ *for all* $i\neq j$, *then* $a/hc \prec r/s$ *where* $c=c_1 \vee \cdots \vee c_n$.

The second group starts with the introduction of the numerical probability $p(a/h)=p(a, h)$, i.e., the number between 0 and 1 forming the basis of the classical theory. This is accomplished as follows:

DEFINITION. *Any set of propositions* $(u_1, \cdots, u_n)$ *shall be called an* n*-scale when they satisfy the conditions* (i) $u_1 \vee \cdots \vee u_n = u \neq 0$; (ii) $u_iu_j=0$ *(all* $i\neq j$*)*; (iii) $u_i/u \approx u_j/u$ *(all* i, j*)*.

ASSUMPTION. *Any positive integer* n *being given, the conceptional existence of at least one* n*-scale may be assumed.*

This is the only principle which need be assumed in addition to the axioms in all the further developments of the theory.[8] It is of a fundamentally different nature from the axioms and might be compared with the assumption so familiar in thermodynamics and other parts of physics of the possibility of a conceptual experiment.

THEOREM. *If* $(u_1, \cdots, u_n)$ *is an* n*-scale and* $(v_1, \cdots, v_m)$ *an* m*-scale,*

$$u_1 \vee \cdots \vee u_\nu/u \;<, \approx, >\; v_1 \vee \cdots \vee v_\mu/v$$

according as $\nu/n <, =, > \mu/m$.

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[8] In the exhibition of certain paradoxes the extension of the assumption regarding the existence of n-scales to $n=\aleph_0$ is required.

Let $t(n)$ be the maximum value of t for which $u_1 \vee \cdots \vee u_t/u \prec a/b$ holds ($t=0$ corresponding to $0/u \prec a/b$) and $T(n)$ the minimum value of T for which $a/b \prec u_1 \vee \cdots \vee u_T/u$. By the previous theorem $t(n)$ and $T(n)$ are independent of the particular n-scale chosen; for fixed a/b they are always defined functions of n. We then prove the following:

THEOREM. *The limits* $p_*(a/h) = \lim_{n\to\infty} t(n)/n$ *and* $p^*(a/h) = \lim_{n\to\infty} T(n)/n$ *always exist, and* $0 \leqq p_*(a/h) \leqq p^*(a/h) \leqq 1$.

The numbers $p_*(a/h)$ and $p^*(a/h)$ may be called the lower and upper numerical probabilities of a/h.

DEFINITION. *If* $p_*(a/h) = p^*(a/h)$, a/h *is said to be appraisable and to have* $p(a/h) = p_*(a/h) = p^*(a/h)$ *as its numerical probability.*

All the classical theorems follow. We give merely the following two examples:

THEOREM. *Let* $a \subset b \subset c$; *if* a/c *and* b/c *are appraisable and if* $p(b/c) \neq 0$, *then* a/b *will be appraisable and* $p(a/c) = p(a/b)p(b/c)$.

THEOREM. *Let* a/h *and* b/h *be appraisable. Then* $a \vee b/h$ *will be appraisable if and only if* ab/h *is appraisable and it will then follow that*

$$p(a/h) + p(b/h) = p(a \vee b/h) + p(ab/h).$$

If now we consider the Boolean ring determined by the totality of propositions considered in a given discussion and assume that every a/h formed in it is appraisable, we are at the threshold of the classical theory. For we are in possession of an additive function obeying all the postulates required for its derivation. Its rôle is thus revealed as a theory of (unfaithful) numerical representation of relations belonging to the more far-reaching logical theory.

Before passing to the third group of results the question as to complete additivity is in order: Our axioms establish only the restricted additivity of numerical probability. The example of the infinite sequence of propositions $a_1, a_2, \cdots$ for which $a_1 \vee a_2 \vee \cdots = 1$, $a_i a_j = 0$ $(i \neq j)$ and $p(a_1/1) = p(a_2/1) = \cdots$ shows that the equation

$$1 = p(a_1 \vee a_2 \vee \cdots /1) = p(a_1/1) + p(a_2/1) + \cdots$$

is impossible.[8] This could be interpreted either by regarding the assumptions concerning $a_1, a_2, \cdots$ as self-contradictory or regarding them to be valid and holding the view that complete additivity is not a general property, but occurs only in an important class of special cases where its validity is a consequence of the physical circumstances and not of the logical aspect of probability. We have adopted the latter position.

Anyone conversant with modern theoretical physics is aware of the fundamental rôle played therein by probability. Statistical mechanics is a familiar example; but even more significant is the case of quantum mechanics, the laws of which can not be stated except in terms of probability. Now if these sciences are to be regarded as affording objective pictures of nature, how can their laws involve in an essential manner the notion of probability if this is indeed a concept of logic—a mode of thought? The answer to this question is immediate: the "probability" of these branches of physics is a misnomer for statistical weight or frequency in a sequence. If an event E in such a theory can have two possible outcomes, "success" (labeled 1) and "failure" (labeled 0), a conceptually infinite sequence of trials under "the same conditions" furnishes an infinite sequence of zeros and ones (α): $(\alpha_1, \alpha_2, \cdots)$ $(\alpha_n = 0, 1)$. The physical assumption that $w = \lim_{n\to\infty} (\alpha_1 + \cdots + \alpha_n)/n$ exists is made and this statistical weight or frequency w is what is designated by the word "probability" of success of E. But the whole objective content of the physical laws in question involves solely the notion of frequency.

Yet the intuitive conception of probability upon which the present work is based plays an essential part in connection with frequency. Its rôle becomes manifest at that very moment when the experimental significance of w is sought—significance, that is, to a pre-named individual in terms of the only phenomena which can come within his ken. Then it is that we become aware that a link is needed between the finite sets of trials—all that we can actually observe—and the mathematical idealization of frequency.[9] Analysis reveals that the only possible link is bound to involve the intuitive idea of probability.[10] Granting then the present theory, are we enabled to solve the problem? That we are indeed able to give a complete and precise solution and to do so without assuming any further principles is the content of the third group of theorems, to which we now turn.

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[9] This remains true even when frequency is thought of as a ratio in a finite sequence containing a larger number of trials than can come before the individual's observation. This is the difficulty which confronts any attempt to dispense with everything of the essence of intuitive probability and replace it by a theory of frequency. It is an attempt often made with the object of freeing the science of subjectivism (sic) and of retaining therein only an account of that which scientists "really observe"; by a strange irony it places the theory of probability completely out of contact with what any given human being could ever observe.

[10] Cf. PS, §1.

Let a_n be the experimental proposition "E succeeds at the nth trial" (so that $\alpha_n = 1$). Let h denote the statement of the common experimental condition at the instance of each trial. Then the following theorem is typical.

THEOREM. *Hypothesis* 1: $\lim_{n\to\infty} (\alpha_1 + \cdots + \alpha_n)/n = w$. *Hypothesis* 2: *For each positive integer* t

$$a_{i_1} \cdots a_{i_t}/h \approx a_{j_1} \cdots a_{j_t}/h,$$

where $(i_1, \cdots, i_t)$ *is any set of* t *distinct positive integers and* $(j_1, \cdots, j_t)$ *a similar set. Conclusion*: a_i/h *is appraisable and* $p(a_i/h) = w$.

The first idea which should enter the mind of the mathematician is that the sequence (α) can (at least when $0 < w < 1$) be reordered so as to yield a different frequency; yet apparently this must still be equal to w; is this not contradictory? The answer consists in examining the precise logical meaning of Hypothesis 1. Firstly, let $W(w, \mu, n)$ denote the assertion: ***h* implies that the number of true propositions in** *the set* $(a_1, \cdots, a_n)$ *is between* $n(w - 1/\mu)$ *and* $n(w + 1/\mu)$. Then Hypothesis 1 becomes: For any given integer μ there exists an m such that for all $n \geqq m$ assertion $W(w, \mu, n)$ is made. In logical symbols this is the familiar

$$\prod_{\mu=1}^{\infty} \sum_{m=1}^{\infty} \prod_{n=m}^{\infty} W(w, \mu, n).$$

This is all clear enough; the ambiguity appears when the logical form for assertion $W(w, \mu, n)$ is sought, for it turns out that there are many. The following is in a certain sense the weakest; it is the one for which the above theorem is proved; it is suitable for relating frequency to probability in physics:

$$W(w, \mu, n): \qquad h \subset \sum_{(p,q)} a_{p_1} \cdots a_{p_t} \sim a_{q_1} \cdots \sim a_{q_f}.$$

Here the $\sum$ calls for the disjunction of all terms where $(p_1, \cdots, p_t, q_1, \cdots, q_f)$ are all possible sets of $t + f$ distinct integers between 1 and n and where t is the least integer $\geqq n(w - 1/\mu)$ and f the least integer $\geqq n(1 - w - 1/\mu)$. But Hypothesis 1 with this form of $W(w, \mu, n)$ is not the one which makes it possible to reorder the sequence so as to produce the contradiction. For this purpose it is necessary to replace it by the entirely different $W'(w, \mu, n)$:

$$W'(w, \mu, n): \qquad \sum_{(p,q)} (h \subset a_{p_1} \cdots a_{p_t} = a_{q_1} \cdots = a_{q_f}).$$

Thus the paradox is resolved as in earlier cases by maintaining the distinction between an assertion of a disjunction and a disjunction of assertions.

The complementary rôle of the hypotheses of this theorem is worthy of note. Hypothesis 1 exhausts the purely objective state of affairs, while Hypothesis 2 which expresses a sort of intuitive random quality, contains the assumption of a subjective nature which allows an actual living being to capture the otherwise inaccessible objective fact and relate it to his own world of possible experience by the agency of intuitive probability. Thus the theorem renders unto objective reality that that is objective, and unto the subjective intuition that which pertains thereto.

No discussion of the bases of probability would be complete at the present day which did not make reference to alleged cases of the violation of certain principles of classical probability by the phenomena of quantum mechanics. The precise form at which we have here arrived makes it particularly simple to subject every such case to minute scrutiny. While we have no time for examples here, we are publishing elsewhere a discussion which shows that it is the physical circumstances to which the laws of probability apply and never the laws themselves which are altered.[11] It would indeed be hard to imagine how it could be otherwise. For insofar as the laws of probability are laws of thought, they are prior to experimental verification in the laboratory. For how indeed can such experiments prove any statement? Firstly, when the statement is an experimental proposition, then a crucial experiment suffices: but this is evidently not the case for the axioms of probability, which are not experimental propositions. Secondly, when the statement introduces harmony and intelligibility into an ensemble of statements proved in the laboratory: but the axioms of probability appear rather in the rôle of the *criteria* of such harmony and intelligibility. To argue, finally, that the axioms repose on *subjective* experiments and hence are experimental in character is beside the point since we are considering quantum mechanics which is based on experiments on electron tubes and things of this sort which are hardly in a class with the subjective experiments by means of which we become aware of our own rational processes.

Having dwelt so long on the positive side, it behooves us to mention a fundamental limitation of these results. The theory cannot prove that the probability of heads on the toss of a coin is $\frac{1}{2}$. More generally, it is as impotent to derive a non-trivial ($\prec$) proposition from a set of propositions of whatever character not containing a ($\prec$) proposition (stated or implied) as are the laws of Newtonian mechanics to predict the position of a particle at a given time when no initial conditions are assigned. For after all, the whole theoretical structure is but the statement *in extenso* of the laws of consistency governing an aggregate of ($\prec$) propositions.

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[11] Cf. PS, §6.

The question is naturally raised whether some further principle of a purely formal-logical nature can be enunciated which will establish ($\prec$) propositions *ab ovo*. Having searched high and low in the literature we become aware that every apparent case of such a principle either contains in some veiled form a ($\prec$) proposition in its hypothesis, or else leads to insurmountable paradoxes, as in the case of the principle of sufficient reason or symmetry of ignorance which has so long sullied the name of a priori intuitive probability. The quest for the first ($\prec$) proposition is epitomized by the attempt to devise an experiment proving the irrelevance of some external condition A in a trial of an event E. Such a statement of irrelevance is of course a ($\prec$) proposition. In order to reason that A is irrelevant to E on one occasion from the results of experiments performed on another, one must assume that certain other unavoidable differences between the two occasions are themselves irrelevant to the situation. The difficulty, exactly contrary to Napoleon's Guard, always retreats but does not expire.

It is in the light of experience such as this that we may well ask whether it is not a principle of epistemology itself that blocks our path; and, even as those who having sought in vain for perpetual motion ended by making a virtue of their failure, so we may hazard the view that in principle the authority for the first ($\prec$) proposition does not reside in any general law of probability, logic, or experimental science. And the notion presents itself that such primary and irreducible assumptions are grounded on a basis as much of the aesthetic as of the logical order.

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ALMOST CYCLIC ELEMENTS AND SIMPLE LINKS OF A CONTINUOUS CURVE[1]

F. B. JONES

Many of the definitions and results concerning connected im kleinen continua become useful only when these continua are locally compact. This is especially true in the cyclic element theory. For if a continuous curve M is not locally compact, it is not necessarily true that a simple closed curve in M belongs to a cyclic element of M. Furthermore, if a continuous curve M is not locally compact, it is not necessarily true that a simple closed curve in M belongs to a simple link in M. However, if M is a continuous curve, there are in M certain subcontinua which strongly resemble both cyclic elements and simple links such that if J is a simple closed curve in M, one of them contains J. It is the purpose of this paper to define these sets, to develop a few of their properties, and to show how they are of considerable interest in spaces where the Jordan curve theorem holds true.

1. **Results for complete Moore spaces.** In this section it is assumed that S, the set of all points, is a complete Moore space, that is, Axioms 0 and 1 of R. L. Moore's *Foundations of Point Set Theory*[2] hold true in S.

DEFINITION. *Suppose that K is a nondegenerate subset of a continuous curve M such that* (1) *if A and B are distinct points of K, there exists a simple closed curve lying in M and containing $A+B$, and* (2) *if X is a point of $M-K$, there is some point O of K such that no simple closed curve lying in M contains both X and O. The set K is said to be a "cyclic nucleus"*[3] *of M.*

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[1] Presented to the Society, December 28, 1939.

[2] American Mathematical Society Colloquium Publications, vol. 13, New York, 1932. Hereinafter, this book will be referred to as *Foundations*. The reader is referred to *Foundations* for the definition of terms used, but not specifically defined, here.

[3] A continuous curve is defined to be a connected im kleinen continuum. It need *not* be locally compact. It is easy to see with the help of Theorems 118 and 120 in Chapter I and the arguments for Theorems 6 and 7 in Chapter II of *Foundations* that if a nondegenerate continuous curve M is regarded as a space and the term "region" is interpreted to mean a connected open subset of M, then with respect to this interpretation of "point" and "region," Axioms 0, 1, and 2 of *Foundations* hold true in M and "limit point" is invariant under this change. Hence, it is possible to apply certain theorems found in Chapter II of *Foundations* and elsewhere to continuous curves. For example, any two points of a connected open subset D of a continuous curve are the extremities of an arc lying wholly in D.

THEOREM 1. *If K is a cyclic nucleus of a continuous curve M, and AB is an arc which lies in M and whose end points belong to K, then AB is a subset*[4] *of K.*

PROOF. Suppose that O is any point of K and that X is a point of AB. There exist in M simple closed curves J_A and J_B containing $O+A$ and $O+B$ respectively. Let H denote $AB+J_A+J_B$. It is easy to see that H is a compact continuous curve and that no point separates O from X in H. By Theorem 40 on page 124 of *Foundations*, H contains a simple closed curve containing $O+X$. Hence X belongs to K.

THEOREM 2. *If K is a cyclic nucleus of a continuous curve M, and AB is an arc which lies in M and whose end points belong to $\bar{K}$, then $AB-(A+B)$ is a subset of K.*

PROOF. Suppose that X is a point of $AB-(A+B)$. There exist two mutually exclusive connected open subsets D_A and D_B of M which contain A and B respectively such that neither contains X. Since both D_A and D_B contain points of K, D_A+AB+D_B contains an arc $A'B'$ whose end points A' and B' belong to $K\cdot D_A$ and $K\cdot D_B$ respectively. But $A'B'$ must contain X. Hence, by Theorem 1, X belongs to K.

THEOREM 3. *If K is a cyclic nucleus of a continuous curve M, and J is a simple closed curve lying in M and containing more than one point of $\bar{K}$, then J is a subset of K.*

Theorem 3 follows at once from Theorem 2.

THEOREM 4. *If K is a cyclic nucleus of a continuous curve M, then neither K nor $\bar{K}$ contains a cut point of itself.*

Theorem 4 may be established with the help of Theorem 3.

THEOREM 5. *If K_1 and K_2 are cyclic nucleae of a continuous curve M, and $\bar{K}_1$ and $\bar{K}_2$ are identical, then K_1 and K_2 are identical.*

PROOF. If X is a point of K_1, X is a subset of a simple closed curve J lying in K_1. Hence J is a subset of $\bar{K}_2$. By Theorem 3, J is a subset of K_2, and X belongs to K_2. Likewise, any point of K_2 is a point of K_1.

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[4] A subset K of a continuous curve M which contains every arc in M whose end points belong to K has been called a *basic set* of M. See W. L. Ayres, *Concerning the arc-curves and basic sets of a continuous curve*, Transactions of this Society, vol. 30 (1928), pp. 567–578, and *Concerning the arc-curves and basic sets of a continuous curve, second paper*, Transactions of this Society, vol. 31 (1929), pp. 595–612.

THEOREM 6. *If K is a cyclic nucleus of a continuous curve M, and C is a component of $M-\overline{K}$, then C has at most one boundary point in $\overline{K}$.*[5]

PROOF. It follows from Theorem 2 on page 89 of *Foundations* that C is a connected open subset of M. Suppose that C has two boundary points in $\overline{K}$. Using the theorem just referred to, it is easy to see that M contains an arc AB such that (1) $AB-(A+B)$ is a subset of C and (2) both A and B belong to $\overline{K}$. By Theorem 2, $AB-(A+B)$ belongs to K, which is a contradiction.

THEOREM 7. *If K is a cyclic nucleus of a continuous curve M, and X is a point of $M-K$, then there is at most one point O of K such that some simple closed curve in M contains $O+X$.*

PROOF. If X is not a point of $\overline{K}$, the conclusion of Theorem 7 follows from Theorem 6. If X is a point of $\overline{K}$ and a simple closed curve in M contains X and a point O of K, then by Theorem 3, X belongs to K, which is a contradiction.

THEOREM 8. *If K is a cyclic nucleus of a continuous curve M, and D is a connected open subset of M containing a point of K, then both $D\cdot K$ and $D\cdot\overline{K}$ are connected.*[6]

PROOF. Suppose that either $D\cdot K$ or $D\cdot\overline{K}$ is not connected. Then it is the sum of two mutually separate sets, K_1 and K_2. Let AB denote an arc in D from a point of K_1 to a point of K_2. By Theorem 2, $AB-(A+B)$ is a subset of K. This involves a contradiction.

THEOREM 9. *If K is a cyclic nucleus of a continuous curve M, $\overline{K}$ is a nondegenerate continuous curve which contains no cut point of itself.*

Theorem 9 may be proved with the help of Theorems 4 and 8.

THEOREM 10. *If K_1 and K_2 are cyclic nucleae of a continuous curve M, then either K_1 and K_2 are identical or $\overline{K}_1$ and $\overline{K}_2$ have at most one point in common.*

PROOF. Suppose that $\overline{K}_1$ contains two points of $\overline{K}_2$. Then it is evident from Theorems 4 and 6 that $\overline{K}_1$ is a subset of $\overline{K}_2$. Conversely, $\overline{K}_2$ is likewise a subset of $\overline{K}_1$. Hence $\overline{K}_1$ and $\overline{K}_2$ are identical. By Theorem 5, K_1 and K_2 are identical.

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[5] Cf. G. T. Whyburn, *Cyclicly connected continuous curves*, Proceedings of the National Academy of Sciences, vol. 13 (1927), p. 33, Theorem 2.

[6] If the word "open" is omitted from the statement of Theorem 8, the resulting proposition is false. Cf. G. T. Whyburn, *Concerning the structure of a continuous curve*, American Journal of Mathematics, vol. 50 (1928), p. 191, Theorem 30.

THEOREM 11. *If J is a simple closed curve lying in a continuous curve M, then there exists one and only one cyclic nucleus of M containing*[7] *J.*

PROOF. Let G denote the collection of all subsets k of M such that (1) k contains J and (2) every two points of k belong to a simple closed curve in M. Let K denote G^* (that is, the sum of the elements of G) and let A and B denote two distinct points of J. If X and Y are two distinct points of K, there exist in M simple closed curves J_{AX}, J_{BX}, J_{AY}, and J_{BY} which contain $A+X$, $B+X$, $A+Y$, and $B+Y$ respectively. Then $J_{AX}+J_{BX}+J_{AY}+J_{BY}$ is a compact continuous curve no point of which separates X from Y; hence it contains a simple closed curve C containing $X+Y$. Consequently K is an element of G.

Suppose that K is not a cyclic nucleus of M. Then $M-K$ contains a point X such that if O is a point of K, M contains a simple closed curve J_{OX} containing both O and X. Again let A and B denote distinct points of J, and let J_{AX} and J_{BX} denote simple closed curves lying in M and containing $A+X$ and $B+X$ respectively. Then $k=J+J_{AX}+J_{BX}$ is a compact continuous curve containing no cut point of itself, and every two of its points belong to a simple closed curve in k which is a subset of M. Hence k is an element of G. But this involves a contradiction. Therefore K is a cyclic nucleus of M which contains J. By Theorem 10, no other cyclic nucleus of M contains J.

THEOREM 12. *In order that a set K be a cyclic nucleus of a continuous curve M, it is necessary and sufficient that K consist of two points A and B which belong to a simple closed curve in M together with all other points X such that* (1) *some simple closed curve lying in M contains $A+X$ and* (2) *some simple closed curve lying in M contains $B+X$.*

PROOF. It is easily seen from Theorem 7 and the definition of a cyclic nucleus of a continuous curve that the condition is necessary. The condition is also sufficient. For let J denote a simple closed curve in M containing $A+B$ and let H denote the cyclic nucleus of M which contains J. Then H is a subset of K. But it is clear from Theorem 7 that K is a subset of H. Hence K is a cyclic nucleus of M.

THEOREM 13. *If A, B and X are three points of a cyclic nucleus K*

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[7] Cf. G. T. Whyburn, *Cyclicly connected continuous curves*, loc. cit., p. 34, Theorem 3.

of a continuous curve M, then there exists in K a simple closed curve containing $A+B+X$.

PROOF. Let J denote a simple closed curve in K containing $A+B$. If X does not belong to J, let J_A and J_B denote simple closed curves in M containing $A+X$ and $B+X$ respectively. Suppose that neither J_A nor J_B intersects J in more than one point. Let AB denote an arc of J from A to B. The set $AB+J_A+J_B$ is a compact continuous curve no point of which separates A from B. By Theorem 40 of page 124 of *Foundations*, $AB+J_A+J_B$ contains a simple closed curve C containing $A+B$. Evidently C intersects each of the curves J_A and J_B in more than one point. So in any case there exist a simple closed curve J_1 in M containing $A+B$ and a simple closed curve J_2 in M containing X such that J_2 intersects J_1 in more than one point. Let T denote the component of $J_2-J_1\cdot J_2$ which contains X. Since T is an arc segment, it is easy to see that J_1+T contains a simple closed curve J_3 containing $A+B+X$. By Theorem 3, J_3 is a subset of K.

THEOREM 14. *In order that a set K of more than two points be a cyclic nucleus of a continuous curve M, it is necessary and sufficient that K consist of two points A and B together with all points X such that some simple closed curve lying in M contains $A+B+X$.*

Theorem 14 follows from Theorems 12 and 13.

DEFINITIONS. *A nondegenerate continuous curve M is said to be "almost cyclicly connected" provided that, if A and B are distinct points of M, and R_A and R_B are regions containing A and B respectively, then M contains a simple closed curve containing both a point of R_A and a point of R_B. A subset H of a continuous curve M is said to be an "almost cyclic element" of M provided that* (1) *H is either a cut point or an end point of M or* (2) *H is a nondegenerate almost cyclicly connected continuous curve which is a subset of M but which is not a proper subset of any other almost cyclicly connected continuous curve which is a subset*[8] *of M.*

THEOREM 15. *If K is a cyclic nucleus of a continuous curve M, then $\bar{K}$ is a nondegenerate almost cyclic element of M.*

PROOF. By Theorem 9, $\bar{K}$ is a nondegenerate continuous curve. Since every two points of K lie together in a simple closed curve which is a subset of K, it is evident that $\bar{K}$ is almost cyclicly connected. Furthermore, it is clear from Theorem 6, that $\bar{K}$ is not a proper subset of an almost cyclicly connected continuous curve lying in M. Hence $\bar{K}$ is an almost cyclic element of M.

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[8] Cf. G. T. Whyburn, *Concerning the structure of a continuous curve*, loc. cit., p. 167.

THEOREM 16. *If H is a nondegenerate almost cyclic element of a continuous curve M, then M contains one and only one cyclic nucleus K such that $\bar{K}=H$.*

PROOF. Let J denote a simple closed curve lying in H. By Theorem 11, M contains one and only one cyclic nucleus K containing J. If H is not a subset of $\bar{K}$, then it follows from Theorem 6 that H contains a cut point of itself. This is impossible. Hence H is a subset of $\bar{K}$. But by Theorem 15, $\bar{K}$ is an almost cyclicly connected continuous curve. Consequently, H is not a proper subset of $\bar{K}$.

THEOREM 17. *No two almost cyclic elements of a continuous curve M have more than one point in common.*

THEOREM 18. *If J is a simple closed curve lying in a continuous curve M, then one and only one almost cyclic element of M contains the curve J.*

DEFINITIONS. *Suppose that P is a point of a continuum M and there do not exist two points A and B of M such that* (1) *P separates A from B in M and* (2) *P is the only point of M which separates A from B in M. Let K denote the set of all points X of M such that no point separates P from X in M. Then K will be called a "simple link of M" and P will be called a "proper point of M."*[9]

THEOREM 19. *Every nondegenerate simple link of a continuous curve M is an almost cyclic element*[10] *of M.*

PROOF. Suppose that K is a nondegenerate simple link of M. Let A and B denote two distinct points of K and let AXB denote an arc in M from A to B containing a point X distinct from A and B. The arc AXB belongs to K, and X does not separate A from B in K. Hence there exists in $M-X$ an arc $A'B'$ having only its end points A' and B' in AXB. Let J denote the simple closed curve contained in $AXB+A'+B'$. Obviously J is a subset of K. By Theorem 18, M contains one and only one almost cyclic element H containing J. Since no two points of K are separated in M by any point of K, it follows from Theorems 6 and 16 that K is a subset of H. On the other hand, K contains a point P together with all other points X of M such that

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[9] See pages 63 and 72 of *Foundations.*

[10] Cf. Theorem 68 on page 148 of *Foundations.*

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X is not separated in M from P by a point of M. Since H contains P and no cut point of itself, it is a subset of K. Consequently K is identical with H.

THEOREM 20. *Every simple link of a continuous curve is itself a continuous curve.*[11]

THEOREM 21. *If a nondegenerate almost cyclic element H of a continuous curve M contains a proper point of M, then H is a simple link*[12] *of M.*

PROOF. If H contains a proper point P of M, then by Theorem 94 on page 68 of *Foundations* M contains one and only one simple link K containing P. Evidently every point of H belongs to K. It follows from Theorems 17 and 19 that H and K are identical.

EXAMPLE. *An almost cyclic element of a continuous curve M need not contain a proper point of M.* Suppose that E is an euclidean 3-space. Let H denote all points $(X, Y, 0)$ of E such that $X^2+Y^2=1$ and let W denote all points (X, Y, Z) of E such that $X^2+Y^2>1$. Now let M denote $H+W$ and define "region" in M as follows: (1) if R is a region in E containing a point of H, then $R\cdot M$ shall be called a "region" in M and (2) if (X, Y, Z) is a point of W and R is a circular region of the plane $Yx-Xy=0$ containing (X, Y, Z) such that $\overline{R}$ contains no point of H, then $R\cdot M$ shall be called a "region" in M. The space M satisfies Axioms 0–2 of *Foundations*. Consequently M is a continuous curve lying in a complete Moore space. It is clear that H is an almost cyclic element of M which contains no proper point of M.

With the theorems and proofs of the preceding pages in mind the reader will find it possible to establish many results for *almost cyclic* elements of a continuous curve in a complete Moore space analogous to those in the literature[13] for *cyclic* elements of a continuous curve in an euclidean n-space by slight changes in the original arguments. For certain results, such as those concerning the number of nondegenerate cyclic elements, it is obviously necessary in complete Moore spaces to assume that the continuous curves involved are separable. One theorem of considerable interest is as follows: *Suppose that M is a continuous curve in a complete Moore space. If the almost cyclic elements of M are regarded as "points" and two such "points" p and q are*

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[11] Cf. Theorem 66 on page 147 of *Foundations.*

[12] Cf. Theorem 69 on page 149 of *Foundations.*

[13] Especially those of the references which have been given in the footnotes of this paper.

regarded as contiguous if and only if one of the two continua p and q is a point of the other, then the set of all such "points" is an acyclic continuous curve.[14]

2. **Spaces in which the Jordan curve theorem holds true.** The next two theorems are obtained by a simple application of the preceding results to spaces in which the Jordan curve theorem holds true.

THEOREM 22. *Suppose that S is a space satisfying Axioms* 0, 1, 2, *and* 4 *of Foundations and that H is a nondegenerate almost cyclic element of S. If H is regarded as a space and the term "region" is interpreted to mean a connected open subset of H, then, with respect to this interpretation of "point" and "region," Axioms* 0–4 *of Foundations are satisfied and "limit point" is invariant under this change.*[15]

PROOF. By Theorem 9 on page 96 of *Foundations*, Axioms 0–2 are satisfied in H and "limit point" is invariant. It follows from Theorems 4 and 16 that Axiom 3 is satisfied. If J is a simple closed curve lying in H, then $S-J$ is the sum of two connected domains, E and I, each having J for its boundary. It follows from Theorems 8 and 16 that $H\cdot E$ and $H\cdot I$, if not vacuous, are connected open subsets of H. If some point P of J is not a limit point of $H\cdot E$, then some segment T of J contains P but no point of $H\cdot \overline{E}$. There exists an arc segment W lying in E whose end points lie in T. By Theorems 2 and 16, W is a subset of H. Hence W is in $H\cdot E$, and the end points of W belong to both $H\cdot \overline{E}$ and T, which is a contradiction. Consequently, every point of J is a limit point of $H\cdot E$. It is now evident that J is the boundary with respect to H of $H\cdot E$. Likewise J is the boundary with respect to H of $H\cdot I$. Hence Axiom 4 holds true in H.

THEOREM 23. *Suppose that S is the set of all points and that Axioms* 0, 1, 2, *and* 4 *of Foundations hold true. Then a nondegenerate simple link of S is a nondegenerate almost cyclic element of S and conversely.*

PROOF. It is obvious from Axiom 4 that no point of a simple closed curve is a cut point of S. Hence every point of a simple closed curve is a proper point of S. The conclusion of Theorem 23 now follows immediately from Theorems 19 and 21.

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[14] Cf. R. L. Moore, *Fundamental theorems concerning points sets*, the Rice Institute Pamphlet, vol. 23 (1936), pp. 1–74; p. 74 in particular.

[15] AXIOM 2. *If P is a point of a region R, there exists a nondegenerate connected domain containing P and lying wholly in R.*

AXIOM 3. *If O is a point, $S-O$ is connected.*

AXIOM 4. *If J is a simple closed curve, $S-J$ is the sum of two mutually separated connected point sets such that J is the boundary of each of them.* (The Jordan curve theorem.)

I wish to point out that Theorems 22 and 23 shed some light on the role played by Axiom 3 in the sequence of axioms in *Foundations.* A space satisfying Axioms 0–4 of *Foundations* is identical with its one almost cyclic element. Hence, by assuming Axiom 3 in addition to Axioms 0, 1, 2, and 4 one has merely confined one's investigation to a single almost cyclic element of the space. In fact with the preceding theorems in mind it is easy to see that many of the theorems in the literature which hold true in spaces satisfying Axioms 0–4 also hold true in spaces satisfying only Axioms 0, 1, 2, and 4. This is true, for instance, of all the theorems in Chapter III of *Foundations* except Theorems 0, 2, 21, 23, and 24.

The University of Texas

THE CONFORMAL NEAR-MOEBIUS TRANSFORMATIONS[1]

EDWARD KASNER AND JOHN DE CICCO

1. **Introduction.** In a previous paper,[2] we discussed the point transformations of the plane with reference to the maximum number of circles preserved. A nonconformal point transformation of the complex plane converts at most $2\infty^2$ circles into circles.[3] A conformal transformation, not of the Moebius type, carries at most $2\infty^1$ circles into circles (excluding the $2\infty^1$ minimal lines which become minimal lines). A Moebius transformation carries the entire family of ∞^3 circles into circles. From these results, we obtain the following two characterizations of the group of Moebius transformations: (1) if $3\infty^2$ circles are carried into circles after a point transformation, then the same is true for all circles, and the point transformation is therefore a Moebius transformation; and (2) any conformal transformation which converts $3\infty^1$ circles into circles is a Moebius transformation.

In this paper, we shall determine the set of all *conformal near-Moebius transformations*. That is, we shall obtain the set of all conformal transformations which convert *exactly* $2\infty^1$ circles into circles. *Any conformal near-Moebius transformation is of the form* M_2TM_1 *where* M_1 *and* M_2 *are Moebius transformations and* T *is any one of the three transformations* e^z, $\log z$, z^n. The two families preserved are two orthogonal pencils of circles.

The conformal near-collineation problem[4] is a special case of our problem. *Any conformal near-collineation is of the form* S_1TS_2 *where* S_1 *and* S_2 *are similitudes and* T *is any one of the three transformations* e^z, $\log z$, z^n. The family preserved is a pencil of straight lines (besides the $2\infty^1$ minimal lines).

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[1] Presented to the Society, February 24, 1940.

[2] Kasner and De Cicco, *Characterization of the Moebius group of circular transformations*, Proceedings of the National Academy of Sciences, vol. 25 (1939), pp. 209–213.

[3] In the previous paper, we derived these results for the point transformations of the *real* cartesian plane. But these same results may easily be derived for the complex cartesian plane without any difficulty. Note that a given family F of geometric configurations in the complex cartesian plane is said to possess ∞^n configurations if each of these is determined uniquely by n complex constants.

[4] Kasner, *The problem of partial geodesic representation*, Transactions of this Society, vol. 7 (1906), pp. 200–206. Also see Kasner, *The characterization of collineations*, this Bulletin, vol. 9 (1903), pp. 545–546; and Prenowitz, *The characterization of plane collineations in terms of homologous families of lines*, Transactions of this Society, vol. 38 (1935), pp. 564–599.

For our purposes, we shall find it convenient to define a point by the minimal coordinates (u, v) instead of the usual cartesian coordinates (x, y). The minimal and cartesian coordinates of the complex cartesian plane are connected by the two independent linear relations

$$u = x + iy, \qquad v = x - iy. \tag{1}$$

2. **The form of the differential equation of the two invariant families ($2\infty^1$) of circles.** In minimal coordinates, the ∞^3 circles (excluding the ∞^2 points and the $2\infty^1$ minimal lines) of the complex plane are represented by the ∞^3 hyperbolas which possess as asymptotes the minimal lines $u = \text{const.}$, and $v = \text{const.}$ Thus the family of ∞^3 circles is given by the equation

$$a_0uv + a_1u + a_2v + a_3 = 0, \tag{2}$$

where a_0, a_1, a_2, a_3 are complex constants such that either $a_0(a_1a_2 - a_3) \neq 0$ or $a_0 = 0$, $a_1a_2 \neq 0$. From this equation, it follows that the differential equation of the third order of the entire family of ∞^3 circles is

$$2pp'' = 3p'^2, \qquad p = dv/du. \tag{3}$$

To obtain the set of conformal (direct and reverse) near-Moebius transformations, it is only necessary to obtain the set of *direct* conformal near-Moebius transformations. For any reverse conformal near-Moebius transformation is the product of a direct conformal near-Moebius transformation by a reflection through the x axis (the straight line $u + v = 0$). Hence, in what follows, we shall only consider the set of direct conformal near-Moebius transformations.

In minimal coordinates, any direct conformal transformation is given by

$$U = \phi(u), \qquad V = \psi(v), \qquad \phi_u\psi_v \neq 0. \tag{4}$$

Upon extending this conformal transformation three times, we obtain

$$\begin{aligned} P &= \frac{\psi_v}{\phi_u} p, \\ P' &= \frac{\psi_v}{\phi_u^2} p' + \frac{\psi_{vv}}{\phi_u^2} p^2 - \frac{\psi_v\phi_{uu}}{\phi_u^3} p, \\ P'' &= \frac{\psi_v}{\phi_u^3} p'' + \frac{3\psi_{vv}}{\phi_u^3} pp' - \frac{3\psi_v\phi_{uu}}{\phi_u^4} p' + \frac{\psi_{vvv}}{\phi_u^3} p^3 - \frac{3\psi_{vv}\phi_{uu}}{\phi_u^4} p^2 \\ &\quad + \left(- \frac{\psi_v\phi_{uuu}}{\phi_u^4} + \frac{3\psi_v\phi_{uu}^2}{\phi_u^5} \right) p. \end{aligned} \tag{5}$$

For those circles which become circles under our conformal transformation, we know that the differential condition (3) must be preserved. Upon applying these conditions to our conformal transformation, we obtain the following theorem.

THEOREM 1. *The only possible circles which become circles under the conformal transformation* (4) (*not of the Moebius type*) *are the* $2\infty^1$ *circles whose differential equation is of the form*

$$p^2 = \frac{\psi_v^2(2\phi_u\phi_{uuu} - 3\phi_{uu}^2)}{\phi_u^2(2\psi_v\psi_{vvv} - 3\psi_{vv}^2)}. \tag{6}$$

We note that the two families ($2\infty^1$) of circles (if they exist) are orthogonal.

3. **The $2\infty^1$ circles whose differential equation is of the form** (6). The two families of circles are given by a differential equation of the form

$$p = \pm\frac{\gamma(u)}{\delta(v)}. \tag{7}$$

We shall find all families of circles whose differential equation is of this form. By means of the Moebius transformations, we shall reduce our results to canonical forms.

For the time being we shall consider only the plus sign. We observe first that *neither* γ *nor* δ *can be zero.* For then the circles would be the minimal lines u or v = const. These are excluded from consideration.

Upon taking the first and second derivatives of (7), we obtain

$$\begin{aligned} p' &= \frac{\gamma_u}{\delta} - \frac{\gamma^2\delta_v}{\delta^3}, \\ p'' &= \frac{\gamma_{uu}}{\delta} - \frac{3\gamma\gamma_u\delta_v}{\delta^3} - \frac{\gamma^3\delta_{vv}}{\delta^4} + \frac{3\gamma^3\delta_v^2}{\delta^5}. \end{aligned} \tag{8}$$

Substituting (7) and (8) into (3), we find that (7) *represents the differential equation of* $2\infty^1$ *circles if and only if*

$$\frac{2\gamma\gamma_{uu} - 3\gamma_u^2}{\gamma^4} = \frac{2\delta\delta_{vv} - 3\delta_v^2}{\delta^4}. \tag{9}$$

This equation will be true if and only if each side is equal to the same complex constant a^2. Hence

$$(10) \qquad 2\gamma\gamma_{uu} - 3\gamma_u^2 = a^2\gamma^4, \qquad 2\delta\delta_{vv} - 3\delta_v^2 = a^2\delta^4.$$

We proceed to discuss the solution of these two differential equations. We shall divide this discussion into several parts.

Case A. *Let* $a=0$. *In this case, the equation* (7) *represents two orthogonal parabolic pencils of circles* (*or their Moebius equivalents*). By a Moebius transformation, these may be reduced to the two orthogonal parallel pencils of straight lines

$$(\mathrm{A}') \qquad v = \pm\, u + \text{const.}$$

Hence the canonical form of the differential equation (7) for Case A is

$$(\mathrm{A}) \qquad p^2 = 1.$$

(I) First let neither γ nor δ be constants. Then (10) may be written in the form

$$(11) \qquad 2\frac{d}{du}\left(\frac{\gamma_u}{\gamma}\right) = \left(\frac{\gamma_u}{\gamma}\right)^2, \qquad 2\frac{d}{dv}\left(\frac{\delta_v}{\delta}\right) = \left(\frac{\delta_v}{\delta}\right)^2.$$

The integration of these yields

$$(12) \qquad \gamma = \frac{b}{(u-u_0)^2}, \qquad \delta = \frac{c}{(v-v_0)^2},$$

where u_0, v_0, b, c are constants. Substituting these into (7) and integrating the resulting differential equation, we find

$$(13) \qquad \frac{c}{v-v_0} = \pm\frac{b}{u-u_0} + \text{const.}$$

These are two orthogonal parabolic pencils of circles. By an appropriate Moebius transformation, we can convert (13) into (A′). Hence the canonical form of our differential equation (7) for this case is (A).

(II) Next let γ and δ be constants. By integrating (7), we find that the two families are two orthogonal parallel pencils of straight lines (which are equivalent by a Moebius transformation to two orthogonal parabolic pencils of circles). By a similitude, we can carry these into (A′). Hence the canonical form of (7) in this case is also equation (A).

(III) Let γ be not constant and δ constant. We find that γ and δ are given by

$$(14) \qquad \gamma = \frac{-b}{(u-u_0)^2}, \qquad \delta = \text{const.},$$

where u_0 and b are constants. Substituting these into (7) and integrating, we obtain

$$\delta v = \pm \frac{b}{u - u_0} + \text{const.} \tag{15}$$

The first family consists of all circles of the same radius $r = (b/\delta)^{1/2}$ and with centers on the minimal line $u = u_0$. The orthogonal family consists of all circles with centers on the minimal line $u = u_0$ and of radius ir. These two families of circles have the same radical axis, namely, their common line of centers $u = u_0$. Also these circles are tangent (and orthogonal) to each other at the point at infinity on this minimal line $u = u_0$.

These two families are equivalent by a Moebius transformation to two orthogonal parabolic pencils of circles. By an appropriate Moebius transformation, these two families (15) can be carried into the two orthogonal parallel pencils of straight lines (A′). Hence the canonical form of (7) in this case is also (A).

(IV) Finally let γ be constant and δ not constant. Then γ and δ are given by

$$\gamma = \text{const.}, \qquad \delta = - \frac{c}{(v - v_0)^2} \cdot \tag{16}$$

Substituting these into (7) and integrating the resulting differential equation, we find

$$\frac{c}{v - v_0} = \pm \gamma u + \text{const.}, \tag{17}$$

where v_0 and c are constants.

The first family consists of all circles of the same radius $r = (c/\gamma)^{1/2}$ and with centers on the minimal line $v = v_0$. The orthogonal family consists of all circles with centers on the minimal line $v = v_0$ and of radius ir. The two families of circles have the same radical axis, namely, their common line of centers $v = v_0$. Also these circles are tangent (and orthogonal) to each other at the point at infinity on this minimal line $v = v_0$.

These two families are equivalent by a Moebius transformation to two orthogonal parabolic pencils of circles. By an appropriate Moebius transformation, these two families (17) can be converted into (A′). Thus the canonical form of (7) in this case is again (A).

Case B. *Let* $a \neq 0$. *In this case, the differential equation* (7) *represents two orthogonal nonparabolic pencils of circles* (*or their Moebius equiva-*

lents). By a Moebius transformation, these may be reduced to the pencil of straight lines with vertex at origin and to the pencil of concentric circles with their common center at the origin

$$v/u = \text{const.}, \qquad uv = \text{const.} \tag{B$'$}$$

Hence the canonical form of the differential equation (7) for this case is

$$p^2 = v^2/u^2. \tag{B}$$

In this case, γ (and δ) cannot be constant. Let $\lambda = \gamma\gamma_u$. The first of equations (10) can be written in the form

$$2\gamma\lambda\lambda_\gamma - 5\lambda^2 = a^2\gamma^6. \tag{18}$$

Multiplying this by γ^{-6} and integrating, we find

$$\lambda^2 = \gamma^5(a^2\gamma + 2b), \tag{19}$$

where a and b are constants. Replacing λ by $\gamma\gamma_u$, this equation (and a similar equation for δ) can be written in the form

$$\gamma_u = \gamma^2(a^2 + 2b/\gamma)^{1/2}, \qquad \delta_v = \delta^2(a^2 + 2c/\delta)^{1/2}, \tag{20}$$

where a, b, c are constants.

(I) Let neither b nor c be zero. The preceding equations then yield

$$\gamma = \frac{2b}{b^2(u - u_0)^2 - a^2}, \qquad \delta = \frac{2c}{c^2(v - v_0)^2 - a^2}. \tag{21}$$

Substituting these into (7) and integrating, we obtain the two families of circles

$$\begin{aligned} \frac{c(v - v_0) - a}{c(v - v_0) + a} &= \text{const.}\, \frac{b(u - u_0) - a}{b(u - u_0) + a}, \\ \frac{c(v - v_0) - a}{c(v - v_0) + a} &= \text{const.}\, \frac{b(u - u_0) + a}{b(u - u_0) - a}. \end{aligned} \tag{22}$$

These two families represent two orthogonal nonparabolic pencils of circles. The circles of the first family are those passing through the two points $(u_0 + a/b,\ v_0 + a/c)$ and $(u_0 - a/b,\ v_0 - a/c)$. The circles of the second family are those passing through the two points $(u_0 + a/b,\ v_0 - a/c)$ and $(u_0 - a/b,\ v_0 + a/c)$.

The two orthogonal nonparabolic pencils of circles (22) have the same center (u_0, v_0). This is the intersection of the two orthogonal radical axes and of the two orthogonal lines of centers. If d is the distance between the two fixed points of the first pencil of circles, then id is the distance between the two fixed points of the second pencil. These properties are characteristic for two orthogonal nonparabolic pencils of circles.

By an appropriate Moebius transformation, the two families (22) can be converted into (B′). Thus the canonical form of the differential equation (7) for this case is (B).

(II) Let $b=c=0$. From (20), we find

$$\gamma = \frac{1}{a(u-u_0)}, \qquad \delta = \frac{1}{a(v-v_0)}. \tag{23}$$

Substituting these into (7) and integrating, we obtain the two families of circles

$$\frac{v-v_0}{u-u_0} = \text{const.}, \qquad (u-u_0)(v-v_0) = \text{const.} \tag{24}$$

The first family is a pencil of straight lines with vertex at (u_0, v_0). The second family is a pencil of concentric circles with the common center at (u_0, v_0). By an appropriate translation, these two families can be carried into (B′). Thus, in this case, the canonical form of the differential equation (7) is (B).

(III) Let $b=0$ and $c\neq 0$. In this case, we have

$$\gamma = \frac{1}{a(u-u_0)}, \qquad \delta = \frac{2c}{c^2(v-v_0)^2 - a^2}. \tag{25}$$

Substituting these into (7) and integrating, we find the two orthogonal pencils of circles

$$\begin{aligned} \frac{c(v-v_0)-a}{c(v-v_0)+a} &= \text{const.}\,(u-u_0), \\ \frac{c(v-v_0)-a}{c(v-v_0)+a} &= \text{const.}\,\frac{1}{u-u_0}. \end{aligned} \tag{26}$$

The circles of the first family are those passing through the point $(u_0, v_0+a/c)$ and the point at infinity on the minimal line $v=v_0-a/c$. The circles of the second family are those passing through the point $(u_0, v_0-a/c)$ and the point at infinity on the minimal line $v=v_0+a/c$.

The radical axis and the line of centers (or the line of centers and the radical axis) of the first (or second) family of circles are respectively the minimal lines $v=v_0+a/c$ and $v=v_0-a/c$.

Under the group of Moebius transformations, these two families are equivalent to any two orthogonal nonparabolic pencils of circles. By an appropriate Moebius transformation, these two families can be converted into (B′). Thus the canonical form of the differential equation (7) for this case is (B).

(IV) Let $b \neq 0$ and $c = 0$. The two families of circles are then

$$(27)\quad v - v_0 = \text{const.}\frac{b(u-u_0)-a}{b(u-u_0)+a}, \quad v - v_0 = \text{const.}\frac{b(u-u_0)+a}{b(u-u_0)-a}.$$

The circles of the first family are those passing through the point $(u_0+a/b, v_0)$ and the point at infinity on the minimal line $u=u_0-a/b$. The circles of the second family are those passing through the point $(u_0-a/b, v_0)$ and the point at infinity on the minimal line $u=u_0+a/b$. The radical axis and the line of centers (or the line of centers and the radical axis) of the first (or the second) family of circles are respectively the minimal lines $u=u_0+a/b$ and $u=u_0-a/b$.

Under the group of Moebius transformations, these two families are equivalent to any two orthogonal nonparabolic pencils of circles. By an appropriate Moebius transformation, these two families can be converted into (B′). Thus the canonical form of the differential equation (7) for this case is (B).

Thus in all cases, we have proved the following theorem:

THEOREM 2. *The only* $2\infty^1$ *circles whose differential equation is of the form* (7) *are two orthogonal pencils of circles. Under the group of Moebius transformations, these may be classified into two distinct types:* (1) *two orthogonal parabolic pencils of circles and* (2) *two orthogonal nonparabolic pencils of circles.*

The canonical forms of the differential equation (7) of types (1) and (2) are respectively (A) and (B).

4. **The conformal near-Moebius transformations.**

We now proceed to find the conformal near-Moebius transformations. First it is seen that *the inverse* N^{-1} *of any conformal near-Moebius transformation* N *is also a conformal near-Moebius transformation.* For since N is a one-to-one correspondence which carries $2\infty^1$ circles into circles, it follows that the transformed circles must be $2\infty^1$ in number. Hence the inverse N^{-1} carries $2\infty^1$ circles into circles, and therefore it must be a conformal near-Moebius transformation.

Let N be any conformal near-Moebius transformation. Then there exist exactly $2\infty^1$ circles which are preserved by N. Denote these circles by γ and their transformed circles under N by Γ. That is,

$N(\gamma)=\Gamma$, and $\gamma=N^{-1}(\Gamma)$. By Theorems 1 and 2, we find that γ *and* Γ *are each two orthogonal pencils of circles.*

Let γ_0 denote the canonical form (A′) and (A), or (B′) and (B) of two orthogonal parabolic, or nonparabolic, pencils of circles. Let M_1 by any Moebius transformation which carries the two orthogonal pencils of circles γ into the canonical form γ_0, and let M_2 be any Moebius transformation which converts the canonical form γ_0 into the two orthogonal pencils of circles Γ. Then the transformation $T=M_2^{-1}NM_1^{-1}$ preserves the canonical form γ_0 of two orthogonal pencils of circles. Hence *any conformal near-Moebius transformation* N *is of the form* M_2TM_1 *where* M_1 *and* M_2 *are Moebius transformations and* T *is any conformal near-Moebius transformation which preserves the canonical form* γ_0 *of two orthogonal pencils of circles.*

Next we shall show that any transformation T which preserves the canonical form γ_0 of two orthogonal pencils of circles must be one of the three types: (1) $U=be^{au}$, $V=ce^{\pm av}$; (2) $U=a\log u+b$, $V=\pm a\log v+c$; and (3) $U=bu^n$, $V=cv^{\pm n}$. Obviously T converts either (I) (A) into (A), or (II) (A) into (B), or (III) (B) into (A), or (IV) (B) into (B).

(I) Let T convert (A) into (A). The differential equation $p^2=1$ must be preserved. For this to be so, we find that T must be of the form $U=au+b$, $V=\pm av+c$ where $a\neq 0$, b, c are constants. This of course is a simple case of the Moebius transformations. Hence we conclude that *there are no conformal near-Moebius transformations which preserve two orthogonal parabolic pencils of circles.*

(II) Let T convert (A) into (B). The differential equation $p^2=1$ is converted into $P^2=V^2/U^2$. For this to be so, we find that T must be of the form $U=be^{au}$, $V=ce^{\pm av}$, where a, b, c are nonzero constants. Hence we conclude that *any conformal near-Moebius transformation* N *which carries two orthogonal parabolic pencils of circles into two orthogonal nonparabolic pencils of circles must be of the form* M_2TM_1 *where* M_1 *and* M_2 *are Moebius transformations and* T *is the transformation* $U=e^u$, $V=e^v$.

(III) Let T convert (B) into (A). The differential equation $p^2=v^2/u^2$ is converted into $P^2=1$. Hence T must be of the form $U=a\log u+b$, $V=\pm a\log v+c$, where $a\neq 0$, b, c are constants. Thus we conclude that *any conformal near-Moebius transformation* N *which carries two orthogonal nonparabolic pencils of circles into two orthogonal parabolic pencils must be of the form* M_2TM_1 *where* M_1 *and* M_2 *are Moebius transformations and* T *is the transformation* $U=\log u$, $V=\log v$.

(IV) Let T convert (B) into (B). The differential equation $p^2=v^2/u^2$ must be carried into the differential equation $P^2=V^2/U^2$.

Hence T must be of the form $U=bu^n$, $V=cv^{\pm n}$ where n, b, c are nonzero constants. Thus *any conformal near-Moebius transformation N which carries two orthogonal nonparabolic pencils of circles into two orthogonal nonparabolic pencils of circles must be of the form M_2TM_1 where M_1 and M_2 are Moebius transformations and T is the transformation $U=u^n$, $V=v^n$ with n a complex nonzero constant.*

We therefore obtain from the preceding results the following:

THEOREM 3. *Any conformal near-Moebius transformation N of the complex cartesian plane is of the form M_2TM_1 where M_1 and M_2 are Moebius transformations and T is any of the three transformations* (1) $U=e^u$, $V=e^v$; (2) $U=\log u$, $V=\log v$; and (3) $U=u^n$, $V=v^n$ (*n a complex nonzero constant*).

In this paper, we have given all the conformal near-Moebius transformations, that is, all the conformal transformatons which preserve *exactly* $2\infty^1$ circles. In a later paper, we shall give the set of all nonconformal near-Moebius transformations, that is, the set of all nonconformal transformations which preserve *exactly* $2\infty^2$ circles.

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A NOTE ON THE DEFINITION OF ARC-SETS[1]

W. L. AYRES

The interesting subsets of a Peano space which were later called the arc-sets or A-sets were defined independently and in a different manner by G. T. Whyburn[2] and the author,[3] and each studied the properties of these sets which have proved so useful in the later theory. The similarity of the properties led shortly to the conjecture that a relation existed between the two definitions, and investigation proved that they were equivalent. These are the properties (1) and (3) of the theorem of the present note. Two more equivalent properties, the properties (2) and (4) of the present note, were observed immediately.[4]

The present note introduces two new properties and shows that all six are equivalent, so that any one may be taken as the definition of the arc-sets or A-sets.

THEOREM. *For a nondegenerate*[5] *subset A of a Peano space P the following six properties are equivalent*:

(1) *There is a set H such that A is the set of points of all arcs of P whose end points belong to H.*

(2) *A contains every arc of P whose end points belong to A.*

(3) *A is a connected collection of cyclic elements of P.*

(4) *For any connected subset C of P, the set $A \cdot C$ is connected.*

(5) *Every set separating*[6] *two points of A in A separates them in P.*

(6) *A is connected, $\bar{A} \cdot \bar{C}$ is a single point for each component C of $P-A$, and this point y belongs to A if $\bar{A}$ contains two continua S and T such that $S \cdot T = y$.*

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[1] Presented to the Society, December 1, 1939.

[2] *Concerning the structure of a continuous curve*, American Journal of Mathematics, vol. 50 (1928), pp. 167–194.

[3] *Concerning the arc-curves and basic sets of a continuous curve*, Transactions of this Society, vol. 30 (1928), pp. 567–578.

[4] Properties (1), (3), and (4) were proved equivalent by W. L. Ayres, *Concerning the arc-curves and basic sets of a continuous curve, second paper*, Transactions of this Society, vol. 31 (1929), pp. 595–612. That (1), (2), (3), and (4) are equivalent was remarked with an indication of proof by C. Kuratowski and G. T. Whyburn, *Sur les éléments cycliques et leurs applications*, Fundamenta Mathematicae, vol. 16 (1930), pp. 305–331. See page 321.

[5] Even in the trivial degenerate case, (1), (2), (4), (5), and (6) are still equivalent if we define a point x as being an arc from x to x.

[6] We use the word *separating* here in the weak or *coupure* sense, that is, a set K *separates* two points x and y in a set A if $x+y \subset A$ and there exists no connected set C such that $x+y \subset C \subset A-K$.

(1)→(2). This has been proved by the author.[7]

(2)→(3). It follows directly from (2) that A is connected, in fact, arc-wise connected. Let E be a cyclic element of P such that $A\cdot E\neq 0$. If $A\cdot E$ is a single point, this point is a cut point of P and thus is itself a cyclic element. Suppose $A\cdot E\supset x+z$ and let y be any other point of E. Then E contains an arc[8] xyz which belongs to A by property (2). Hence E belongs to A.

(3)→(4). This has been proved by Kuratowski and Whyburn.[9]

(4)→(5). Let K denote a set separating x and y in A. If there exists a connected set C such that $x+y\subset C\subset P-K$, then $C\cdot A$ is a connected subset of $A-K$ containing $x+y$ by (4). Then K does not separate x and y in A.

(5)→(6). The set A is connected, for otherwise the null set would separate two points of A in A, which it could not in P.

If $C=y$ is a degenerate component of $P-A$, $\overline{C}\cdot\overline{A}=y$. As A is connected, any degenerate component of $P-A$ is a non-cut point of P. Then y is an end point or belongs to some simple closed curve J of P. As y is a point component of $P-A$, J contains two points x and z of A. Let α and β denote the subarcs xy and yz of the arc xyz of J. There exists an open set U such that $U\supset\alpha-y$ and $\overline{U}\cdot\beta=y$. This open set may be obtained by summing all open spheres whose centers are points w of $\alpha-y$ and whose radii are less than the distance from w to β. Then $K=F(U)-y$ separates x and z in A but does not separate them in P as it contains no point of the arc xyz. Hence every degenerate component of $P-A$ is an end point. As y is an end point of P, there do not exist in P, and thus not in $\overline{A}$, two continua S and T such that $S\cdot T=y$.

Suppose that C is nondegenerate and $\overline{A}\cdot\overline{C}\supset x+z$. As C is connected, $C-x-z\neq 0$. We shall show that P contains an arc xyz, where $y\subset C$. If there is a point y in P which separates x and z, then the connectivity of C gives $C\supset y$ and any arc xz of P contains y. If no point of P separates x and z, there is a true cyclic element $E\supset x+z$. As each component of $P-E$ is an open set and has just one limit point in E, it follows that $P-E+x+z$ contains no connected subset containing $x+z$. But as $C+x+z$ is such a connected set, C must contain a point y of E distinct from x and from z. Then E contains[8] an arc xyz.

As $x+z\subset\overline{A}\cdot\overline{C}$, each neighborhood of x and z respectively contains points x' and z' of A. If these neighborhoods are chosen sufficiently small, there exist small arcs xx' and zz' such that the set $xx'+xyz+zz'$ contains an arc $x'yz'$. Then just as above we may construct an open set U such that $F(U)-y$ separates x' and z' in A but not in P. Thus $\bar{A}\cdot\bar{C}$ is a single point for every component C of $P-A$.

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[7] *Arc-curves*, Theorem 3.

[8] W. L. Ayres, *Continuous curves which are cyclicly connected*, Bulletin de l'Académie Polonaise des Sciences et des Lettres, 1928, pp. 127–142, Theorem 3.

[9] Loc. cit., Theorem 8.5.

Suppose continua S and T exist such that $S+T\subset\bar{A}$ and $S\cdot T=y$. Suppose $y\not\subset A$. Then S contains a point x of A, for otherwise S belongs to a component C_1 of $P-A$ and $\bar{A}\cdot\bar{C}_1\supset S$, which is not a single point. Similarly T contains a point z of A. By the same process as outlined above, there exists an open set U such that $U\supset S-y$, $\bar{U}\cdot T=y$. Then $F(U)-y$ separates x and z in A but not in P as it contains no point of the connected $S+T$.

(6)→(1). Let $H=A$. Then the set of points of all arcs of P whose end points belong to H certainly contains A. Suppose there is an arc xyz such that $x+z\subset H=A$ and $y\not\subset A$. Let C denote the component of $P-A$ containing y. The subarc xy contains a point x of A and a point y of C, and thus contains a point of $\bar{A}\cdot\bar{C}$. Similarly the subarc yz contains a point of $\bar{A}\cdot\bar{C}$. But $\bar{A}\cdot\bar{C}$ is a single point. Hence $\bar{A}\cdot\bar{C}=y$.

As y was any interior point of the arc which belonged to $P-A$ and we showed that $\bar{A}\supset y$, we have $\bar{A}\supset$ arc xyz. Let $S=$subarc xy and $T=$subarc yz. We have $S+T\subset\bar{A}$, $S\cdot T=y$, and hence $y\subset A$ by property (6).

THE UNIVERSITY OF MICHIGAN

ON REARRANGEMENTS OF SERIES[1]

RALPH PALMER AGNEW

1. **Introduction.** Let E denote the metric space in which a point x is a permutation $x_1, x_2, x_3, \cdots$ of the positive integers and the distance (x, y) between two points $x \equiv \{x_1, x_2, \cdots\}$ and $y \equiv \{y_1, y_2, \cdots\}$ of E is given by the Fréchet formula

$$(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{| x_n - y_n |}{1 + | x_n - y_n |}.$$

The space E is of the second category (Theorem 2).

Let $c_1 + c_2 + \cdots$ be a convergent series of real terms for which $\sum |c_n| = \infty$. To simplify typography, we write $c(n)$ for c_n. To each $x \varepsilon E$ corresponds a rearrangement $c(x_1) + c(x_2) + \cdots$ of the series $\sum c_n$. By a well known theorem of Riemann, $x \varepsilon E$ exists such that $c(x_1) + c(x_2) + \cdots$ converges to a preassigned number, diverges to $+\infty$ or to $-\infty$, or oscillates in a preassigned fashion.

The set A of $x \varepsilon E$ for which $c(x_1) + c(x_2) + \cdots$ converges is therefore a proper subset of E, and M. Kac has proposed the problem of determining whether $E - A$ is of the second category. The following theorem shows not only that A is of the first category (and hence that $E - A$ is of the second category) but also that the set of $x \varepsilon E$ for which the series $c(x_1) + c(x_2) + \cdots$ has unilaterally bounded partial sums is of the first category.

THEOREM 1. *For each $x \varepsilon E$ except those belonging to a set of the first category,*

$$\liminf_{N \to \infty} \sum_{n=1}^{N} c(x_n) = -\infty, \qquad \limsup_{N \to \infty} \sum_{n=1}^{N} c(x_n) = \infty.$$

2. **Proof of Theorem 1.** The fact that the "coordinates" x_n and y_n of two points x and y of E are integers implies roughly that, if N is large, then $x_n = y_n$ for $n = 1, 2, \cdots, N$ if and only if (x, y) is near 0. To make this precise, let $x \varepsilon E$, $r > 0$, and let $S(x, r)$ denote the set of points y such that $(x, y) < r$, so that $S(x, r)$ is an open sphere with center at x and radius r. It is easy to show that if x and y are two points of E such that $y \varepsilon S(x, 2^{-N-1})$ then $x_n = y_n$ when $n = 1, 2, \cdots, N$; and that if x and y are such that $x_n = y_n$ when $n = 1, 2, \cdots, N$ then $y \varepsilon S(x, 2^{-N})$.

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[1] Presented to the Society, October 28, 1939.

To prove Theorem 1, let B denote the set of $x \varepsilon E$ for which

$$\limsup_{N\to\infty} \sum_{n=1}^{N} c(x_n) < \infty;$$

and, for each $h>0$, let B_h denote the set of $x \varepsilon E$ for which

$$\underset{N=1,2,3,\cdots}{\text{l.u.b.}} \sum_{n=1}^{N} c(x_n) < h.$$

Then

$$B = B_1 + B_2 + B_3 + \cdots.$$

We show that B is the first category by showing that B_h is nondense for each $h>0$. Suppose $h>0$ exists such that the closure $\overline{B}_h$ of B_h contains a sphere S' with center at $x' \equiv \{x_1', x_2', \cdots\}$ and radius $r>0$. Choose m so great that $2^{-m-1}+2^{-m-2}+\cdots<r/2$. Let $x_n''=x_n'$ when $1\leqq n\leqq m$; and define x_n'' for $n>m$ in such a way that $\sum c(x_n'')$ diverges to $+\infty$. Then $(x', x'')<r/2$ so that $x'' \varepsilon S'$. Choose an index q such that

$$c(x_1'') + c(x_2'') + \cdots + c(x_q'') > h,$$

and then choose $\delta>0$ such that $x_k=y_k$ for $k=1, 2, \cdots, q$ whenever $x, y \varepsilon E$ and $(x, y)<\delta$.

If x is a point within the sphere S'' with center at x'' and radius δ (that is, if $(x, x'')<\delta$), then $c(x_1)+c(x_2)+\cdots+c(x_q)>h$ and x is not in B_h. Thus B_h contains no point of S'' and consequently $\overline{B}_h$ does not contain x''. This contradicts the assumption that $\overline{B}_h$ contains S', and hence proves that B_h is nondense and B is of the first category. Similar considerations show that the set C of $x \varepsilon E$ for which $c(x_1)+\cdots+c(x_N)$ has inferior limit greater than $-\infty$ is of the first category. Since the union of two sets B and C of the first category is itself of the first category, Theorem 1 is established.

If $z_1+z_2+\cdots$ is a convergent series of complex terms for which $\sum|z_n|=\infty$, it is easy to apply our theorem to the series of real and imaginary parts of z_n to show that the set of $x \varepsilon E$ for which $z(x_1)+z(x_2)+\cdots$ has bounded partial sums is a set of the first category.

3. **The space** E.

In this section we obtain some properties of E and prove the following result.

THEOREM 2. *The space E is of the second category at each of its points.*

That the space E is not complete was pointed out to the author by Professor L. M. Graves. In fact if $x^{(n)}$ is the point

$$x^{(n)} \equiv \{2, 3, \cdots, n-1, n, 1, n+1, n+2, \cdots\}$$

of E, then $x^{(n)}$ is a Cauchy sequence in E which does not converge to a point of E. If $\mathcal{E}$ is the space in which a point is a sequence of positive integers not necessarily a permutation of all positive integers, and the distance between two points of $\mathcal{E}$ is given by the Fréchet formula, then $\mathcal{E}$ is complete and E is a subspace of $\mathcal{E}$. It is easy to show that the closure of E in $\mathcal{E}$ is the space $\mathcal{E}_1$ in which a point is a sequence of positive integers containing each positive integer *at most once*, and hence that $\mathcal{E}_1$ is the least complete subspace of $\mathcal{E}$ which contains E. For example, $\{2, 4, 6, 8, \cdots\}$ is a point of $\mathcal{E}_1$ which is not a point of E.

If $\mathcal{E}_x\{x_n=k\}$ denotes, for each n, $k=1, 2, \cdots$, the set of all $x \varepsilon \mathcal{E}$ for which $x_n=k$, then

$$\mathcal{E}_2 \equiv \prod_{k=1}^{\infty} \sum_{n=1}^{\infty} \mathcal{E}_x\{x_n = k\}$$

is the subset of $\mathcal{E}$ in which a point is a sequence containing each positive integer *at least once*. Since $\mathcal{E}_x\{x_n=k\}$ is an open subset of $\mathcal{E}$ for each n, $k=1, 2, \cdots$, $\mathcal{E}_2$ is the intersection of a countable set of open sets (that is, $\mathcal{E}_2$ is a G_δ) in $\mathcal{E}$. Since $\mathcal{E}_1$ is a closed subset of $\mathcal{E}$ and $E = \mathcal{E}_1\mathcal{E}_2$, it follows that E is a G_δ in the complete space $\mathcal{E}$.

Therefore, by a fundamental theorem whose proof is an easy extension of the familiar proof that a complete metric space is of the second category, E is of the second category at each of its points and Theorem 2 is proved.

CORNELL UNIVERSITY

ABELIAN GROUPS THAT ARE DIRECT SUMMANDS OF EVERY CONTAINING ABELIAN GROUP[1]

REINHOLD BAER

It is a well known theorem that an abelian group G satisfying $G=nG$ for every positive integer n is a direct summand of every abelian group H which contains G as a subgroup. It is the object of this note to generalize this theorem to abelian groups admitting a ring of operators, and to show that the corresponding conditions are not only sufficient but are at the same time necessary. Finally we show that every abelian group admitting a ring of operators may be imbedded in a group with the above mentioned properties; and it is possible to choose this "completion" of the given group in such a way that it is isomorphic to a subgroup of every other completion.

Our investigation is concerned with abelian groups admitting a ring of operators. A ring R is an abelian group with regard to addition, its multiplication is associative, and the two operations are connected by the distributive laws. As the multiplication in R need not be commutative, we ought to distinguish left-, right- and two-sided ideals. Since, however, only left-ideals will occur in the future, we may use the term "ideals" without fear of confusion. Thus an ideal in R is a non-vacuous set M of elements in R with the property:

If m', m'' are elements in M, and if r', r'' are elements in R, then $r'm' \pm r''m''$ is an element in M.

An abelian group G whose composition is written as addition is said to admit the elements in the ring R as operators (or shorter: G is an *abelian group over* R), if with every element r in R and g in G is connected their uniquely determined product rg so that this product is an element in G and so that this multiplication satisfies the associative and distributive laws. If G is an abelian group over R, then its subgroups M are characterized by the same property as the ideals M in R.

We assume finally the existence of an element 1 in R so that $g=1g$ for every g in G and $r\cdot 1=1\cdot r=r$ for every r in R.

If x is any element in the abelian group G over R, then its *order* $N(x)$ consists of all the elements r in R which satisfy $rx=0$. One verifies that every order $N(x)$ is an ideal in R, and that $N(x)=R$ if, and only if, $x=0$.

If M is an ideal in R, and if x is an element in G, then a subgroup of G is formed by the elements mx for m in M; and this subgroup may be denoted by Mx. (It is a subgroup of the cyclic group generated by x.) The correspondence between the element m in M and the element mx in Mx is a special case of the homomorphisms of M into G. Here a *homomorphism* ϕ of the ideal M in R into the abelian group G over R is defined as a single-valued function m^ϕ of the elements m in M with values in G which satisfies

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[1] Presented to the Society, February 24, 1940.

$$(r'm' \pm r''m'')^\phi = r'(m'^\phi) \pm r''(m''^\phi)$$

for m', m'' in M and r', r'' in R.

We are now ready to state and prove our main result.[2]

THEOREM 1. *The following two properties of an abelian group G over the ring R are each a consequence of the other.*

(a) *If G is a subgroup of the abelian group H over R, then G is a direct summand of H.*

(b) *To every ideal M in R and to every homomorphism ϕ of M into G there exists some element v in G so that $m^\phi = mv$ for every m in M.*

PROOF. Assume first that (a) is satisfied by G. If M is an ideal in R, and if ϕ is a homomorphism of M into G, then there exists one and essentially only one group H over R which is generated in adjoining to G an element h, subject to the relations

$$mh = m^\phi \qquad \text{for every } m \text{ in } M.$$

It is a consequence of (a) that H is the direct sum of G and of a suitable subgroup K of H so that every element in H may be represented in one and only one way in the form: $g+k$ for g in G and k in K. This applies in particular to the element h so that $h=v+w$ for uniquely determined elements v and w in G and K respectively. Then $m^\phi = mh = mv + mw$ is for every element m in M an element in G, and so is mv and $m^\phi - mv = mw$. Since mw is an element in K, it follows that $mw = 0$ or $m^\phi = mv$ for every m in M; and this shows that (b) is a consequence of (a).

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[2] The following is a remark by the referee: "It is perhaps of some interest to observe that Theorem 1 contains a generalization of the theorem that every representation of a semisimple algebra is fully reducible. Indeed, how does one characterize those rings R such that *every* abelian group G admitting R as an operator ring has the property (a)? The answer is that every left-ideal in R must be generated by an idempotent element, and this is equivalent to saying that R is semisimple (both chain conditions and no radical).

"The sufficiency of this condition is proved by showing that G has the property (b). Let $m \rightarrow m^\phi$ be any homomorphism of a left-ideal M of R into G. Since $M = Re$ with $e^2 = e$, then $me = m$ for every m in M, and if we take $v = e^\phi$ we have $m^\phi = (me)^\phi = me^\phi = mv$.

"The necessity is proved by observing that every left-ideal M of R is itself an abelian group over R, and is a subgroup of R. Hence to each M there exists a complementary left-ideal N. From $1 = e + e'$ with unique e in M, e' in N, one concludes in the usual way that $e^2 = e$ and $M = Re$."

Assume now conversely that (b) is satisfied by the abelian group G over R, and that G is a subgroup of the abelian group H over R. Then there exists a greatest subgroup K of H whose meet with G is 0. The subgroup S of H which is generated by G and K is their direct sum; and hence it suffices to prove that $H = S = G + K$.

If $S \neq H$, then there exists in H an element w that is not contained in S. The coset $W = S + w$ is an element in the quotient-group H/S and its order $N(W)$ is an ideal in R. If m is any element in $N(W)$, then mw is an element in S; and it follows from the construction of S that there exist uniquely determined elements $g(m)$ and $k(m)$ in G and K respectively so that

$$mw = g(m) + k(m) \qquad \text{for } m \text{ in } N(W).$$

Thus a homomorphism of $N(W)$ into G is defined in mapping the element m in $N(W)$ upon the element $g(m)$ in G. There exists therefore by condition (b) an element v in G so that $mv = g(m)$ for every m in $N(W)$. The element $w' = w - v$ consequently satisfies

$$S + w' = S + w = W$$

and

$$mw' = k(m)$$

for every element m in $N(W)$. Since w is not an element in S, neither is w'. Since K is a greatest subgroup of H whose meet with G is 0, and since w' is not an element in S and therefore not in K, adjoining w' to K generates a subgroup whose meet with G is different from 0. Hence there exists an element k in K, and an element r in R so that

$$k + rw' = g \neq 0$$

is an element in G. Since $rw' = g - k$ is an element in S, it follows that r is an element in $N(W)$ so that $g = k + k(r)$ is an element in the meet of G and K. This contradicts, however, the construction of K and the choice of g. Our hypothesis $S \neq H$ has thus led us to a contradiction; and this completes the proof.

If M is an ideal in the ring R, and if G is an abelian group over R, then G is termed *M-complete*, if there exists to every homomorphism ϕ of M into G an element v in G so that $m^\phi = mv$ for every m in M. In this terminology, condition (b) of Theorem 1 states that G is M-complete for every ideal M in R. It is our object to characterize the M-complete groups, provided M is a principal ideal. For this end we need several notations.

If M is an ideal in R, then denote by G_M the set of all the elements g in G which satisfy $mg=0$ for every m in M. Note that G_M need not be a subgroup of the abelian group G over R, though it is closed with regard to addition and subtraction.

If p is any element in R, then pG consists of all the elements pg for g in G. Note again that pG need not be a subgroup of the abelian group G over R.

The principal ideal in R, generated by the element p in R, consists of all the elements rp for r in R and may therefore be denoted by Rp; and $N(p)$ consists of all the elements r in R so that $rp=0$. $N(p)$ is clearly an ideal in R.

THEOREM 2. *The abelian group G over R is Rp-complete if and only if $G_{N(p)} \leqq pG$.*

PROOF. Suppose first that G is Rp-complete. If x is an element in $G_{N(p)}$, then $r'p=r''p$ implies $r'x=r''x$, since the first equation is equivalent to the fact that $r'-r''$ is an element in $N(p)$. Thus a homomorphism of Rp into G is defined in mapping rp upon rx. Since G is Rp-complete, there exists an element v in G so that $rpv=rx$ for every r in R. This implies in particular that $pv=x$, that is, our condition is necessary.

Suppose conversely that our condition be satisfied by G. If ϕ is a homomorphism of Rp into G, then p^ϕ is an element in $G_{N(p)}$, since we have $r(p^\phi)=(rp)^\phi=0$ for elements r in $N(p)$. Hence there exists an element q in G so that $p^\phi=pq$; and clearly $(rp)^\phi=r(p^\phi)=r(pq)=(rp)q$ for every r in R so that G is Rp-complete.

COROLLARY 1. *If p is an element in R so that $N(p)=0$, then $G=pG$ is a necessary and sufficient condition for Rp-completeness of the abelian group G over R.*

This is a consequence of Theorem 2, since $G_0=G$.

COROLLARY 2. *If $N(p)=0$ for every element $p\neq 0$ in the ring R, and if every ideal in R is a principal ideal Rp, then $G=pG$ for every $p\neq 0$ in R is a necessary and sufficient condition for the abelian group G over R to be a direct summand of every abelian group H over R which contains G as a subgroup.*

This is an obvious consequence of Theorem 1 and Corollary 1.

If, in particular, R consists of the rational integers, then the hypotheses of Corollary 2 are satisfied. In this case the sufficiency of the condition of the Corollary 2 has been known for a long time.[3]

An abelian group G over the ring R is termed *complete*, if it is M-complete for every ideal M in R. Thus the complete groups are just the groups satisfying the properties (a) and (b) of Theorem 1.

THEOREM 3. *Every abelian group over the ring R is a subgroup of a complete abelian group over the ring R.*

PROOF. If G is an abelian group over the ring R, M an ideal in R, and ϕ a homomorphism of M into G, then there exists an abelian group H over R which contains G as a subgroup and which contains an element x so that $mx=m^\phi$ for every m in M.

By repetition of the construction of the preceding paragraph one may show that if G is an abelian group over the ring R, then there exists an abelian group G' over the ring R which contains G as a subgroup and which satisfies the following condition:

(3.1) If M is an ideal in R, and if ϕ is a homomorphism of M into G, then there exists an element v in G' so that $mv=m^\phi$ for every m in M.

Denote now by λ an ordinal number which is a limit-ordinal and whose cardinal number is greater than the number of elements in R. Then it follows from the second paragraph of the proof that there exists for every ordinal ν with $0\leqq\nu\leqq\lambda$ an abelian group G_ν over R with the following properties:

(i) $G_0=G$;

(ii) $G_\nu\leqq G_\mu$ for $\nu<\mu$;

(iii) G_ν is for limit-ordinals ν the set of all the elements contained in groups G_μ for $\mu<\nu$;

(iv) G_ν and $G_{\nu+1}$ satisfy condition (3.1).

Suppose now that M is an ideal in R and that ϕ is a homomorphism of M into $H=G_\lambda$. Then there exists an ordinal $\sigma<\lambda$ so that G_σ contains all the elements m^ϕ; and there exists therefore an element v in $G_{\sigma+1}$ so that $mv=m^\phi$ for every m. H is therefore complete.

THEOREM 4. *To every subgroup G of the complete abelian group K over the ring R there exists a complete subgroup G^* of K which contains G as a subgroup and which satisfies the following condition:*

(E) *Every isomorphism of G upon a subgroup of a complete abelian group H over R is induced by an isomorphism of G^* upon a subgroup of H.*

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[3] For a comparable proof see Annals of Mathematics, (2), vol. 37 (1936), pp. 766–767, (1; 1).

Proof. If T is an abelian group over the ring R, if M is an ideal in R, then the homomorphism ϕ of M into T is termed *reducible* in T, if there exists an ideal M' in R and a homomorphism ϕ' of M' into T so that $M<M'$ and so that ϕ and ϕ' coincide on M. If ϕ is not reducible, then it is irreducible in T.

(4.1) *The abelian group T over the ring R is complete, if there exists to every ideal M in R and to every irreducible homomorphism ϕ of M into T an element v in T so that $mv=m^\phi$ for every m in M.*

To prove this statement let J be an ideal in R and γ a homomorphism of J into T. Then there exists a greatest ideal M in R so that $J\leqq M$ and so that γ is induced in J by a homomorphism ϕ of M into T. It is clear that ϕ is irreducible in T. Hence there exists an element v in T so that $mv=m^\phi$ for every m in M. This implies however that $nv=n^\phi=n^\gamma$ for every n in J, that is, T is complete.

It is a consequence of (4.1) and of the completeness of K that there exists an ascending chain of subgroups G_ν for $0\leqq\nu\leqq\lambda$ with the following properties:

(i) $G=G_0$;

(ii) $G_\nu\leqq K$ for $\nu\leqq\lambda$;

(iii) $G_{\nu+1}$ is generated by adjoining to G_ν an element g_ν with the following properties:

(iii′) The homomorphism of $N(G_\nu+g_\nu)$ into G_ν which is defined by mapping the element m in $N(G_\nu+g_\nu)$ upon the element mg_ν in G_ν is irreducible in G_ν.

(iii″)[4] G_ν does not contain any element x so that $mx=mg_\nu$ for every m in $N(G_\nu+g_\nu)$.

(iv) G_ν is for limit-ordinals ν the set of all the elements contained in groups G_μ for $\mu<\nu$.

(v) $G_\lambda=G^*$ is complete.

We are now going to prove that this subgroup G^* of K satisfies condition (E). Thus assume that ρ is an isomorphism of G upon the subgroup $G'=G^\rho$ of the complete group H. We are going to construct subgroups G_ν' of H and isomorphisms ρ_ν of G_ν upon G_ν' with the following properties:

(1) $G'=G_0'$, $\rho=\rho_0$;

(2) $G_\nu'\leqq G_\mu'$ for $\nu\leqq\mu$;

(3) ρ_ν and ρ_μ coincide on G_ν for $\nu\leqq\mu$.

In order to prove the possibility of this construction it suffices to show the existence of $G_{\nu+1}'$, $\rho_{\nu+1}$ under the hypothesis of the existence of G_ν', ρ_ν.

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[4] This condition (iii″) is not really needed for the proof, though it is convenient for the construction of the chain G_ν.

A homomorphism irreducible in G'_ν of $M=N(G_\nu+g_\nu)$ into G'_ν is defined by mapping the element m in M upon the element $m^\phi=(mg_\nu)^{\rho_\nu}$. Since H is complete, there exists an element h in H so that $m^\phi=mh$ for every m in M. If $M'=N(G'_\nu+h)$, then it is clear that $M\leqq M'$. If m is in M', then mh is an element in G'_ν. Thus a homomorphism γ of M' into G_ν is defined by mapping the element m in M' upon the element $m^\gamma=(mh)^{\rho_\nu^{-1}}$. If, in particular, m is an element in M, then $m^\gamma=mg_\nu$; and it follows from (iii′) that $M=M'$. Suppose now that g' is an element in G'_ν and u an element in R so that $g'+uh=0$. Then u is an element in $M=M'$ and it follows from the above considerations that $-g'=uh=u^{\gamma\rho_\nu}=(ug_\nu)^{\rho_\nu}$. Hence there exists one and only one isomorphism $\rho_{\nu+1}$ of $G_{\nu+1}$ upon the group $G'_{\nu+1}$, generated by G'_ν and h, which isomorphism induces ρ_ν in G_ν and maps g_ν upon h.

Thus there exists finally an isomorphism ρ_λ of $G^*=G_\lambda$ upon G'_λ which induces ρ in G; and this completes the proof.

COROLLARY. *Assume that K is a smallest complete abelian group over the ring R containing the subgroups G_i. Then G_1 and G_2 are isomorphic if, and only if, there exists an automorphism of K mapping G_1 upon G_2.*

This is an obvious consequence of Theorem 4. It should be noted, however, that the complete group G^*, satisfying (E) and containing G, whose existence is assured by Theorems 3 and 4, is only "essentially smallest," but need not be "actually smallest."

UNIVERSITY OF ILLINOIS

CLOSURE OF PRODUCTS OF FUNCTIONS[1]

D. G. BOURGIN

This note presents some natural theorems on the characterizations of certain closed (*or complete*) sets of functions with separable variables. In order to motivate the developments of the paper we treat a simple case first in elaborate detail. The proof is so formulated that it holds with trifling modifications for the more general situations in Theorems 3 and 4. The result in Theorem 5 belongs to a slightly different range of ideas.

Let $s\sim(s_1, \cdots, s_m)$ and $t\sim(t_1, \cdots, t_n)$ here stand for points in the euclidean spaces R_m and R_n. The term "interval" designates the generalized rectangular parallelopipedon open on the left.[2] We shall make use of the intervals $I_s \subset R_m$, $I_t \subset R_n$ and $I_2 = I_s \times I_t \subset R_{n+m}$. We are first interested in $L_2(I)$, the space of complex valued functions of summable square over I. The norm and scalar product are defined as usual by

$$\|f(s, t) - g(s, t)\| = \left[\int_{I_t}\int_{I_s} |f(s, t) - g(s, t)|^2 dI_s dI_t\right]^{1/2}, \tag{1}$$

$$(f(s, t), g(s, t)) = \int_{I_t}\int_{I_s} f(s, t)\bar{g}(s, t) dI_s dI_t, \tag{2}$$

where $\bar{g}(s, t)$ is the conjugate of $g(s, t)$. The subscript I_s or I_t will indicate that the left-hand functionals are on the corresponding intervals.

We shall understand closure of the sequence of functions[3] $\{\phi_\gamma(t)\psi_\mu(s)\}$, $\gamma, \mu = 0, 1, \cdots$, to mean that for every $f(s, t) \varepsilon L_2(I_2)$ and arbitrary $\epsilon > 0$ there exists a finite sequence of complex constants $\{\beta_{\gamma\mu}\}$ and integers A and B such that

$$\left\|f(s, t) - \sum_0^A \sum_0^B \beta_{\gamma\mu}\phi_\gamma(t)\psi_\mu(s)\right\| < \epsilon. \tag{3}$$

It is well known that with the adjunction of the scalar product defined in (2), $L_2(I_2)$ is a complex Hilbert space and that closure and completeness are equivalent concepts.

THEOREM 1. *If* $\{\phi_\gamma(t)\psi_\mu(s)\}$, $\gamma, \mu = 0, 1, \cdots$, *is a sequence of com-*

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[1] Presented to the Society, December 2, 1939.

[2] S. Saks, *Theory of the Integral*, English edition, p. 57.

[3] Curly brackets, $\{\ \}$, will always denote sequences.

plex valued functions in $L_2(I_2)$, then a necessary and sufficient[4] *condition for closure is that $\{\phi_\gamma(t)\}$ and $\{\psi_\mu(s)\}$ be closed in the spaces $L_2(I_t)$ and $L_2(I_s)$ respectively.*

We deal with the sufficiency demonstration first. Suppose the denumerable set of all subintervals, with rational end points, of I_t to be ordered according to $0, 1, 2, \cdots$. We designate by $h_\rho(t)$ the characteristic function[5] of the ρth subinterval divided by its norm. The function $g_\nu(s)$ is similarly defined for the range I_s. Thus

$$\| h_\rho(t) \|_{I_t} = \| g_\nu(s) \|_{I_s} = 1. \tag{4}$$

It is well known that $\{h_\rho(t) g_\nu(s)\}$, $\rho, \nu = 0, 1, \cdots$, has the closure property in $L_2(I_2)$. Hence for $f(s, t) \in L_2(I_2)$ and arbitrary $\epsilon > 0$ we can find integers M and N and MN complex constants $\{a_{\rho\nu}\}$ such that

$$\left\| f(s, t) - \sum_0^M \sum_0^N a_{\rho\nu} h_\rho(t) g_\nu(s) \right\| < \epsilon/2. \tag{5}$$

Let

$$\delta \leqq \min\left(\frac{\epsilon}{4MN} \max |a_{\rho\nu}|, 1\right). \tag{6}$$

Thus

$$2\delta \sum_0^M \sum_0^N | a_{\rho\nu} | < \epsilon/2. \tag{6.1}$$

In view of the assumed closure properties of $\{\phi_\gamma(t)\}$ and $\{\psi_\mu(s)\}$, integers A and B and complex constants $\{d_\mu^{(\nu)}\}$, $\{e_\gamma^{(\rho)}\}$, $\rho = 0, 1, \cdots, M$ and $\nu = 0, 1, \cdots, N$, exist which yield the simultaneous inequalities

$$\left\| g_\nu(s) - \sum_{\mu=0}^B d_\mu^{(\nu)} \psi_\mu(s) \right\|_I < \delta/2, \tag{7}$$

$$\left\| h_\rho(t) - \sum_0^A e_\gamma^{(\rho)} \phi_\gamma(t) \right\|_{I_t} < \delta/2. \tag{7.1}$$

Hence

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[4] A special case amounting to the assertion of sufficiency, only, for the subspace of $L_2(I_2)$ composed of real continuous functions, when $\{\phi_\gamma(t)\}$ and $\{\psi_\mu(s)\}$ are restricted to be orthogonal sets of functions, has been given by Courant: Courant-Hilbert, *Methoden der mathematischen Physik*, vol. 1, 1st edition, p. 90. Another special sufficiency proof is given in A. Zymund, *Trigonometrical Series*, p. 13.

[5] Saks, loc. cit., p. 6.

$$(7.2)\qquad \left\|\sum_0^B d_\mu^{(\nu)}\psi_\mu(s)\right\|_{I_s} \leqq \|g_\nu(s)\|_{I_t} + \left\|g_\nu(s) - \sum_0^B d_\mu^{(\nu)}\psi_\mu(s)\right\|_{I_s} \leqq 2.$$

Let $\beta_{\gamma\mu}=\sum_{\rho=0}^{M}\sum_{\nu=0}^{N} a_{\rho\nu}e_\gamma^{(\rho)}d_\mu^{(\nu)}$. The triangle inequality for norms yields, in view of (6), (7), (7.11), and (7.2)

$$(8)\qquad \begin{aligned} &\left\| h_\rho(t)g_\nu(s) - \sum_0^A\sum_0^B e_\gamma^{(\rho)}d_\mu^{(\nu)}\phi_\gamma(t)\psi_\mu(s)\right\| \\ &\leqq \left\| h_\rho(t)\left(g_\nu(s) - \sum_0^B d_\mu^{(\nu)}\psi_\mu(s)\right)\right\| \\ &\qquad + \left\| \sum_0^B d_\mu^{(\nu)}\psi_\mu(s)\left(h_\rho(t) - \sum_0^A e_\gamma^{(\rho)}\phi_\gamma(t)\right)\right\| \\ &\leqq \|h_\rho(t)\|_{I_t}\left\| g_\nu(s) - \sum_0^B d_\mu^{(\nu)}\psi_\mu(s)\right\|_{I_s} \\ &\qquad + \left\|\sum_0^B d_\mu^{(\nu)}\psi_\mu(s)\right\|_{I_s}\left\| h_\rho(t) - \sum_0^A e_\gamma^{(\rho)}\phi_\gamma(t)\right\|_{I_t} \\ &\leqq 2\delta, \qquad \text{for } \rho = 0, 1, \cdots, M, \nu = 0, 1, \cdots, N. \end{aligned}$$

On combining the various inequalities above

$$(9)\qquad \begin{aligned} &\left\| f(s, t) - \sum_0^A\sum_0^B \beta_{\gamma\mu}\phi_\gamma(t)\psi_\mu(s)\right\| \\ &\leqq \left\| f(s, t) - \sum_0^M\sum_0^N a_{\rho\nu}h_\rho(t)g_\nu(s)\right\| \\ &\qquad + \left\| \sum_0^M\sum_0^N a_{\rho\nu}\left(h_\rho(t)g_\nu(s) - \sum_0^A\sum_0^B e_\gamma^{\rho}d_\mu^{\nu}\phi_\gamma(t)\psi_\mu(s)\right)\right\| \\ &\leqq \epsilon/2 + \sum_0^M\sum_0^N\left(|a_{\rho\nu}|\left\| h_\rho(t)g_\nu(s) - \sum_0^A\sum_0^B e_\gamma^{(\rho)}d_\mu^{(\nu)}\phi_\gamma(t)\psi_\mu(s)\right\|\right) \\ &\leqq \epsilon/2 + 2\delta\sum_0^M\sum_0^N|a_{\rho\nu}| \leqq \epsilon. \end{aligned}$$

This asserts the closure property for $\{\phi_\gamma(t)\psi_\mu(s)\}$.

The necessity demonstration is equally direct. A trivial application of Fubini's theorem indicates that $\phi_\gamma(t) \varepsilon L_2(I_t)$, $\psi_\mu(s) \varepsilon L_2(I_s)$ when $\phi_\gamma(t)\psi_\mu(s) \varepsilon L_2(I_2)$. No generality is lost if we assume that $\{\psi_\mu(s)\}$ is a linearly independent set of functions. Suppose $\{\psi_\mu(s)\}$ does not have the closure property. Then $f(s) \varepsilon L_2(I_s)$ exists for which for all R and b_μ

$$\text{G.L.B.}\left\| f(s) - \sum_0^R b_\mu\psi_\mu(s)\right\|_{I_s} = c > 0, \qquad b_\mu = b'_\mu + ib''_\mu. \tag{10}$$

A fundamental result of Riesz guarantees the existence of *minimal* constants,[6] $\{\bar{b}_\mu^R\}$, such that for $b_\mu \neq \bar{b}_\mu^R$, $\mu \leqq R$,

$$\left\| f(s) - \sum_0^R \bar{b}_\mu^R \psi_\mu(s)\right\|_{I_s} \leqq \left\| f(s) - \sum_0^R b_\mu\psi_\mu(s)\right\|_{I_s}. \tag{11}$$

The corresponding minimal constants for $Af(s)$ are evidently $\{A\bar{b}_\mu^R\}$. Hence[7]

$$\left\| F(t)f(s) - \sum_0^R \bar{b}_\mu^R F(t)\psi_\mu(s)\right\|_{I_s} \leqq \left\| F(t)f(s) - \sum_0^R b_\mu(t)\psi_\mu(s)\right\|_{I_s}, \quad t \,\varepsilon\, I_t, \tag{12}$$

when $F(t) \,\varepsilon\, L_2(I_t)$ is a fixed function of positive norm. We write

$$b_\mu(t) = \sum_0^Q a_{\gamma\mu}\phi_\gamma(t), \qquad Q < \infty. \tag{13}$$

In view of (12) we have

$$\begin{aligned} 0 < c\|F(t)\|_{I_t} &\leqq \left\| f(s)F(t) - \sum_0^R \bar{b}_\mu^R F(t)\psi_\mu(s)\right\| \\ &= \left[\int_{I_t} \left\| f(s)F(t) - \sum_0^R \bar{b}_\mu^R F(t)\psi_\mu(s)\right\|_{I_s}^2 dI_t\right]^{1/2} \\ &\leqq \left[\int_{I_t} \left\| f(s)F(t) - \sum_0^R\sum_0^Q a_{\gamma\mu}\phi_\gamma(t)\psi_\mu(s)\right\|_{I_s}^2 dI_t\right]^{1/2} \\ &= \left\| f(s)F(t) - \sum_0^R\sum_0^Q a_{\gamma\mu}\phi_\gamma(t)\psi_\mu(s)\right\|. \end{aligned} \tag{14}$$

Since (14) is in contradiction with the assumed closure property of $\{\phi_\gamma(t)\psi_\mu(s)\}$ our necessity proof is complete.

We denote by $h'_\rho(t)$ and $g'_\nu(s)$ the step functions in R_n and R_m analogous to $h_\rho(t)$ and $g_\nu(s)$. According to a classical result, $\{h'_\rho(t)g'_\nu(s)\}$, ρ, $\nu = 0, 1, \cdots$, have the closure property in $L_2(E_2)$ when the s, t integration is over R_{n+m} or any Lebesgue measurable subset E_2. Accordingly Theorem 1 and its demonstration remain formally valid in detail when I_s, I_t and I_2 are replaced either by R_n, R_m and R_{n+m} or by

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[6] F. Riesz, Acta Mathematica, vol. 41 (1916), p. 77, Lemma 3.

[7] With the choice $F(t)f(s)$, the method of proof of the necessity condition remains valid when sets of infinite measure are included.

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the sets E_s, E_t and $E_2 = E_s \times E_t$ of finite or infinite Lebesgue measure.

THEOREM 2. *If* $\{F_\rho(s, t)\}$, $\rho = 0, 1, \cdots$, *is closed in* $L_2(E_2)$, *then the sequence is also closed in* $L_2(E_s)$ *except possibly for a* t *set of zero measure.*[8]

Suppose a lower bound of approximation to $f(s) \varepsilon L_2(E_s)$, by linear combinations of $\{F_\rho(s, t)\}$, is $c(t) \varepsilon L_2(E_t)$, where $\infty > c(t) > 0$ for $t \varepsilon G \subset E_t$. Let $F(t) \varepsilon L_2(E_t)$ differ from 0 on G (say $F(t) = c(t)$). The analogue of (14) is

$$\|c(t)F(t)\|_G \leqq \|c(t)F(t)\|_{E_t} \leqq \left\| f(s)F(t) - \sum_0^R b_\mu F_\mu(s, t) \right\|. \tag{14'}$$

Hence G has zero measure. Let $\{f^{(\sigma)}(s)\}$ be closed in $L_2(E_s)$ and denote the corresponding G sets, defined above, by $\{G^\sigma\}$. The denumerable sum $\mathfrak{S}G^\sigma$ is plainly of measure zero. Thus $\{F_\rho(s, t)\}$ is closed in $L_2(E_s)$ for all $t \varepsilon E_t - \mathfrak{S}G^\sigma$.

We now abstract the properties needed in the foregoing proofs. Let $T(E)$ denote a Banach space[9] of real functions on E. A set G, $G \subset E$, will be called a ***non-significant*** set if $f(z) \varepsilon T(E)$ may be arbitrarily changed on G without affecting the value of $\|f(z)\|_E$. The postulates below hold for $T(E)$. When (d) and (e) are omitted we write $T_-(E)$.

(a) If $f(s, t) \varepsilon T(E_2)$ then $f(s, t) \varepsilon T(E_s)$ and $f(s, t) \varepsilon T(E_t)$ for all save a non-significant set of t or s values respectively. If $f(s) \varepsilon T(E_s)$, $F(t) \varepsilon T(E_t)$ then $f(s)F(t) \varepsilon T(E_2)$.

(b) $\|f(s, t)\|_{E_2} = \| \, \|f(s, t)\|_{E_s} \|_{E_t}$.

(c) If, neglecting non-significant sets, $|f_1(t)| > |f_2(t)|$, then $\|f_1(t)\|_{E_t} > \|f_2(t)\|_{E_t}$.

(d) There exists a sequence $\{h_\rho(t)g_\nu(s)\}$, $\rho, \nu = 0, 1, \cdots$, with the closure property in $T(E_2)$, where $h_\rho(t) \varepsilon T(E_t)$ and $g_\nu(s) \varepsilon T(E_s)$.

(e) Denumerable sums of non-significant sets are non-significant sets.

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[8] A sharper result follows from Fatou's lemma. Suppose $F(t) \varepsilon L_2(E_t)$ differs from 0 for almost all $t \varepsilon E_t$. Now

$$0 = \mathrm{L}_{N\to\infty} \Big\| f(s)F(t) - \sum_0^N b_\rho^{(N)} F_\rho(s, t) \Big\|_{E_2}^2 \geqq \int_{E_t} \mathrm{L}_{N\to\infty} \Big\| f(s)F(t) - \sum^N b_\rho^{(N)} F_\rho(s, t) \Big\|_{E_s}^2 dE_t.$$

Thus a suitable sequence $\{\sum^{N_i} b_\rho^{(i)} F_\rho(s, t)\}$, with *constant coefficients* $\{b_\rho^{(i)}\}$, converges strongly to $f(s)$ in $L_2(E_s)$ for almost all $t \varepsilon E_t$. Moreover if E_t is of finite measure, the Egoroff theorem guarantees uniform convergence for $t \varepsilon D_\delta \subset E_t$ where the measure of $E_t - D_\delta$ is inferior to arbitrary δ. A closed sequence $\{f_\sigma(s)\}$ is introduced as above.

[9] S. Banach, *Théorie des Opérations Linéaires*, pp. 53, 58. Banach uses *fundamental* in the sense of our *closed.*

THEOREM 3. *(α) If $\{\phi_\gamma(t)\}$ and $\{\psi_\mu(s)\}$ are closed in $T(E_t)$, $T(E_s)$ then $\{\phi_\gamma(t)\psi_\mu(s)\}$ is closed in $T(E_2)$. (β) If $\{\phi_\gamma(t)\psi_\mu(s)\}$ is closed in $T_-(E_2)$, then $\{\psi_\mu(s)\}$ is closed in $T_-(E_s)$. (γ) If $\{F_\rho(s, t)\}$ is closed in $T(E_2)$, then $\{F_\rho(s, t)\}$ is closed in $T(E_s)$ for all but a non-significant set of t values in E_t.*

The demonstrations of Theorems 1 and 2 apply without change in form.[10] The space[11] $L_p(E, \mu)$, $p \geqq 1$, is included in $T(E)$. This is the space of measurable functions whose pth powers are summable over the measurable set E, where the Lebesgue-Radon-Stieltjes integral is equally admissible with the usual Lebesgue integral. Thus the symbol $\mu(E)$ denotes either the Lebesgue measure, or the Radon measure determined by a non-negative additive function of intervals. In all cases $\mu_2(E_2) = \mu_s(E_s)\mu_t(E_t)$, and the sets of zero measure constitute the non-significant sets. The norm is

$$\|f(s, t)\| = \left[\int_E |f|^p d\mu(E)\right]^{1/p}. \tag{15}$$

The verification of the main postulates is implied by the Fubini theorem, the Hölder-Minkowski inequalities and the denseness of the step functions. The functions $\{h_\rho(t)\}$, $\{g_\nu(s)\}$ or $\{h_\rho'(t)g_\nu'(s)\}$ as defined in Theorem 1 are again available.[12]

The space $C(E)$ of continuous functions is another special case of $T(E)$. We assume $E_s \subseteqq R_m$, $E_t \subseteqq R_n$ and $E_2 \subseteqq R_{n+m}$ are bounded closed sets. The null set is the only non-significant set. The norm is

$$\|f(s, t)\| = \max_{s,t \in E_2} |f(s, t)|. \tag{16}$$

The sequences $h_\rho(t)$ and $g_\nu(s)$ are the ordered products of the elements $1, t_1, \cdots, t_n$ and of $1, s_1, \cdots, s_m$ respectively.

Postulates (b) and (c) may be replaced by the weaker

(b′) $\|f(w, z)\|_{E_2} < \epsilon$ implies $\|f(w, z)\|_{E_w} < \eta(\epsilon)$, where $\mathrm{L}_{\epsilon \to 0}\eta(\epsilon) = 0$ except possibly for non-significant z sets.

(c′) $\|G(z)\|_{E_z} = 1$, $\|H(w)\|_{E_w} < \epsilon$ imply $\|G(z)H(w)\|_{E_2} < \eta(\epsilon)$.

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[10] For (α), postulate (d) may be replaced by the assumption that each $f(s, t) \in T(E_2)$ is the strong limit of some (not necessarily fixed) sequence $\{h_\rho^f(t)g_\nu^f(s)\}$, where $h_\rho^f(t) \in T(E_t)$ and $g_\nu^f(s) \in T(E_s)$.

[11] Saks, loc. cit. (1928), chap. 3, or J. Radon, Sitzungsberichte der Akademie der Wissenschaften, Vienna, class IIa, vol. 122 (1913). The Lebesgue case admits sets of infinite measure.

[12] For $p > 1$ a valid theorem on *completeness* is obtained from Theorem 3 if closure (in $L_p(E, \mu)$) is replaced by completeness in $L_{p/(p-1)}(E, \mu)$ where E refers to E_s, E_t and E_2 in turn,

These modifications will be connoted by writing $T'(E)$ and $T'_-(E)$. Consider, for instance, $C^1(E)$, the space of functions continuous together with their first partial derivatives on[13] E. We restrict ourselves now to *closed* linear intervals I_s, I_t and the rectangle I_2: $I_s \times I_t$. The norms in $C^1(I_2)$ and $C^1(I_s)$ are,[14] with $f_s \equiv \partial f/\partial s$,

$$\tag{17} \begin{aligned} \|f(s, t)\| &= \max_{I_2} |f(s, t)| + \max_{I_2} |f_s(s, t)| + \max |f_t(s, t)|, \\ \|f(s)\| &= \max_{I_s} |f(s)| + \max_{I_s} |f_s(s)|. \end{aligned}$$

It is well known that $C^1(I_s)$ (and $C^1(I_t)$) is complete. It is easy to show that $C^1(I_2)$ also is complete. Indeed if $\{f^{(n)}(s, t)\}$ is a Cauchy sequence in $C^1(I_2)$, then $f^{(n)}(s, t), f_s^{(n)}(s, t)$ and $f_t^{(n)}(s, t)$ converge ***uniformly*** in I_2 and hence define an element of $C^1(I_2)$.

Since

(b′) $\|F(s, t)\|_{I_2} \geqq \max_{t \varepsilon I_t} \|f(s, t)\|_{I_s}$ (t and s are interchangeable),

(c′) $\|G(s)H(t)\|_{I_2} \leqq \|G(s)\|_{I_s} \|H(t)\|_{I_t}$.

it is clear that (b′) and (c′) are satisfied.

THEOREM 4. *The conclusions in* (α), (β), (γ) *of Theorem* 3 *remain valid when* $T'(E)$ *and* $T'_-(E)$ *replace* $T(E)$ *and* $T_-(E)$.

For (α) we now choose δ small enough in (7) and (7.1) to yield $\eta(\delta)$ inferior to the right side of (6). Then (6.1) is valid with $\eta(\delta)$ written in place of δ. On making use of (c′) it is easily shown that the left side of (8) is smaller than $2\eta(\delta)$ and the final inequality in (9) is again obtained. For (β) we need only change (14) slightly. Indeed, by reference to (b′) and (13)

$$\epsilon \geqq \|f(s)F(t) - \sum \sum a_{\gamma\mu}\phi_\gamma(t)\psi_\mu(s)\|_{I_2}$$

would imply the contradiction

$$\tag{14''} \eta(\epsilon) \geqq |c| \text{ true max } |F(t)| > 0.$$

The ***true maximum*** is defined just as in the analogous case of measurable functions and implies neglect of non-significant t sets. Evidently (γ) also may be maintained. Indeed the argument in footnote 8, for example, is easily amended to yield the desired result.

If the closure property of the sequence $\{\phi_\rho(z)\}$ in $T_-(E)$ or $C^1(I)$ is unaffected by the omission of $\phi_\sigma(z)$, then we shall say $\{\phi_\rho(z)\}$ is a

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[13] The sets used in $C(E)$ are available for $C^1(E)$ also.

[14] Even if $f(s, t)$ and $g(s)h(t) \varepsilon C^1(E_2)$, $\| \|f(s, t)\|_{I_s}\|_{I_t}$ and $\| \|g(s)h(t)\|_{I_s}\|_{I_t}$ need not exist. Thus $C^1(E)$ is not included under $T(E)$.

"*redundant*" sequence and $\phi_\sigma(z)$ is a "*superfluous*" function. If $\{\phi_{\sigma_k}\}$, $k=1, 2, \cdots, K$, is superfluous, then for arbitrary ϵ we can satisfy

$$\left\| \phi_{\sigma_k} - \sum_0^{N}{}' c_j \phi_j(z) \right\| < \epsilon, \quad j \neq \sigma_l, l = 1, \cdots, K. \tag{18}$$

LEMMA 1. *In $T_-(E)$ or $T'_-(E)$ if $\{f_\mu(z)\}$ is closed and non-redundant, then for any $F(z)$, $\overline{\mathrm{L}}_{\epsilon\to 0}|d_1(\epsilon)| \leqq D < \infty$ where $d_1(\epsilon)$ is consistent with $\|F(z) - d_1(\epsilon)f_1(z) - \sum_2^N d_j f_j(z)\| < \epsilon$.*

In the contrary case

$$\begin{aligned} \epsilon + \|F(z)\| &\geqq \|F(z)\| + \left\| F(z) - d_1(\epsilon)f_1(z) - \sum_2^N d_j f_j(z) \right\| \\ &\geqq |d_1(\epsilon)| \left\| f_1(z) - \sum_2^N \frac{d_i}{d_1} f_i(z) \right\|. \end{aligned} \tag{19}$$

Now

$$\left\| f_1(z) - \sum_2^N \frac{d_i}{d_1} f_i(z) \right\| \geqq c > 0, \tag{20}$$

for all N and d_i, since $f_1(z)$ is not superfluous. For all sufficiently small ϵ, (19) and (20) imply

$$|d_1(\epsilon)| \leqq 2\|F(z)\|/c \tag{21}$$

in contradiction with the hypothesized non-boundedness of $d_1(\epsilon)$.

THEOREM 5. *If $\{\phi_\mu(t)\psi_\mu(s)\}$ is closed in $T_-(E_2)$ or C^1E, then* (I) *$\{\psi_\mu(s)\}$ is closed in $T_-(E_s)$ (or $C^1(I_s)$);* (II) *every finite subsequence of $\{\psi_\mu(s)\}$ is superfluous.*[15]

Evidently (I) is a special case of Theorem 3(β). In view of (I) if $\phi_\sigma(T)\psi_\sigma(s)$, $\sigma=1, \cdots, q$, is superfluous, then $\psi_\sigma(s)$, $\sigma=1, \cdots, q$, is superfluous. Accordingly we may restrict ourselves to non-redundant sequences $\{\phi_\mu(t)\psi_\mu(s)\}$.

We demonstrate (II) by induction. Suppose $\psi_1(s), \cdots, \psi_{n-1}(s)$ are superfluous. Since no finite basis exists in $T_-(E)$ or $C^1(I)$, we may find a function $F(t)$ such that the set $F(t), \phi_\sigma(t)$, $\sigma=1, \cdots, n$, is linearly independent. Suppose $\psi_n(s)$ is not superfluous. Then

$$\left\| \psi_n(s) - \sum_{n+1}^N k_i \psi_i(s) \right\|_E \geqq c > 0, \tag{22}$$

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[15] Evidently $\{\psi_\mu(s)\}$ need not be *dense closed* in the sense that any infinite subsequence is closed.

for all k_i and N. By hypothesis sequences $\{a_i^{(\rho)}\}$ and a constant N exist for arbitrary ϵ such that

$$\left\| \psi_\rho(s) - \sum_{i=n}^{N} a_i^{(\rho)} \psi_i(s) \right\|_{E_s} \leqq \epsilon, \qquad \rho = 1, \cdots, n-1. \tag{22.1}$$

Moreover

$$\begin{aligned} &\left\| \psi_n(s)F(t) - \sum_{\rho=1}^{n} d_\rho \phi_\rho(t)\psi_\rho(s) - \sum_{n+1}^{N} d_i\phi_i(t)\psi_i(s) \right\| \\ &\qquad + \left\| \sum_{\rho=1}^{n-1} d_\rho\phi_\rho(t) \left(\psi_\rho(s) - \sum_{n}^{N} a_i^{(\rho)} \psi_i(s) \right) \right\| \\ &\qquad \geqq \left\| \psi_n(s) \left[F(t) - \sum_{\rho=1}^{n-1} d_\rho a_n^{\rho} \phi_\rho(t) - d_n\phi_n(t) \right] \right. \\ &\qquad\qquad \left. - \sum_{n+1}^{N} d_i\phi_i(t)\psi_i(s) - \sum_{\sigma=1}^{n-1} \sum_{n+1}^{N} a_i^{(\sigma)} d_\sigma \phi_\sigma(t)\psi_i(s) \right\|. \end{aligned} \tag{23}$$

The right side of this inequality, by an argument similar in all details to that involved in the passage from (12) to (14), dominates

$$c \left\| F(t) - \sum_{1}^{n-1} d_\sigma a_n^{(\sigma)} \phi_\sigma(t) - d_n\phi_n(t) \right\|_{I_t} \text{ in } T_-(E_2) \tag{23.1}$$

or (cf. (b''))

$$\max_{t \in I_t} c \left| F(t) - \sum_{1}^{n-1} d_\sigma a_n^{\sigma} \phi_\sigma(t) - d_n\phi_n(t) \right| \text{ in } C^1(I_2). \tag{23.2}$$

In (23.2) we note $\phi_j(t) \varepsilon C^1(I_t)$ implies $\phi_j(t) \varepsilon C(I_t)$. Hence again by the Riesz theorem the expressions in (23.1) and (23.2) have a positive lower bound, denoted by $K>0$. In view of (22.1) closure of $\{\phi_\mu(t)\psi_\mu(s)\}$ and postulates (b) or (c''), constants N, d_i and $a_i^{(\rho)}$ exist such that the left side of (23) is inferior to

$$\epsilon + \sum_{1}^{n-1} |d_\rho| \, \|\phi_\rho(t)\|_{I_t}\epsilon. \tag{24}$$

Hence by Lemma 1 applied to each d_ρ the upper bound in (24) approaches 0 with ϵ in contradiction with the conclusion $K>0$. Thus $\psi_n(s)$ is superfluous.

This type of argument may be used to show that the non-redundancy of $\{\phi_\mu(t)\psi_\mu(s)\}$ implies that $\psi_1(s)$ is superfluous. The induction is thus complete and part (II) of our theorem is established. It is an easy matter to extend the theorem to $T'_-(E)$ spaces.

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THE SPACES L^p WITH $0<p<1$[1]

MAHLON M. DAY[2]

If we have a Lebesgue measurable set E in any n-dimensional euclidean space, and have any positive number q, we define $L^q(E)$ to be the class of all real-valued Lebesgue measurable functions f on E for which $\int_E |f|^q < \infty$. As is well known, whenever $q \geqq 1$ the class of functions $L^q(E)$ is a Banach space with the norm $\|f\|_q = (\int_E |f|^q)^{1/q}$. When $0<p<1$, the function $\|f\|_p$ no longer satisfies the triangle inequality $\|f_1+f_2\|_p \leqq \|f_1\|_p + \|f_2\|_p$ but in general only the weaker condition $\|f_1+f_2\|_p \leqq 2^\nu[\|f_1\|_p+\|f_2\|_p]$, where $\nu=(1-p)/p$. (This can be shown by considering the function $(1+x^p)/(1+x)^p$.) If we consider such an L^p space as a linear topological space in which the neighborhoods of a point f_0 are the spheres of radius $\epsilon>0$, $E_{f \varepsilon L^p}[\|f-f_0\|_p<\epsilon]$, it follows from theorems of Hyers and Wehausen[3] that this topology can be given by an equivalent Fréchet metric. This suggests that while many theorems on Banach spaces which can be applied to the spaces $L^p(E)$ with $p \geqq 1$ may fail to hold in those spaces with $0<p<1$, there may still remain many theorems on Fréchet spaces and pseudo-normed spaces which may be applicable. However, Theorem 1 shows that almost no results depending on the use of linear (that is, additive and continuous) functionals can be usefully applied in these spaces.

THEOREM 1. *Any linear functional on $L^p(E)$ is identically zero.*

The proof of this and of some additional results is given in a series of lemmas using a more general set of assumptions. We assume as a background some knowledge of the first chapter of Saks[4] book, in which he deals with what he calls "the integral," a completely additive integral having many of the properties of the Lebesgue integral. We consider a set Y of elements y, an additive family[5] $\mathfrak{X}$ of subsets of Y and an additive, non-negative[6] set-function μ on $\mathfrak{X}$ such that $\mu(Y)<\infty$. This last condition will be imposed from here until we reach Theorem 4. We shall finally remove it altogether and get the same type of theorem giving the linear functionals on any $L_\mu^p(Y)$. We shall also find the conjugate space of the sequence spaces related to these. For each $p>0$ let $L^p = L_\mu^p(Y)$ be the class of real-valued, $\mathfrak{X}$-measurable[7] functions f on Y such that $\|f\|_p = (\int_Y |f|^p d\mu)^{1/p} < \infty$. It is clear, as in the spaces with $p \geqq 1$, that if U is a linear functional on L^p, there is a $K \geqq 0$ such that $|U(f)| \leqq K$ whenever $\|f\|_p \leqq 1$. We let $\|U\|$ be the smallest such K; that is, $\|U\| = \sup_f |U(f)|$ where the sup is taken over those f with $\|f\|_p = 1$. Similarly if an additive functional is bounded on the unit sphere it is easily seen to be continuous and hence linear.

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[1] Presented to the Society, February 24, 1940.

[2] Corinna Borden Keen Research Fellow of Brown University.

[3] D. H. Hyers, *A note on linear topological spaces*, this Bulletin, vol. 44 (1938), pp. 76–80, and J. V. Wehausen, *Transformations in linear topological spaces*, Duke Mathematical Journal, vol. 4 (1938), pp. 157–169.

[4] S. Saks, *Theory of the Integral*, Warsaw, 1937.

[5] A class $\mathfrak{X}$ of subsets of Y is called an additive family if (a) $X \varepsilon \mathfrak{X}$ implies $Y-X \varepsilon \mathfrak{X}$, and (b) $X_n \varepsilon \mathfrak{X}$ implies $\sum_{n<\infty} X_n \varepsilon \mathfrak{X}$.

[6] A set-function μ is additive if whenever X_n are disjoint sets of $\mathfrak{X}$ then $\mu(\sum_{n<\infty} X_n) = \sum_{n<\infty} \mu(X_n)$; μ is non-negative if $\mu(X) \geqq 0$ for every $X \varepsilon \mathfrak{X}$.

In the series of lemmas immediately following we shall consider a fixed number p with $0<p<1$ and a fixed family $\mathfrak{X}$ and measure μ.

LEMMA 1. *If $f_i, f \in L^1$ and $\|f_i - f\|_1 \to 0$, then $\|f_i - f\|_p \to 0$.*

PROOF. Let $E_i = E_{y \in Y}[|f_i(y) - f(y)| > 1]$, and let $g_i(y) = f(y)$ in E_i, $g_i(y) = f_i(y)$ in $Y - E_i$. Then

$$\int_Y |f_i - f|^p d\mu = \int_{E_i} |f_i - f|^p d\mu + \int_Y |g_i - f|^p d\mu.$$

We have $\int_{E_i} |f_i - f|^p \leqq \int_{E_i} |f_i - f|\, d\mu \leqq \int_Y |f_i - f|\, d\mu$; this integral tends to zero by hypothesis. Also $\|g_i - f\|_1 \to 0$, so g_i converges in measure to f; moreover, $|g_i(y) - f(y)| \leqq 1$ for all $y \in Y$, so, by the Lebesgue convergence theorem, $\int_Y |g_i - f|^p d\mu \to 0$.

LEMMA 2. *Every linear functional on L^1 can be expressed in the form $U(f) = \int_Y fu\,d\mu$ where u depends on U, is $\mathfrak{X}$-measurable and, except possibly on a set of μ-measure zero, is bounded.*

The proof follows that given in Banach[8] for the special case $Y = (0, 1)$. If we let f_E be the characteristic function of $E \in \mathfrak{X}$ and let $\Phi(E) = U(f_E)$, the set-function Φ thus defined on $\mathfrak{X}$ is additive, $|\Phi(E)| \leqq \|U\| \mu(E)$ and $U(f) = \int_Y f d\Phi$ for each $f \in L^1$. By the theorem of Radon and Nikodym (Saks, p. 36) there is an $\mathfrak{X}$-measurable function u such that $\Phi(E) = \int_E u\,d\mu$ and $\|U\| = \operatorname{ess\,sup}_{y \in Y} |u(y)|$. By a theorem on change of measure (Saks, p. 37) we have $U(f) = \int_Y fu\,d\mu$.

LEMMA 3. *If U is linear on L^p, there is an $\mathfrak{X}$-measurable, essentially bounded function u on Y such that whenever $f \in L^1$, $U(f) = \int_Y fu\,d\mu$, while if $f \in L^p - L^1$ there exists a sequence $\{f_i\}$ of functions in L^1 such that*

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[7] A function f defined on Y is $\mathfrak{X}$-measurable if for each real a the set $E_{y \in Y}[f(y) > a]$ is in $\mathfrak{X}$.

[8] S. Banach, *Théorie des Opérations Linéaires*, Warsaw, 1932.

$\|f_i - f\|_p \to 0$, and for every such sequence $U(f) = \lim_{i\to\infty} U(f_i) = \lim_{i\to\infty} \int_Y f_i u\, d\mu$.

This follows immediately from the first two lemmas and from the obvious fact that the elements of L^1 form a dense set in L^p.

Lemma 4. *If for each $E \varepsilon \mathfrak{X}$ with $\mu(E) > 0$ there is a function $f_0 \varepsilon L^p - L^1$ such that $f_0(y) = 0$ when $y \varepsilon Y - E$, and if U is linear on L^p, then $U(f) = 0$ for every $f \varepsilon L^p$.*

Proof. If U is not identically zero, the function u of Lemma 3 cannot be almost everywhere zero; hence there is an $\alpha > 0$ and a set $E \varepsilon X$ such that $|u(y)| > \alpha$ if $y \varepsilon E$. Let $f'(y) = f_0(y)$ signum $f_0(y) \cdot u(y)$ where signum $s = |s|/s$ when $s \neq 0, = 0$ when $s = 0$; for each n let $E_n = E_{y \varepsilon Y}[|f'(y)| \leqq n]$, and let $f_n(y) = f'(y)$ on E_n, $f_n(y) = 0$ otherwise. Then $\|f_n - f'\|_p \to 0$; also

$$U(f_n) = \int_Y u f_n d\mu = \int_{E_n} |f_0 u|\, d\mu \geqq \alpha \int_{E_n} |f_0|\, d\mu,$$

but $E_n \uparrow E$ and $\int_E |f_0|\, d\mu = \infty$; so $U(f_n) \uparrow \infty$, contradicting continuity.

Lemma 5. *If $q > 1$, $E \varepsilon \mathfrak{X}$ and $\mu(E) > 0$, and if no function f which is zero on $Y - E$ belongs to $L^1 - L^q$, then every $f \varepsilon L^1$ is essentially bounded on E.*

Proof. If there is a function not essentially bounded on E, there is such a function f_0 which at the same time is zero on $Y - E$. Hence there exists a countably infinite sequence of sets $E_n \varepsilon \mathfrak{X}$ with $\mu(E_n) = b_n > 0$, $E_n \subset E$, E_n disjoint, $|f_0(y)| \geqq n$ on E_n, and, since $f_0 \varepsilon L^q$, $\sum n^q b_n < \infty$. Let α be so chosen that $1 < \alpha < 1/q + q - 1$ and let $a_n = 1/b_n n^\alpha$; let $f(y) = a_n$ if $y \varepsilon E_n$, $f(y) = 0$ if $y \varepsilon Y - \sum_{n<\infty} E_n$. Then $\int_Y |f|\, d\mu = \sum_{n<\infty} a_n b_n = \sum n^{-\alpha} < \infty$, so $f \varepsilon L^1$. On the other hand $\int_Y |f|^q d\mu = \sum a_n^q b_n = \sum b_n^{1-q} n^{-\alpha q}$. Since $\sum n^q b_n < \infty$ there must be a $K > 0$ such that $n^q b_n < K$ for all n; this implies that $b_n^{1-q} > n^{q(q-1)} K^{1-q}$ so $\int_Y |f|^q d\mu \geqq K^{1-q} \sum n^{q(q-1)} n^{-\alpha q} \geqq K^{1-q} \sum n^{-1} = \infty$, and $f \not\varepsilon L^q$ contrary to hypothesis.

For convenience we introduce a property of sets of $\mathfrak{X}$: a set $E \varepsilon \mathfrak{X}$ has property A relative to μ (in symbols $E \varepsilon \mathfrak{A}$) if $0 < \mu(E) < \infty$ and for every $E' \varepsilon \mathfrak{X}$ either[9] $\mu(E \cdot E') = 0$ or $\mu(E - E') = 0$.

There are several obvious properties of the sets of $\mathfrak{A}$.

(1) If E has only one element and $\mu(E) > 0$, then $E \varepsilon \mathfrak{A}$.

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[9] The referee has called to the writer's attention the fact that S. Saks calls such sets "singular" in his paper, *Addition to the note on some functionals*, Transactions of this Society, vol. 35 (1933), pp. 965–970.

(2) If $E \varepsilon \mathfrak{A}$ and f is $\mathfrak{X}$-measurable, then f is equal to a constant $\bar{f}$ almost everywhere on E.

(3) If $E_1 \varepsilon \mathfrak{A}$ and $E_2 \varepsilon \mathfrak{X}$, either $\mu(E_1E_2)=0$ or $\mu(E_1E_2)=\mu(E_1)$.

(4) $\mathfrak{A}$ is empty if and only if given $\epsilon>0$ it is possible to cover any set $E \varepsilon \mathfrak{X}$ with a countable collection of sets $S_i \varepsilon \mathfrak{X}$ with $\mu(S_i)<\epsilon$ for all i.

(5) If $E \varepsilon \mathfrak{A}$ and $\mu(E')=\mu(E'')=0$, then $E_0=E+E'-E'' \varepsilon \mathfrak{A}$ also.

THEOREM 2. *There is a nonzero linear functional on L^p if and only if $\mathfrak{A}$ is not empty.*

PROOF. If $\mathfrak{A}$ is empty and $E \varepsilon \mathfrak{X}$ with $\mu(E)>0$, there is a sequence of disjoint subsets of E, each of positive measure. Hence there is a function in L^1 essentially unbounded on E; therefore, by Lemma 5 with $q=1/p$, there is an $f \varepsilon L^p-L^1$ with $f(y)=0$ in $Y-E$; by Lemma 4 every linear functional on L^p is identically zero.

On the other hand if there is a set $E \varepsilon \mathfrak{A}$, and if $\|f_n-f\|_p\to 0$, f_n converges in measure to f; that is, for each $\epsilon>0$, $\mu(E_{y\varepsilon Y}[|f_n(y)-f(y)|>\epsilon])\to 0$ as $n\to\infty$. If $\bar{f}_n$ and $\bar{f}$ are the values of f_n and f almost everywhere on E, it follows that $\bar{f}_n\to\bar{f}$. Hence for any real number k setting $U(f)=k\bar{f}$ defines a linear functional U on L^p and if $k\neq 0$ then U is not identically zero.

For the case in which $m(E)$, the Lebesgue measure of E, is finite, Theorem 1 follows directly from Theorem 2 since every Lebesgue measurable set can be split into two subsets, each of measure half as great, so $\mathfrak{A}$ is empty for Lebesgue measure; the same conclusion could be derived from (4) above. The general case follows from this and from Theorem 6 below.

There are a large number of measures μ for which $\mathfrak{A}$ is empty. For example, to get one such class of measures, let X be a separable metric space and μ a Carathéodory measure on the class $\mathfrak{X}$ of μ-measurable sets; the class $\mathfrak{X}$, as is well known, contains all the Borel sets in X. If Y is a measurable subset of X, we say that μ is a uniformly continuous function on Y if the conditions $E_n \varepsilon \mathfrak{X}$, $E_n \subset Y$, and the sequence of diameters[10] $\delta(E_n)\to 0$ together imply that $\mu(E_n)\to 0$. It is easy to show from this condition that given $\epsilon>0$ there is a $\delta>0$ such that $\mu(E_n)<\epsilon$ whenever $E \varepsilon \mathfrak{X}$, $E\subset Y$ and $\delta(E)<\epsilon$.

LEMMA 6. *If μ is a Carathéodory measure on the separable space X, if $X=\sum Y_j$ with each $Y_j \varepsilon \mathfrak{X}$ and $\mu(Y_j)<\infty$, and μ is uniformly continuous on each Y_j, then the class $\mathfrak{A}$ is empty.*

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[10] $\delta(E)=\sup_{x,y}\rho(x, y)$ where the sup is taken over $x \varepsilon E$ and $y \varepsilon E$ and $\rho(x, y)$ is the distance in X from x to y.

Proof. If $E \varepsilon \mathfrak{A}$ and $\mu(E)=b>0$, we have by (3) that $\mu(EY_i)=0$ if $i\neq j$, while $\mu(EY_j)=b$; then by the separability of X we can cover Y_j with spheres of arbitrarily small radius. Hence by uniform continuity and (4) $\mu(EY_j)=0$ also and $\mu(E)=0$, so E cannot be in $\mathfrak{A}$.

Corollary. *For such* X *and* μ, *if* $\mu(X)<\infty$ *there are no nonzero linear functionals on* $L^p_\mu(X)$, $0<p<1$.

We remark that a later theorem allows us to remove the restriction that $\mu(X)<\infty$.

The interesting spaces $L^p_\mu(Y)$, then, are those for which the set $\mathfrak{A}$ is not empty; in such a case the structure of $\mathfrak{A}$ is rather simple.

Lemma 7. *If* $\mathfrak{A}$ *is not empty, there exists a finite or countable sequence of sets* $\{E_i\}\subset\mathfrak{A}$ *such that every* $E \varepsilon \mathfrak{A}$ *differs from just one* E_i *only by sets of measure zero.*

Proof. Well-order $\mathfrak{A}$, and define E_α to be the first set in the ordering which does not overlap any E_β, $\beta<\alpha$, on a set of positive measure. Since $\mu(Y)<\infty$, the number of these sets E_α for which $\mu(E)>1/n$ must be finite for each n; hence the sequence $\{E_\alpha\}$ is finite or countable.

In what follows we shall let $\{E_i\}$ be the family of sets of $\mathfrak{A}$ of the preceding lemma, let $b_i=\mu(E_i)$ and let $\bar{f}_i$ be the value of the measurable function f almost everywhere on E_i.

Theorem 3. *Any linear functional on* L^p *is identically zero, if* $\mathfrak{A}$ *is empty, or can be expressed in the form* $U(f)=\sum_i u_i\bar{f}_i$ *where* $\|U\|=\sup_i |u_i| b_i^{-1/p}$. *If* u_i *are given so that* $|u_i|<Kb_i^{1/p}$ *for all* i, *the functional* $U(f)=\sum_i u_i\bar{f}_i$ *is linear on* L^p.

Proof. It is easily seen from Theorem 2, applied to $L^p_\mu(Y-\sum_i E_i)$, that $U(f)$ is independent of the values of f on $Y-\sum_i E_i$; hence no generality is lost if we assume $Y=\sum_i E_i$ for simplicity. Let $f_{(i)}$ be the characteristic function of E_i (that is, $f_{(i)}(y)=1$ if $y \varepsilon E_i$, $f_{(i)}(y)=0$ otherwise), and take any $f \varepsilon L^p$; then $\|\sum_{i\leqq n}\bar{f}_i f_{(i)}-f\|_p\rightarrow 0$ as $n\rightarrow\infty$; so, by continuity, $U(f)=\lim_n U(\sum_{i\leqq n}\bar{f}_i f_{(i)})=\lim_n \sum_{i\leqq n}\bar{f}_i U(f_{(i)})=\sum_i u_i\bar{f}_i$ where $u_i=U(f_{(i)})$. Postponing the computation of $\|U\|$ for a moment, let us assume u_i given such that $|u_i|\leqq Kb_i^{1/p}$, and take $\|f\|_p=1$. Then $|\bar{f}_i b_i^{1/p}|\leqq\|f\|_p\leqq 1$, so that $|\bar{f}_i| b_i^{1/p}\leqq|\bar{f}_i|^p b_i$ since $0<p<1$. Therefore $|U(f)|\leqq\sum_i|u_i\bar{f}_i|\leqq K\sum_i|b_i^{1/p}\bar{f}_i|\leqq K\sum_i|\bar{f}_i|^p b_i\leqq K$, so U is bounded; hence it is continuous, and $\|U\|\leqq K$. It follows, if $K=\sup_i |u_i| b_i^{-1/p}$, that $\|U\|\geqq K$ also for $|U(b_i^{-1/p}f_{(i)})|=|u_i| b_i^{-1/p}\leqq\|U\|$; therefore $\sup_i|u_i| b_i^{-1/p}\leqq\|U\|$ also. We note that this shows that the series $\sum_i|u_i\bar{f}_i|$ converges.

We now turn to the case in which $\mu(Y)$ need not be finite; for greater convenience we introduce another property of sets relative to μ; Y has property B (in symbols $Y \varepsilon \mathfrak{B}$) if Y can be expressed as $\sum_{j<\infty} Y_j$ with $Y_j \varepsilon \mathfrak{X}$ and $\mu(Y_j) < \infty$ for each j.

Clearly Lemma 7 holds if $Y \varepsilon \mathfrak{B}$ as well as if $\mu(Y) < \infty$; we let E_i, b_i and $\bar{f}_i$ have their previous meanings.

THEOREM 4. *If $Y \varepsilon \mathfrak{B}$, a functional U on $L^p_\mu(Y)$ is linear if and only if* (a) *it is identically zero and $\mathfrak{A}$ is empty, or* (b) *$\mathfrak{A}$ is not empty and U can be expressed in the form $U(f) = \sum_i u_i \bar{f}_i$ with $\|U\| = \sup_i |u_i| b_i^{-1/p}$.*

PROOF. The sets Y_j which exist by property B can obviously be taken disjoint; we let U_j be a linear functional on $L^p_\mu(Y)$ defined by $U_j(f) = U(f'_j)$ where $f'_j(y) = f(y)$ when $y \varepsilon Y_j$, $f'_j(y) = 0$ otherwise. Then U_j is linear on $L^p_\mu(Y_j)$; so by Theorems 2 and 3 either U_j is identically zero or $U_j(f) = \sum_i u_{ji} \bar{f}_{ji}$. Now $\lim_n \|\sum_{i \leqq n} f'_i - f\|_p = 0$, so $U(f) = \lim_n \sum_{j \leqq n} U(f'_j) = \sum_j U_j(f)$. Moreover there is an $f_0 \varepsilon L^p$ such that $|f_0(y)| = |f(y)|$ for all $y \varepsilon Y$ and $U_j(f_0) = |U_j(f)|$ for all j; hence $|U_j(f)| \leqq \sum_j U_j(f_0) = U(f_0) \leqq \|U\| \cdot \|f_0\| = \|U\| \cdot \|f\|$, so $\|U_j\| \leqq \|U\|$ and the series $\sum_j U_j(f)$ converges absolutely for all $f \varepsilon L^p_\mu$. Then $|u_{ji}| b_{ji}^{-1/p} \leqq \|U_j\| \leqq \|U\|$ and $U(f) = \sum_j \sum_i u_{ji} \bar{f}_{ji}$ unless all U_j are identically zero, that is, unless $\mathfrak{A} = 0$. Rearranging this into a simple series, permissible since it converges absolutely, we get $U(f) = \sum u_i f_i$. The other conclusions follow as in Theorem 3.

The general form of Theorem 1 follows directly from this, since the whole euclidean space is the sum of a countable set of finite intervals.

There remains the case in which Y is not in $\mathfrak{B}$. As we did in the proof of Lemma 7, we can define a well-ordered set $\{E_\gamma\}$ of sets of $\mathfrak{A}$, disjoint up to sets of measure zero and such that every $E \varepsilon \mathfrak{A}$ is, except for sets of measure zero, equal to some E_γ; however we have no assurance that the sequence will be countable. As before we let $b_\gamma = \mu(E_\gamma)$ and $\bar{f}_\gamma$ be the value of f almost everywhere on E_γ. We need also a known lemma about the set in which a function in $L^p_\mu(Y)$ is not equal to 0.

LEMMA 8. *For every $q > 0$ and any $f \varepsilon L^q_\mu(Y)$, $E_0 = E_y[f(y) \neq 0] \varepsilon \mathfrak{B}$.*

PROOF. Since $f \varepsilon L^q$ means $|f|^q \varepsilon L^1$, we need only consider $L^1_\mu(Y)$ and may suppose that $f(y) > 0$ for all $y \varepsilon Y$. By definition of the integral (see Saks, p. 19 ff.) $\int_Y f d\mu = \sup_g \int_Y g d\mu$ where the sup is taken over those functions g which are linear combinations of characteristic functions of sets of $\mathfrak{X}$ and which at the same time satisfy the inequalities $0 \leqq g(y) \leqq f(y)$ for almost all $y \varepsilon Y$. But such a function can be different from zero only on a set of finite measure if it is to have a finite integral. There is a sequence $\{g_i\}$ of these functions such that $g_i \uparrow f$ almost everywhere in Y and $\int_Y f d\mu = \lim_i \int_Y g_i d\mu$; hence the set where $f > 0$ is, except for a set of measure zero, the sum of the sets on which the $g_i > 0$, and each of these sets is of finite measure.

By this lemma it is possible to put each function f of $L^p_\mu(Y)$ into at least one class C_E such that $f(y) = 0$ in $Y - E$ and $E \varepsilon \mathfrak{B}$; then C_E is equivalent to $L^p_\mu(E)$, and, if we set $U_E(f) = U(f)$ for $f \varepsilon C_E$, U_E is linear on $L^p_\mu(E)$ and therefore can be expressed as before where the E_i of Theorem 4 will be those E_γ which, except for sets of measure zero, lie in E. But $f = 0$ on $Y - E$; so we can write $U(f) = \sum_\gamma u_\gamma \bar{f}_\gamma$ if it is not identically zero, with the convention that the sum of any number of terms in which $\bar{f}_\alpha$ or u_α is zero shall be zero. Since this can be done for each $f \varepsilon L^p_\mu(Y)$, we get our final result, including the previous theorems as special cases.

THEOREM 5. *A functional U on $L^p_\mu(Y)$ is linear if and only if* (a) $\mathfrak{A}$ *is empty and U is identically zero or* (b) $\mathfrak{A}$ *is not empty and U can be expressed in the form $U(f) = \sum_\gamma u_\gamma \bar{f}_\gamma$ and $\|U\| = \sup_\gamma |u_\gamma| b_\gamma^{-1/p}$.*

If we let L^{p*} be the space of linear functionals on $L^p_\mu(Y) = L^p$, it is clear that with the given norm L^{p*} is a Banach space. To make the study of its structure simpler we introduce the class Γ of ordinal numbers as follows; If $\mathfrak{A}$ is empty, Γ is also empty; if $\mathfrak{A}$ is not empty, Γ is the class of ordinals used in ordering the sets E_γ of $\mathfrak{A}$ defined above. We use $\mathbf{x} = \{x_\gamma\}$ to stand for a real-valued function on Γ and define the special function $\mathbf{x}_\gamma$ to have the value 1 at γ and 0 elsewhere. Let $\textsc{m} = \textsc{m}(\Gamma)$ be the class of bounded functions $\mathbf{x}$ with $\|\mathbf{x}\|_M = \sup_{\gamma \varepsilon \Gamma} |x_\gamma|$; for any $q > 0$ let $\textsc{l}^q = \textsc{l}^q(\Gamma)$ be the class of all $\mathbf{x}$ such that $\|\mathbf{x}\|_q = (\sum_{\gamma \varepsilon \Gamma} |x_\gamma|^q)^{1/q} < \infty$.

It is clear that the space $\textsc{l}^q(\Gamma)$ here defined is also the space $L^q_{\mu_0}(\Gamma)$ where μ_0 is the trivial measure assigning the measure 1 to each set containing just one point γ and $\mathfrak{X}$ is the smallest additive family on the countable sets in Γ. From Theorem 5 we have this result:

THEOREM 6. *If Γ is not empty, a functional U on $\textsc{l}^p$ is linear if and only if it can be expressed as $U(\mathbf{x}) = \sum_{\gamma \varepsilon \Gamma} u_\gamma x_\gamma$ where $\|U\| = \sup_{\gamma \varepsilon \Gamma} |u_\gamma|$.*

Considering again the space L^{p*}, it is clear that when $\mathfrak{A}$ is not empty the transformation which associates the element $U \varepsilon L^{p*}$ defined by $U(f) = \sum_{\gamma \varepsilon \Gamma} u_\gamma \bar{f}_\gamma$ with the element $u = \{u_\gamma b_\gamma^{-1/p}\} \varepsilon \textsc{m}$ is one-to-one, linear both ways and norm-preserving between all of L^{p*} and all of $\textsc{m}$; but this is precisely the definition of equivalence of two Banach spaces (Banach, p. 180); hence:

(6) L^{p*} and $\textsc{m}$ are equivalent Banach spaces.

This holds even for the case in which $\mathfrak{A}$ is empty if we define $m(\Gamma)$ for Γ empty to be the Banach space with just one element, the zero element.

Slightly modifying our previous usage, we let $f_{(\gamma)}$ be $b_\gamma^{-1/p}$ on E_γ and be zero elsewhere. If we extend the definition of equivalence of Banach spaces in the obvious way to these L^p spaces we get this result:

THEOREM 7. *$L_\mu^p(\sum_{\gamma\varepsilon\Gamma}E_\gamma)$ is equivalent to $l^p(\Gamma)$ under the linear transformation for which $Tf=\mathbf{x}$, where $\mathbf{x}=\{\mathbf{x}_\gamma\}=\{\bar{f}_\gamma b_\gamma^{1/p}\}$; that is, which takes $f_{(\gamma)}$ to $\mathbf{x}_\gamma$.*

PROOF. By linearity if $f=\sum_\gamma x_\gamma f_{(\gamma)}$, $Tf=\mathbf{x}$ must be $\sum_\gamma x_\gamma \mathbf{x}_\gamma$. Now $\|f\|_p=(\sum|x_\gamma|^p)^{1/p}$ and $\|\mathbf{x}\|_p=(\sum|x_\gamma|^p)^{1/p}$; so the transformation is one-to-one on all L^p and l^p and $\|Tf\|=\|f\|$ for all $f\,\varepsilon\,L^p(\sum_\gamma E_\gamma)$; so the spaces are equivalent.

COROLLARY. *If Γ' is any set of ordinals and $\Gamma'\subset\Gamma$, then $l^p(\Gamma')$ is equivalent to the subspace $L_\mu^p(\sum_{\gamma\varepsilon\Gamma'}E)$ of $L_\mu^p(Y)$.*

Since, as is well known, $l^{1*}=m$, the following relation between L^p and l^1 is not entirely unexpected.

(7) If we define on the subspace $L_\mu^p(\sum_{\gamma\varepsilon\Gamma}E_\gamma)$ of $L_\mu^p(Y)$ the transformation T by associating to $f=\sum_{\gamma\varepsilon\Gamma}x_\gamma f_{(\gamma)}$ the function $Tf=\mathbf{x}=\sum_\gamma x_\gamma\mathbf{x}_\gamma$, then T is linear on $L_\mu^p(\sum_{\gamma\varepsilon\Gamma}E_\gamma)$ to l^1 and $\|T\|=1$.

In general, if Γ is not a finite set, the set of all Tf with $f\,\varepsilon\,L_\mu^p(\sum_{\gamma\varepsilon T}E_\gamma)$ is not all of l^1.

We conclude with some elementary remarks about the cases when Γ is a finite set; then the spaces $l^q(\Gamma)$, $q>0$, can be gotten by giving a new pseudo-norm to an n-dimensional euclidean space. When $q\geqq 1$ this pseudo-norm is actually a norm and the "spheres" $E_{\mathbf{x}}[\|\mathbf{x}\|\leqq\epsilon]$ are convex point sets, and we have some of the well known Minkowski spaces. When $0<q<1$ these sets are not convex but each contains an ordinary euclidean sphere and is contained in another such sphere, so the topology in all these $l^q(\Gamma)$ is the same as that in, say, $l^2(\Gamma)$, the space with euclidean metric. Convergence in $m(\Gamma)$ is also the same as that in $l^2(\Gamma)$ when Γ is finite. However these spaces are not in general equivalent as Banach spaces; if $\Gamma=\{1\}$, all are equivalent Banach spaces; if $\Gamma=\{1, 2\}$, $l^1(\Gamma)$ and $m(\Gamma)$ are equivalent; none of the others are.

INSTITUTE FOR ADVANCED STUDY

A FORMAL EXPANSION THEORY FOR FUNCTIONS OF ONE OR MORE VARIABLES[1]

NEWMAN A. HALL

It is a familiar property of the expansion of a function in series of functions that the coefficients often may be expressed in terms of the coefficients of the Taylor series for the original function. This has been done explicitly for many specific cases with functions of a single variable.[2] In this paper there is presented a method of obtaining more general results of this nature for functions of any number of variables defined by power series.

The umbral calculus introduced by Blissard in his *Theory of generic functions*[3] has been used by Lucas and Bell among others as a convenient instrument in the manipulation of generating functions. The algebra of the umbrae has been discussed by Bell[4] and some of the simplest properties of these will be used in the theory presented below.

A function $f(x)$ defined by a power series[5]

$$f(x) = \sum_n a_n \frac{x^n}{n!}$$

may be equally well defined by the matrix

$$a = \left| \, a_0, a_1, \cdots, a_n, \cdots \right|.$$

The umbral calculus admits the equality $a^n = a_n$, that is, the nth power of the matrix is equal to the nth term. From this it follows that $f(x) = e^{ax}$.

Functions of several variables suggest a similar notation. The function

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[1] Presented to the Society, November 27, 1937, under the title *A formal expansion theory for functions defined by two variable power series.*

[2] N. Nielsen, *Fonctions Métasphériques*, chap. 4. N. Nielsen, *Recherches sur le développement d'une fonction analytique en series de fonctions hypergéométriques*, Annales Scientifiques d'École Normale Supérieure, (3), vol. 30 (1913), p. 12. S. Pincherle, *Alcuni teoremi sopra gli sviluppi en serie per funzioni analitiche*, Rendiconti dell' Istituto Lombardo di Scienze e Lettere, (2), vol. 15 (1882), p. 224. J. M. Whittaker, *Interpolatory Function Theory*, Cambridge, 1937.

[3] John Blissard, Quarterly Journal of Mathematics, vols. 4–6 (1861–1864).

[4] E. T. Bell, *Algebraic Arithmetic*, American Mathematical Society Colloquium Pubications, vol. 7, New York, 1927, pp. 146–159.

[5] All summations are to extend from 0 to ∞. In place of a repeated summation, $\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} \cdots \sum_{n_k=0}^{\infty} A_{n_1,n_2,\cdots,n_k}$ we shall write $\sum_{n_1,\cdots,n_k} A_{n_1,\cdots,n_k}$.

$$f(x) = f(x_1, \cdots, x_k) = \sum_{n_1, \cdots, n_k} a_{n_1, \cdots, n_k} \frac{x_1^{n_1} \cdots x_k^{n_k}}{n_1! \cdots n_k!}$$

is defined also by the k-dimensional matrix

$$a = a'a'' \cdots a^{(k)} = \left| a_{i_1, \cdots, i_k} \right|.$$

The exponential property of the one-dimensional matrices may be extended by setting $a'^{n_1}a''^{n_2} \cdots a^{(k)\,n_k} = a_{n_1, n_2, \cdots, n_k}$ so that

$$f(x) = \exp \left\{ a'x_1 + a''x_2 + \cdots + a^{(k)}x_k \right\}.$$

This use of an umbral notation puts the primary emphasis on the number of the term in the series and on the coefficient rather than on the variable.

We shall call two sets of functions

$$P_n(x) = P_{n_1, \cdots, n_k}(x_1, \cdots, x_k), \qquad Q_n(y) = Q_{n_1, \cdots, n_k}(y_1, \cdots, y_k)$$

associate, if they satisfy the relation

$$\exp \left\{ x_1y_1 + x_2y_2 + \cdots + x_ky_k \right\} = \sum_{n_1, \cdots, n_k} \frac{P_n(x)Q_n(y)}{n_1! \cdots n_k!}.$$

In this discussion $P_n(x)$ and $Q_n(y)$ shall be restricted to functions defined by power series or by the matrices

$$p_n = \left| p_{n_1, \cdots, n_k; i_1, \cdots, i_k} \right|, \qquad q_n = \left| q_{n_1, \cdots, n_k; i_1, \cdots, i_k} \right|.$$

According to the theorem below, in order to expand $f(x)$ in a series of functions $P_n(x)$, it is only necessary to know the associate function $Q_n(y)$.

THEOREM I. *If $P_n(x)$ and $Q_n(y)$ are associate functions and $f(x)$ is a function defined by the matrix $a'a'' \cdots a^{(k)}$, the expansion of $f(x)$ in the series of functions $P_n(x)$ is*

$$\begin{aligned} f(x) &= \exp \left\{ a'x_1 + \cdots + a^{(k)}x_k \right\} \\ &= \sum_{n_1, \cdots, n_k} \frac{P_{n_1, \cdots, n_k}(x_1, \cdots, x_k)Q_{n_1, \cdots, n_k}(a', \cdots, a^{(k)})}{n_1! \cdots n_k!}. \end{aligned}$$

A discussion for functions of two variables will illustrate the argument required in general. From the definition of the associate functions,

$$e^{x_1y_1+x_2y_2} = \sum_{m_1, m_2} \frac{P_{m_1, m_2}(x_1, x_2)Q_{m_1, m_2}(y_1, y_2)}{m_1!m_2!}$$

or

$$\sum_{n_1,n_2} \frac{x_1^{n_1} y_1^{n_1} x_2^{n_2} y_2^{n_2}}{n_1!n_2!} = \sum_{m_1,m_2} \frac{1}{m_1!m_2!} \sum_{r_1,r_2} \frac{p_{m_1,m_2:r_1,r_2}}{r_1!r_2!} x_1^{r_1} x_2^{r_2}$$

$$\cdot \sum_{s_1,s_2} \frac{q_{m_1,m_2:s_1,s_2}}{s_1!s_2!} y_1^{s_1} y_2^{s_2}.$$

If $y_1^h y_2^k$ is replaced throughout by $a_{h,k}$, the equality exhibited is not affected in any way, since the variables in effect serve only to number the terms in the series. The equality follows from relations between the coefficients and is not dependent upon the variables. With this substitution made the theorem follows.

When one set of associate functions is known, any number of others may be derived by the use of the following result.

THEOREM II. *If* $P_n(x)$ *and* $Q_n(y)$ *are a set of associate functions and* $c'c'' \cdots c^{(k)} = c_{n_1,\cdots,n_k}$ *is an arbitrary matrix, then*

$$P_n(x_1c', x_2c'', \cdots, x_kc^{(k)}), \qquad Q_n(y_1/c', y_2/c'', \cdots, y_k/c^{(k)})$$

are associate functions.

Again the proof for two variables follows from an examination of the defining identity as given in the discussion of Theorem I. It is only necessary to replace $x_1^{r_1}x_2^{r_2}$ by $x_1^{r_1}x_2^{r_2}c_{r_1,r_2}$ and $y_1^{s_1}y_2^{s_2}$ by $y_1^{s_1}y_2^{s_2}/c_{s_1,s_2}$, which by reasoning as above will not affect the equality nor will it affect the left side since c_{n_1,n_2} will cancel $1/c_{n_1,n_2}$.

The associate functions may be obtained directly from the relation between the coefficients which follow from the identity defining the associate functions or more conveniently by applying Theorem II to certain basic sets. In the illustrations below it will be shown that a great number of expansions in functions of one or two variables will follow in this latter manner from two basic sets.

The Hermite polynomials suggest the more general class of functions $H_n^p(x)$, defined by the generating function

$$e^{zx-z^p/p} = \sum_n \frac{z^n}{n!} H_n^p(x),$$

so that

$$H_n^p(x) = \sum_s (-)^{s(p+1)} \frac{(-n, ps)}{(1, s)} \frac{x^{n-ps}}{p^s}.$$

The associate function $N_n^p(y)$ is readily determined from the generating function, since

$$e^{xy} = e^{xy-y^p/p+y^p/p} = \sum_n \frac{H_n^p(x)y^n e^{y^p/p}}{n!},$$

so that

$$N_n^p(y) = y^n e^{y^p/p} = \sum_r \frac{y^{n+pr}}{p^r(1, r)}.$$

The well known Neumann expansions in Bessel functions are special cases of the expansions obtained from the associate functions

$$B_n^k(\alpha, x) = \sum_r \frac{x^{n+kr}}{(\alpha + 2n/k + 1, r)r!},$$

$$L_n^k(\alpha, y) = \sum_s (-)^{ks} \frac{(-n, ks)y^{n-ks}}{(-\alpha - 2n/k + 1, s)s!}.$$

We show that these are associate by referring to the basic definition

$$\begin{aligned}
\sum_n \frac{B_n^k(\alpha, x)L_n^k(\alpha, y)}{n!}
&= \sum_{n,r,s} (-)^{ks} \frac{(-n, ks)x^{n+kr}y^{n-ks}}{(\alpha + 2n/k + 1, r)(-\alpha - 2n/k + 1, s)n!r!s!} \\
&= \sum_p \frac{x^p y^p}{p!} \\
&\quad \cdot \sum_{r,s} (-)^{k(r+s)} \frac{(-p, kr)(-p + kr, ks)y^{-kr-ks}}{(-\alpha - 2p/k + 2r + 1, s)(\alpha + 2p/k - 2r + 1, r)r!s!} \\
&= \sum_{p,q} \frac{x^p y^{p-kq}}{p!q!} \\
&\quad \cdot \sum_r (-)^{kr} \frac{(-q, r)(-p, kr)(-p + kr, kq - kr)}{(-\alpha - 2p/k + 2r + 1, q - r)(\alpha + 2p/k - 2r + 1, r)r!} \\
&= \sum_{p,q} (-)^{kq} \frac{(-p, kq)x^p y^{p-kq}}{(-\alpha - 2p/k + 1, q)p!q!} \\
&\quad \cdot \sum_r \frac{(-q, r)(-\alpha - 2p/k + 1, 2r)(-\alpha - 2p/k, r)}{(-\alpha - 2p/k + q + 1, r)(-\alpha - 2p/k, 2r)r!} \\
&= \sum_{p,q} (-)^{kq} \frac{(-p, kq)x^p y^{p-kq}}{(-\alpha - 2p/k + 1, q)p!q!} \cdot \frac{(-\alpha - 2p/k + 1, q)(0, q)}{[(\frac{1}{2}\{-\alpha - 2p/k + 1\}, q)]^2} \\
&= \sum_p \frac{x^p y^p}{p!} = e^{xy}.
\end{aligned}$$ [6]

Thus B_n and L_n fulfill the definition of associate functions.

---

[6] W. N. Bailey, *Generalized Hypergeometric Series*, Cambridge, 1935, p. 25.

The Neumann series of the first kind gives expansions in

$$
\begin{aligned}
(\tfrac{1}{2}x)^{-v}J_{v+n}(x) &= \sum_r (-)^r \frac{(\tfrac{1}{2}x)^{n+2r}}{\Gamma(v+n+r+1)r!} \\
&= \frac{1}{\Gamma(v+n+1)} \sum_r (-)^r \frac{(\tfrac{1}{2}x)^{n+2r}}{(v+n+1, r)r!} \\
&= \frac{(-i)^n}{\Gamma(v+n+1)} B_n^2(v; \tfrac{1}{2}ix).
\end{aligned}
$$

The associate function is then

$$
\begin{aligned}
(i)^n\Gamma(v+n+1)L_n^2(v, -2ix) &= i^n\Gamma(v+n+1)\sum_s \frac{(-n, 2s)(-2i)^{n-2s}y^{n-2s}}{(-v-n+1, s)(1, s)} \\
&= \Gamma(v+n+1)\sum_s (-)^s \frac{(-n, 2s)2^{n-2s}}{(-v-n+1, s)s!} y^{n-2s}
\end{aligned}
$$

so that

$$
\begin{aligned}
f(x) &= \sum_n a_n \frac{x^n}{n!} \\
&= \sum_n (\tfrac{1}{2}x)^{-v}J_{v+n}(x) \cdot \frac{\Gamma(v+n+1)}{n!} \sum_s (-)^s \frac{(-n, 2s)2^{n-2s}}{(-v-n+1, s)s!} a_{n-2s}.
\end{aligned}
$$

The Jacobi polynomial may be similarly considered. The general form is

$$
\begin{aligned}
{}_2F_1(-n, \alpha+n; \gamma; x) &= \sum_r \frac{(-n, r)(\alpha+n, r)}{(\gamma, r)(1, r)} x^r \\
&= \sum_r (-)^n \frac{(-n, r)(\alpha+n, n-r)}{(\gamma, n-r)(1, r)} x^{n-r} \\
&= (-)^n(\alpha+n, n) \\
&\quad \cdot \sum_r (-)^r \frac{(-n, r)x^{n-r}}{(-\alpha-2n+1, r)(\gamma, n-r)r!} \\
&= (-)^n(\alpha+n, n)L_n^1(\alpha, cx), \qquad c_h = 1/(\gamma, h),
\end{aligned}
$$

so that according to Theorem II the associate function is

$$
\frac{(-)^n}{(\alpha+n, n)} B_n^1(\alpha, y/c) = \frac{(-)^n}{(\alpha+n, n)} \sum_s \frac{(\gamma, n+s)y^{n+s}}{(\alpha+2n+1, s)s!}.
$$

The Gegenbauer function, $C_n^v(x)$, includes the Legendre polynomial as the special case $v=\frac{1}{2}$. By definition, $(1-2hx+h^2)^{-v}=\sum_n h^n\cdot C_n^v(x)$. Hence

$$C_n^v(x) = \frac{(v,\, n)}{(1,\, n)}\sum_s \frac{(-\,n,\, 2s)(2x)^{n-2s}}{(-\,v-n+1,\, s)s!} = \frac{(v,\, n)}{(1,\, n)}\overset{2}{L}_n(v,\, 2x).$$

The associate function is

$$\frac{(1,\, n)}{(v,\, n)}\overset{2}{B}_n(v,\, \tfrac{1}{2}y) = \frac{(1,\, n)}{(v,\, n)}\sum_r \frac{(\frac{1}{2}y)^{n+2r}}{(v+n+1,\, r)r!}$$

and

$$f(x) = \sum_n a_n\frac{x^n}{n!} = \sum_n \frac{C_n^v(x)}{(v,\, n)}\sum_r \frac{a_{n+2r}}{2^{n+2r}(v+n+1,\, r)r!}\,.$$

The Hermite polynomial

$$H_n(x) = \sum_s \frac{(-\,n,\, 2s)}{(1,\, s)}\,\frac{x^{n-2s}}{2^s} = \overset{2}{H}_n(x)$$

has the associate function $N_n^2(y)=\sum_r y^{n+2r}/2^r(1,\, r)$, so that

$$f(x) = \sum_n a_n\frac{x^n}{n!} = \sum_n \frac{H_n(x)}{(1,\, n)}\sum_r \frac{a_{n+2r}}{2^r(1,\, r)}\,.$$

The Bessel functions of the third order introduced by P. Humbert[7] provide a direct extension of the Neumann expansions. By definition,

$$\begin{aligned}
&(\tfrac{1}{3}x)^{-\alpha-\beta}J_{\alpha+2n/3,\beta+n/3}(x)\\
&\quad= \frac{1}{\Gamma(\alpha+\frac{2}{3}n+1)\Gamma(\beta+\frac{1}{3}n+1)}\\
&\qquad\cdot\sum_r(-)^r\frac{(\frac{1}{3}x)^{n+3r}}{(\alpha+\frac{2}{3}n+1,\, r)(\beta+\frac{1}{3}n+1,\, r)r!}\\
&\quad= \frac{1}{\Gamma(\alpha+\frac{2}{3}n+1)}\sum_r(-)^r\frac{(\frac{1}{3}x)^{n+3r}}{\Gamma(\beta+\frac{1}{3}(n+3r)+1)(\alpha+\frac{2}{3}n+1,\, r)r!}\\
&\quad= \frac{e^{-n\pi i/3}}{\Gamma(\alpha+\frac{2}{3}n+1)}\overset{3}{B}_n(\alpha,\, cx), \qquad c_h = \frac{(\frac{1}{3}e^{\pi i/3})^h}{\Gamma(\beta+\frac{1}{3}h+1)}\,.
\end{aligned}$$

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[7] P. Humbert, Comptes Rendus de l'Académie des Sciences, Paris, vol. 190 (1930), p. 59.

Hence the associate function is

$$
\begin{aligned}
&e^{n\pi i/3}\Gamma(\alpha + \tfrac{2}{3}n + 1)L_n^3(\alpha, y/c) \\
&\quad = e^{n\pi i/3}\Gamma(\alpha + \tfrac{2}{3}n + 1) \\
&\quad\quad \cdot \sum_s (-)^s \frac{(-n, 3s)y^{n-3s}}{(-\alpha - \tfrac{2}{3}n + 1, s)s!} \frac{\Gamma(\beta + \tfrac{1}{3}(n + 3s) + 1)}{(\tfrac{1}{3}e^{\pi i/3})^{n-3s}} \\
&\quad = \Gamma(\alpha + \tfrac{2}{3}n + 1)\Gamma(\beta + \tfrac{1}{3}n + 1) \\
&\quad\quad \cdot \sum_s (-)^s \frac{(-n, 3s)y^{n-3s}}{(-\alpha - \tfrac{2}{3}n + 1, s)(-\beta - \tfrac{1}{3}n + 1, s)s!} .
\end{aligned}
$$

The simplest two-variable expansions are those in products of one-variable functions. For example, the Neumann series of Bessel functions can be extended. Thus, if

$$f(x, y) = \sum_{m,n} a_{m,n} \frac{x^m}{(1, m)} \frac{y^n}{(1, n)},$$

then

$$f(x, y) = \sum_{m,n} \alpha_{m,n} \frac{(\tfrac{1}{2}x)^{-v}(\tfrac{1}{2}y)^{-u}}{m!n!} J_{v+m}(x)J_{u+n}(y),$$

where

$$
\begin{aligned}
\alpha_{m,n} &= \Gamma(v + m + 1)\Gamma(u + n + 1) \\
&\quad \cdot \sum_{r,s} (-)^{r+s} \frac{(-m, 2r)(-n, 2s)2^{n+m-2r-2s}}{(-v - m + 1, r)(-u - n + 1, s)m!n!} a_{m-2r,n-2s}.
\end{aligned}
$$

We have here made use of the associate functions for the Bessel functions as mentioned above.

There are expansions in mixed products. Thus, for $f(x, y)$ above, we have

$$f(x, y) = \sum_{m,n} \frac{\beta_{m,n}}{m!n!} H_m(x)C_n^v(y),$$

where

$$\beta_{m,n} = \frac{(1, n)}{(v, n)} \sum_{r,s} \frac{a_{m+2r,n+2s}}{2^{n+r+2s}(v + n + 1, s)r!s!},$$

giving the expansion in products of Hermite and Gegenbauer polynomials.

P. Humbert[8] introduced the set of confluent hypergeometric functions of two variables, special cases of which can be variously interpreted as extensions of the Sonine and Jacobi polynomials as well as the Bessel functions. Many of these provide expansions for two-variable functions in accordance with our theory. One of the simplest will illustrate. The polynomial

$$\begin{aligned}\phi_2(-m, -n; \gamma; x_1, x_2) &= \sum_{r,s} \frac{(-m, v)(-n, s)}{(\gamma, r+s)r!s!} x_1^r x_2^s, \\ &= (-)^{m+n} H_m^1(c' x_1) H_n^1(c'' x_2), \\ &\qquad c'^r c''^s = 1/(\gamma, r+s),\end{aligned}$$

has the associate function

$$\begin{aligned}(-)^{m+n} N_m^1\left(\frac{y_1}{c'}, \frac{y_2}{c''}\right) &= \sum_{r,s} (-)^{m+n} \frac{y_1^{m+r} y_2^{n+s}}{c'^{m+r} c''^{n+s} r!s!} \\ &= \sum_{r,s} (-)^{m+n} \frac{(\gamma, r+s+m+n)}{(1, r)(1, s)} y_1^{m+r} y_2^{n+s}\end{aligned}$$

so that

$$\begin{aligned}f(x_1, x_2) &= \sum_{m,n} a_{m,n} \frac{x_1^m x_2^n}{m!n!} \\ &= \sum_{m,n} (-)^{m+n} \frac{\phi_2(-m, -n; \gamma; x_1, x_2)}{(1, m)(1, n)} \\ &\quad \cdot \sum_{r,s} \frac{(\gamma, r+s+m+n)}{(1, r)(1, s)} a_{m+r,n+s}.\end{aligned}$$

In addition to expansions derived from the two basic sets above there are many expansions in functions whose associate functions may be obtained by other means. As one example of this, we mention the two-variable Hermite polynomials[9] $H_{m,n}(x_1, x_2)$ defined by the expansion

$$\begin{aligned}\exp\left\{h(ax_1 + bx_2) + k(bx_1 + cx_2) - \tfrac{1}{2}(ah^2 + 2bhk + ck^2)\right\} \\ = \sum_{m,n} \frac{h^m k^n}{m!n!} H_{m,n}(x_1, x_2).\end{aligned}$$

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[8] P. Humbert, Proceedings of the Royal Society of Edinburgh, vol. 41 (1921), p. 73.

[9] Ch. Hermite, *Oeuvres*, vol. 2, pp. 293–308.

If we write $ha+kb=y_1$, $hb+kc=y_2$, then

$$e^{x_1y_1+x_2y_2}=\sum_{m,n}\frac{H_{m,n}(x_1,x_2)}{m!n!}h^mk^n\exp\left\{\tfrac{1}{2}(ah^2+2bhk+ck^2)\right\},$$

so that the function associate to $H_{m,n}(x_1,x_2)$ is

$$N_{m,n}(y_1,y_2)=\frac{(cy_1-by_2)^m(-by_1+ay_2)^n}{\Delta^{m+n}}\cdot\exp\left\{\frac{cy_1^2+2by_1y_2+ay_2^2}{2\Delta}\right\}$$

where $\Delta=ac-b^2\neq 0$.

Restrictions of space prevent our continuing with the almost endless variety of illustrations of the theory. These methods can be particularly useful in obtaining the explicit forms of new expansions. The detailed properties of the sets of functions will suffice to answer questions of convergence.

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TOTAL REGULARITY OF GENERAL TRANSFORMATIONS[1]

HENRY HURWITZ, JR.

A method of summation is said to be regular if it assigns to every convergent series its actual value. If it also assigns the value $+\infty$ (or $-\infty$) to every series which diverges to $+\infty$ (or $-\infty$) it is said to be totally regular. The conditions for regularity are well known, and those for total regularity have been worked out for triangular matrix transformations by W. A. Hurwitz.[2] We here obtain necessary and sufficient conditions for total regularity for a more general type of transformation. (The former conditions, though still sufficient, are not necessary.)

Suppose $x_1, x_2, x_3, \cdots$ is the sequence of partial sums of the original series which is assumed real. A value Y is assigned to this sequence in the following way:

$$Y = \lim_{D(t)\to 0} y(t); \qquad y(t) = \sum_{k=1}^{\infty} a_k(t) x_k.$$

t is a variable ranging over some point set, $D(t)$ is a positive real function, and the functions $a_k(t)$ are real, but not necessarily continuous. We assume the transformation is regular so that the three Silverman-Toeplitz conditions are satisfied:

(1) $\sum_{k=1}^{\infty} |a_k(t)|$ is bounded for $D(t)$ sufficiently small;

(2) $\lim_{D(t)\to 0} \sum_{k=1}^{\infty} a_k(t) = 1$;

(3) $\lim_{D(t)\to 0} a_k(t) = 0$ for all k.

We then ask when Y will be positively infinite if $\lim_{k\to\infty} x_k = +\infty$. (We demand that for $D(t)$ sufficiently small $y(t)$ will be defined although it may be positively infinite.)

First it may be seen that for sufficiently advanced t (that is, t for which $D(t)$ is sufficiently small) there can be only a finite number of negative coefficients $a_k(t)$ in each row (that is, for each t) if the transformation is to be totally regular. Otherwise a sequence t_n with $D(t_n)\to 0$ could be picked out such that for each t_n there would be an infinite number of negative coefficients. Then a sequence x_k could be defined so that $x_k \to \infty$ and

$$\sum_{k=1}^{\infty} a_k(t_n) x_k$$

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[1] Presented to the Society, February 24, 1940, under the title *Total regularity of infinite matrix transformations.*

[2] W. A. Hurwitz, Proceedings of the London Mathematical Society, vol. 26 (1927), p. 231.

is for all t_n, if defined at all, certainly less than, say, zero. This would be accomplished by defining x_k unusually large for some k for which $a_k(t_1)<0$, then for a k for which $a_k(t_2)<0$, and so on in the order $t_1, t_2, t_1, t_2, t_3, t_1, t_2, t_3, t_4, \cdots$, each time taking a greater k. Imposing this necessary condition considerably limits the possibilities and assures that for any $x_k\to\infty$, $y(t)$ will be defined for t sufficiently advanced.

DEFINITION. *The guard of the coefficient $a_k(t_0)$ with respect to the finite number of rows $t_1, t_2, \cdots, t_n$, written*

$$Ga_k(t_0)]_{t_1t_2\cdots t_n},$$

is equal to $+1$ if $a_k(t_0)\geqq 0$. If $a_k(t_0)<0$ it is equal to the maximum value of

$$\frac{a_k(t')}{|a_k(t_0)|}$$

for $t'=t_1, t_2, \cdots, t_n$.

THEOREM 1. *If a real regular transformation is to be totally regular, it is necessary and sufficient that*

(a) *for $D(t)$ sufficiently small there is only a finite number of negative coefficients in each row;*

(b) *there does not exist a sequence $t_1, t_2, \cdots$ such that* $\lim_{n\to\infty} D(t_n)=0$ *and for every finite number of rows $t_1, t_2, \cdots, t_F$*

$$\limsup_{n\to\infty} Ga_{k^*}(t_n)]_{t_1t_2\cdots t_F}\leqq 0$$

for some sequence $k_1^*, k_2^*, \cdots$ such that $k_n^*\to\infty$, where k_n^* may depend on F.

THEOREM 2. *Theorem* 1 *holds with* $\limsup_{n\to\infty} G$ *replaced by* $\liminf_{n\to\infty} G$.

The equivalence of these two theorems may be demonstrated directly, but it also follows by proving that the conditions of Theorem 1 are sufficient, and those of Theorem 2 are necessary for total regularity.

To prove the sufficiency part of Theorem 1, we show that if a transformation satisfies the conditions there cannot exist a sequence $x_k\to\infty$ such that there is a sequence t_n with $D(t_n)\to 0$ and $y(t_n)<M$ for all n. M is some large positive number. Suppose such sequences x_k and t_n do exist. We choose K_n such that

$$\sum_{k=1}^{K_n} |a_k(t_n)x_k| < M, \qquad \sum_{k=1}^{K_n} |a_k(t_n)| < \tfrac{1}{3}.$$

By the third S-T condition we may have $K_n \to \infty$, and indeed may assume without loss of generality that all the K_n's are so large that if k is greater than any one of them $x_k > 9M$. By the second S-T condition, we may assume that

$$\left| \sum_{k=1}^{\infty} a_k(t_n) - 1 \right| < \tfrac{1}{3}$$

for all n. By hypothesis there is some integer F such that there is no sequence $k_n^* \to \infty$ for which

$$\limsup_{n\to\infty} G a_{k_n^*}(t_n)]_{t_1 t_2 \cdots t_F} \leqq 0.$$

It then follows that at least for an infinite subsequence t_{n_l} every coefficient $a_k(t_{n_l})$ in the row t_{n_l} with $k > K_n$ has

$$G a_k(t_{n_l})]_{t_1 t_2 \cdots t_F} > \beta$$

where β is some positive number, unique for the whole subsequence. Since $y(t_{n_1}) < M$, we must have

$$J_1 = \sum_{K_{n_1}+1}^{\infty}{}^{*} a_k(t_{n_1}) x_k < -M$$

where the asterisk indicates that the summation is to be extended over only those k's for which $a_k(t_{n_1}) < 0$. Because the guard with respect to the rows $t_1, t_2, \cdots, t_F$ of each of the coefficients appearing in the expression for J_1 is greater than β, we have

$$\sum_{J_1} a_k(t_{i_k}) x_k > \beta M$$

where the summation is over all the k's appearing in the expression for J_1 and the i_k's are suitably chosen integers ranging between 1 and F. We next pick from the sequence n_l an element n_2' such that $K_{n_2'}$ is greater than any k appearing in J_1. As before we have

$$J_2 = \sum_{K_{n_2'}+1}^{\infty} {}^{*} a_k(t_{n_2'}) x_k < -M$$

so that

$$\sum_{J_1, J_2} a_k(t_{i_k}) x_k > 2\beta M$$

where the summation is now over all the k's appearing in either J_1 or J_2. Proceeding in this way we can prove that it is possible to pick from the rows $t_1, t_2, \cdots, t_F$ a series diverging to positive infinity. But since there is only a finite number of negative terms in each row it is then impossible that

$$y(t_i) < M$$

for $i=1, \cdots, F$, so that an x_k sequence of the type assumed cannot exist.

To prove that the conditions of Theorem 2 are necessary, we assume that a sequence t_n of the kind forbidden by (b) of Theorem 1, with lim sup G replaced by lim inf G, exists and actually construct a sequence $x_k \to \infty$ such that

$$\lim_{D(t)\to 0} y(t) \neq +\infty .$$

By the second S-T condition we may assume

$$\sum_{k=1}^{\infty} | a_k(t_n) | < T > 1$$

for all n. Let $n_1 = 1$. Take K_1 such that

$$\sum_{k=K_1+1}^{\infty} | a_k(t_{n_1}) | < \frac{1}{2\cdot 2^2} = \frac{1}{8} \cdot$$

Define

$$x_k = \frac{1}{T}, \qquad k = 1, 2, \cdots, K_1.$$

By the assumption we have in general, for $s>1$,

$$\liminf_{n\to\infty} Ga_{k_n^{(s-1)}}(t_n) \,\big]_{t_{n_1} t_{n_2} \cdots t_{n_{s-1}}} \leqq 0$$

for some sequence $k_n^{(s-1)}$ which becomes infinite with n. We can therefore choose $n_s > n_{s-1}$ so that

(a) $Ga_{k_{n_s}^{(s-1)}}(t_{n_s}) \,\big]_{t_{n_1} t_{n_2} \cdots t_{n_{s-1}}} < 1/s\cdot 2^s$,
(b) $k_{n_s}^{(s-1)} > K_{s-1}$,
(c) $D(t_{n_s})$ is so small that $\sum_{k=1}^{K_{s-1}} | a_k(t_{n_s}) x_k | < 1$.
Call $k_{n_s}^{(s-1)}$ simply k_s. Take $K_s > k_s$ such that

$$\sum_{k=K_s+1}^{\infty} | a_k(t_{n_p}) | < \frac{1}{(s+1)2^{s+1}} \cdot$$

for $p=1, 2, \cdots, s$. Define in general

$$x_k = s/T, \qquad K_{s-1}+1 \leqq k \leqq K_s \quad (k \neq k_s), \qquad x_{k_s} = X_s$$

where X_s is the greater of $s/|a_{k_s}(t_{n_s})|$, s/T. It is apparent that the x_h sequence thus defined approaches infinity. Also, since $a_{k_s}(t_{n_s})$ is negative and has a small guard, it may be verified that $y(t_{n_s})$ remains bounded for all s, so that the transformation is not totally regular.

THEOREM 3. *Theorems* 1 *and* 2 *hold with the modification that the sequence* k_n^* *must be independent of* F.

To prove this it must be shown that if a sequence t_n of the type forbidden by Theorems 1 or 2 exists, then one with k_n^* independent of F can be found. But this can easily be done by taking a subsequence $t_{n_1}, t_{n_2}, t_{n_3}, \cdots$ from the sequence $t_1, t_2, t_3, \cdots$ so that some negative coefficient $a_{k_l'}(t_{n_l})$ with large k_l' in row t_{n_l} has a small guard with respect to all the rows $t_{n_1}, t_{n_2}, \cdots, t_{n_{l-1}}$. The k_l' and t_{n_l} sequences thus obtained will be of the type forbidden by Theorem 3. The same line of reasoning also serves to prove a second modification:

THEOREM 4. *Theorem* 3 *will hold with*

$$\limsup_{n\to\infty} Ga_{k_n^*}(t_n)\,]_{t_1 t_2 \cdots t_{n-1}} \leqq 0$$

instead of

$$\limsup_{n\to\infty} Ga_{k_n^*}(t_n)\,]_{t_1 t_2 \cdots t_F} \leqq 0.$$

(*The same thing holds with inferior limits.*)

The most powerful combination of these four theorems is obtained by using the necessary conditions of Theorem 2 and the sufficient conditions (in terms of superior limits) of Theorem 4. The theorem obtained by W. A. Hurwitz for triangular matrix transformations follows quite easily from these results.

THEOREM 5. *It is necessary* (*but not sufficient*) *for a real regular transformation to be totally regular that there does not exist a sequence of coefficients* $a_{k_n}(t_n)$, $n=1, 2, \cdots$, *such that* (a) $a_{k_n}(t_n)<0$ *for all* n; (b) *the maximum number of* n*'s for which* k_n *has any particular value is finite*; (c) $\lim_{n\to\infty} D(t_n)=0$; (d) $\sum_{n=1}^{\infty} a_{k_n}(t_n) = -\infty$.

THEOREM 6. *It is necessary for a real regular transformation to be totally regular that*

$$\lim_{D(t)\to 0} \sum_{k=1}^{\infty} [\,|a_k(t)| - a_k(t)] = 0.$$

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A NEW APPROACH TO THE CRITICAL VALUE THEORY

W. MAYER

The well known inequalities of M. Morse have as algebraical foundation the rank-equations

$$\text{(I)} \quad \begin{aligned} rB_i(\Sigma - \Sigma_1) = rB_i(\Sigma) - rB_i(\Sigma_1) + rD_i(\Sigma_1, \Sigma) + rD_{i-1}(\Sigma_1, \Sigma), \\ i = 0, 1, 2, \cdots . \end{aligned}$$

These formulas hold for any topological group-system Σ, any subsystem Σ_1 of Σ, and the difference-system $\Sigma - \Sigma_1$. (W. Mayer, *Topologische Gruppensysteme*, Monatshefte für Mathematik und Physik, vol. 47 (1938); henceforth referred to as M, TG.) Here $B_i(\Sigma)$ denotes the i-dimensional Betti group of Σ, while $B_i(\Sigma_1)$ and $B_i(\Sigma - \Sigma_1)$ are these groups for Σ_1 and $\Sigma - \Sigma_1$ respectively. The symbol $r(\ )$, of course, stands for the rank of the group in the parentheses. By $D_i(\Sigma_1, \Sigma)$ we mean the subgroup of $B_i(\Sigma_1)$ containing all the classes of this group whose elements bound in Σ.

The formula (I) was first derived for the case of a complex in Lefschetz' *Topology*, 1930 (p. 150), and independently for the complex modulo 2 by J. Rybarz, Monatshefte für Mathematik und Physik (1931).

In the generality needed here the proof of (I) is given in M, TG (pp. 54–57), under the assumption, of course, that all the ranks appearing in (I) are finite, since otherwise the formula would be meaningless. But the proof there given shows also that

(a) $rB_i(\Sigma - \Sigma_1) = \infty$ implies that either $rB_i(\Sigma)$ or $rD_{i-1}(\Sigma_1, \Sigma)$, or both, are infinite;

(b) $rB_i(\Sigma - \Sigma_1)$ finite implies $rD_{i-1}(\Sigma_1, \Sigma)$ finite, and if in addition $rB_i(\Sigma_1)$ is finite then $rB_i(\Sigma)$ is finite too; and

(c) $rB_i(\Sigma - \Sigma_1) = 0$ implies $rD_{i-1}(\Sigma_1, \Sigma) = 0$ and if in addition $rB_i(\Sigma_1)$ is finite, then $rB_i(\Sigma_1) = rB_i(\Sigma) + rD_i(\Sigma_1, \Sigma)$.

As an immediate consequence of equations (I) we notice the inequality

$$\text{(I}'\text{)} \qquad rB_i(\Sigma) \leqq rB_i(\Sigma_1) + rB_i(\Sigma - \Sigma_1),$$

which is true whenever the terms on the right are finite (remark (b)) and trivial otherwise. The next step in attaining the Morse inequalities is the application of (I) to $m+2$ topological group-systems satisfying the inclusion relations

$$\text{(1)} \qquad \Sigma_m \supset \Sigma_{m-1} \supset \cdots \supset \Sigma_0 \supset \Sigma_{-1}$$

with Σ_{-1} empty (all $L_i(\Sigma_{-1})$ being zero-groups). Assuming that all ranks involved are finite, we get by summation on k in

$$\begin{aligned} rB_i(\Sigma_k - \Sigma_{k-1}) = rB_i(\Sigma_k) - rB_i(\Sigma_{k-1}) + rD_i(\Sigma_{k-1}, \Sigma_k) \\ + rD_{i-1}(\Sigma_{k-1}, \Sigma_k) \end{aligned} \tag{2}$$

the formula

$$\begin{aligned} &\sum_{k=0}^{m} rB_i(\Sigma_k - \Sigma_{k-1}) \\ &\qquad = rB_i(\Sigma_m) + \sum_{k=1}^{m} rD_i(\Sigma_{k-1}, \Sigma_k) + rD_{i-1}(\Sigma_{k-1}, \Sigma_k), \end{aligned} \tag{3}$$

since all the $rB_i(\Sigma_{-1})$, $rD_i(\Sigma_{-1}, \Sigma_0)$ are zero, Σ_{-1} being empty.

Let Σ_m be a neighborhood-space and let Σ_k, $k=m-1, \cdots, -1$, be subspaces of Σ_m satisfying the inclusion relation (1). With the introduction of singular simplices and chains modulo 2 each space Σ_k gives rise to a topological group system, which we also denote by Σ_k. (The i-dimensional complexes K^i are the finite chains of singular simplices.) The relations (1) are then inclusion relations for the so-constructed group-systems Σ_k, and for these systems relation (3) will hold provided the ranks involved are finite. The finiteness of all these ranks will follow from the finiteness of those of the left side of (3), that is, of the

$$rB_i(\Sigma_k - \Sigma_{k-1}).$$

If these are finite, we see from (2) that $rB_i(\Sigma_0)$ $(=rB_i(\Sigma_0 - \Sigma_1))$ is finite, and thus from (2) and remark (b) we conclude that $rB_i(\Sigma_1)$ is finite. So, step-by-step, using (2) and remark (b) we find that all the ranks $rB_i(\Sigma_k)$ are finite. Since $D_i(\Sigma_{k-1}, \Sigma_k) \subset B_i(\Sigma_{k-1})$, the ranks of the groups D_i are finite too. We have then the result:

The equations (3) *hold for the group-systems* Σ_k *if only the ranks appearing on the left sides of these equations are finite.*

It is an interesting fact, in view of its geometrical implications, that the finiteness of the ranks $rB_i(\Sigma_k - \Sigma_{k-1})$ has as a consequence the finiteness of the Betti numbers $rB_i(\Sigma_k)$. As an additional remark it may be noticed that equations (3) can be written in the form of Morse's inequalities (of the strong type) if (3) holds for all indices i. Denoting by M_i the left sides of (3):

$$M_i = \sum_{k=0}^{m} rB_i(\Sigma_k - \Sigma_{k-1}), \tag{4}$$

and successively eliminating the sums with terms rD_i, we get

$$\begin{aligned} M_0 &\geqq R_0, \\ M_1 - M_0 &\geqq R_1 - R_0, \\ M_2 - M_1 + M_0 &\geqq R_2 - R_1 + R_0, \\ &\cdots\cdots\cdots\cdots\cdots ; \\ M_n - M_{n-1} + \cdots \pm M_0 &= R_n - R_{n-1} + \cdots \pm R_0, \end{aligned} \tag{3'}$$

where n is the dimension of the topological system Σ_m and R_i stands for $rB_i(\Sigma_m)$, the ith Betti number of Σ_m. The inequalities (3′), however, are not fully equivalent to equations (3) since they result only if (3) holds for all the dimensions i from zero up to n, the dimension of Σ_m. In addition, in (3) the difference

$$(M_i - M_{i-1} + \cdots \pm M_0) - (R_i - R_{i-1} + \cdots \pm R_0)$$

is shown to be $\sum rD_i(\Sigma_{k-1}, \Sigma_k)$, which fact, of course, is of importance since it readily yields the conditions for the equality sign in (3′).

A subdivision of the space Σ_m by subspaces Σ_k of Σ_m can be obtained by the use of a real bounded function $\mathfrak{J}$ defined on Σ_m. Let σ_0 and σ_m be the greatest lower and least upper bound of $\mathfrak{J}$ respectively, so that

$$\sigma_m \geqq \mathfrak{J} \geqq \sigma_0; \tag{5}$$

then intermediate values σ_k, $k=1, \cdots, m-1$, and $\sigma_{-1}<\sigma_0$ may be introduced and subjected to the inequalities

$$\sigma_m > \sigma_{m-1} > \cdots > \sigma_1 > \sigma_0 > \sigma_{-1} \tag{6}$$

if $\mathfrak{J}$ is not constant on Σ_m. If $\mathfrak{J}$ is constant, (6) can be reduced to $\sigma_0 > \sigma_{-1}$.

We now define the subspace Σ_k as the set of all points P of Σ_m such that $\mathfrak{J}(P) \leqq \sigma_k$, that is, symbolically,

$$\Sigma_k \equiv \{\mathfrak{J} \leqq \sigma_k\}. \tag{7}$$

(We observe that in this notation the subspace Σ_m defined in (7) coincides with the space originally so denoted.)

Constructing the corresponding topological group-systems, denoted likewise by Σ_m (with Σ_{-1} consisting of zero groups only), we see that equations (3) hold if only the sum M_i of (4) is finite. This shows the full generality of these equations, in which only the concepts of neighborhood space and its partition by subspaces (not necessarily defined by a function $\mathfrak{J}$) are used. (Later on the validity of formula

(3) will be shown for a partition of the space by a countable set of subspaces.)

Let Σ_m be a compact Riemannian manifold and $\mathfrak{F}$ a function of class C^2 defined on Σ_m and having a finite number only of stationary points. Corresponding to these points there are a finite number of stationary values σ_k, $k=0, 1, \cdots, m$, with σ_0 and σ_m the absolute minimum and maximum respectively. Using these values in defining the Σ_k as previously described and denoting by the same letters Σ_k the corresponding group-systems, we again arrive at equations (3), given the finiteness of their left sides M_i.

Denote by $\Sigma_k^{\cdot}$ the space of all points P of Σ_m with the property $\mathfrak{F}(P) < \sigma_k$; then obviously $\Sigma_k^{\cdot} \subset \Sigma_k$.

By definition (Seifert-Threlfall, *Variationsrechnung im Grossen*, §4; we refer to this henceforth as S.T.) the type numbers $m_i(\sigma)$ for any value σ of $\mathfrak{F}$ are the ranks of the Betti groups of $\Sigma_{(\sigma)} - \Sigma_{(\sigma)}^{\cdot}$, where $\Sigma_{(\sigma)}$ and $\Sigma_{(\sigma)}^{\cdot}$ are the point sets $\{\mathfrak{F} \leqq \sigma\}$ and $\{\mathfrak{F} < \sigma\}$ respectively. The value σ is called critical if some of the $m_i(\sigma)$ are different from zero.

Only a stationary value (that is, a value belonging to a stationary point) can be critical, so that only for the stationary values σ_k, $k=0, 1, \cdots, m$, are the type numbers

$$m_i(\sigma_k) = rB_i(\Sigma_k - \Sigma_k^{\cdot}) \tag{8}$$

not all necessarily zero. In consequence of our assumption all the stationary points are *isolated* and thus their contribution to the corresponding $m_i(\sigma_k)$ will be finite (S.T., §10). Since only a finite number of stationary points belong to a stationary value σ_k, $m_i(\sigma_k)$ *will be finite* (S.T., p. 87).

If therefore we can prove that

$$rB_i(\Sigma_k - \Sigma_k^{\cdot}) = rB_i(\Sigma_k - \Sigma_{k-1}), \tag{9}$$

then not only is (3) verified, but its left side is shown to be the sum of all the type numbers of dimension i. Thus, in proving isomorphisms between the Betti groups of the systems

$$\Sigma_k - \Sigma_k^{\cdot} \tag{α_k}$$

and

$$\Sigma_k - \Sigma_{k-1} \tag{β_k}$$

for $k=0, 1, \cdots, m$, we likewise prove (9) and thereby (3).

The above isomorphisms are included in the following two statements:

(a) Each class of $B_i(\Sigma_k - \Sigma_{k-1})$ lies in a definite class of $B_i(\Sigma_k - \Sigma_k^{\cdot})$.

(b) Each class of $B_i(\Sigma_k - \Sigma'_k)$ contains one and only one class of $B_i(\Sigma_k - \Sigma_{k-1})$.

Indeed, (a) and (b) establish a correspondence between the classes of the above groups which is obviously an isomorphism.

As to the proof of the statements, (a) is almost self-evident, since an i-cycle of $\Sigma_k - \Sigma_{k-1}$ is an i-cycle of $\Sigma_k - \Sigma'_k$, and an i-cycle of $\Sigma_k - \Sigma_{k-1}$ which is homologous to zero in $\Sigma_k - \Sigma_{k-1}$ is easily seen to be an i-cycle of $\Sigma_k - \Sigma'_k$ homologous to zero in $\Sigma_k - \Sigma'_k$.

We now prove the first part of (b), that is, each class of $B_i(\Sigma_k - \Sigma'_k)$ contains a class of $B_i(\Sigma_k - \Sigma_{k-1})$. This is by no means evident, and to prove it we must make use of $\mathfrak{J}$-deformations, which in the case we are considering will exist. Suppose given $\{Z^i\} \subset B_i(\Sigma_k - \Sigma'_k)$; then there exists a $K^{i-1} \subset \Sigma'_k$ such that

$$(10) \qquad R(Z^i) = K^{i-1}.$$

But, on the compact point set K^{i-1}, the continuous function $\mathfrak{J}$ has somewhere a maximum which, of course, is smaller than σ_k, $(K^{i-1} \subset \Sigma'_k)$. Thus K^{i-1} lies in some Σ^* defined by $\{\mathfrak{J} \leqq \sigma_k - \epsilon\}$, where $\epsilon > 0$ may be so chosen that $\sigma_k - \epsilon > \sigma_{k-1}$. Then (S.T., p. 87) an $\mathfrak{J}$-deformation exists which carries the point set $\{\mathfrak{J} \leqq \sigma_k - \epsilon\}$ into the point set $\{\mathfrak{J} \leqq \sigma_{k-1}\}$. Thus there exist a K^i on $\{\mathfrak{J} \leqq \sigma_k - \epsilon\}$ and a K_1^{i-1} on $\{\mathfrak{J} \leqq \sigma_{k-1}\}$ such that

$$(11) \qquad R(K^i) = K^{i-1} + K_1^{i-1}.$$

Adding (10) to (11) we get

$$(12) \qquad R(Z^i + K^i) = K_1^{i-1},$$

thus showing that $Z^i + K^i$, a cycle of the class $\{Z^i\}$ of $B_i(\Sigma_k - \Sigma'_k)$, is a cycle of $\Sigma_k - \Sigma_{k-1}$. Hence $\{Z^i\}$ contains this cycle of $\Sigma_k - \Sigma_{k-1}$, and, according to (a), the class of $B_i(\Sigma_k - \Sigma_{k-1})$ it represents.

We finally prove the second part of (b), namely: Each class of $B_i(\Sigma_k - \Sigma'_k)$ contains only one class of $B_i(\Sigma_k - \Sigma_{k-1})$. This again follows from the statement: If a cycle Z^i of $\Sigma_k - \Sigma_{k-1}$ considered as a cycle of $\Sigma_k - \Sigma'_k$ is homologous to zero in $\Sigma_k - \Sigma'_k$, then this cycle will be homologous to zero in $\Sigma_k - \Sigma_{k-1}$. For this proof one must again make use of the $\mathfrak{J}$-deformation mentioned above. Let Z^i be a cycle of $\Sigma_k - \Sigma_{k-1}$ homologous to zero in $\Sigma_k - \Sigma'_k$; then there exist a K^{i+1} in Σ_k and a K^i in Σ'_k such that

$$(13) \qquad R(K^{i+1}) = Z^i + K^i,$$

which shows that $Z^i \sim 0$ in $\Sigma_k - \Sigma'_k$. Furthermore there will exist a K^{i-1} in Σ_{k-1} such that

$$R(Z^i) = K^{i-1}, \tag{14}$$

since Z^i is a cycle of $\Sigma_k - \Sigma_{k-1}$. From (13) and (14) follows

$$R(K^i) = K^{i-1}. \tag{15}$$

The compact set K^i of Σ_k, $(\mathfrak{J} < \sigma_k)$, will lie in some $\{\mathfrak{J} \leqq \sigma_k - \epsilon\}$. Using the $\mathfrak{J}$-deformation carrying $\{\mathfrak{J} \leqq \sigma_k - \epsilon\}$ into $\{\mathfrak{J} \leqq \sigma_{k-1}\}$, we establish the existence of a chain K_1^{i+1} of $\{\mathfrak{J} \leqq \sigma_k - \epsilon\}$ and two chains K_1^i and L_1^i of $\{\mathfrak{J} \leqq \sigma_{k-1}\}$ satisfying the relation

$$R(K_1^{i+1}) = K^i + K_1^i + L_1^i, \tag{16}$$

where L_1^i, as the deformation chain of $R(K^i) = K^{i-1}$ (which lies in Σ_{k-1}) will lie in Σ_{k-1}, by definition of an $\mathfrak{J}$-deformation. Adding the two relations (13) and (16), we get finally

$$R(K^{i+1} + K_1^{i+1}) = Z^i + K_1^i + L_1^i, \tag{17}$$

where $K_1^i + L_1^i \subset \Sigma_{k-1}$ and $K^{i+1} + K_1^{i+1} \subset \Sigma_k$. This proves that $Z^i \sim 0$ in $\Sigma_k - \Sigma_{k-1}$.

We add that in proving (9) we made no use of the fact that either of the values σ_k or σ_{k-1} was stationary. We used only the existence of the above mentioned $\mathfrak{J}$-deformations, and these will always exist provided only that between σ_k and σ_{k-1} there are no stationary values. Given this condition, (9) always holds (of course, in the form $0=0$ when σ_k is not a stationary value).

The theory so far developed is unable to deal with the concept of the "variational calculus in the large," since in this case the function $\mathfrak{J}$ defined on Σ is unbounded above. We now adapt the theory to that case.

First we prove the following lemma:

LEMMA. *For three group-systems* Σ_α, $\alpha = 1, 2, 3$, *satisfying the inclusion relations*

$$\Sigma_1 \subset \Sigma_2 \subset \Sigma_3, \tag{18}$$

the relation

$$rD_i(\Sigma_1, \Sigma_3) \leqq rD_i(\Sigma_1, \Sigma_2) + rD_i(\Sigma_2, \Sigma_3) \tag{19}$$

holds.

PROOF. Each class of $D_i(\Sigma_1, \Sigma_3)$ is contained in a definite class of $D_i(\Sigma_2, \Sigma_3)$ since each cycle of Σ_1 bounding in Σ_3 is a cycle of Σ_2 bounding in Σ_3 and each cycle of Σ_1 bounding in Σ_1 is a cycle of Σ_2 bounding in Σ_2. By correlating with an element of $D_i(\Sigma_1, \Sigma_3)$ that element of $D_i(\Sigma_2, \Sigma_3)$ which contains it, we define a homomorphism,

$$D_i(\Sigma_1, \Sigma_3) \to D_i(\Sigma_2, \Sigma_3), \tag{20}$$

whose kernel group is the subgroup of $D_i(\Sigma_1, \Sigma_3)$ consisting of all classes bounding in Σ_2, hence $D_i(\Sigma_1, \Sigma_2)$. Denoting the map-group of the homomorphism by $D_i'(\Sigma_2, \Sigma_3)$ we have the isomorphism

$$D_i(\Sigma_1, \Sigma_3) - D_i(\Sigma_1, \Sigma_2) \approx D_i'(\Sigma_2, \Sigma_3) \tag{21}$$

and the rank-equation

$$rD_i(\Sigma_1, \Sigma_3) = rD_i(\Sigma_1, \Sigma_2) + rD_i'(\Sigma_2, \Sigma_3), \tag{22}$$

whence (19) follows (since $D_i'(\Sigma_2, \Sigma_3) \subset D_i(\Sigma_2, \Sigma_3)$).

Combining (I) and (19) we derive

$$rB_i(\Sigma_3 - \Sigma_1) \leqq rB_i(\Sigma_3 - \Sigma_2) + rB_i(\Sigma_2 - \Sigma_1), \tag{23}$$

thus showing that in subdividing a given partition we never decrease the sum M_i.

As for the validity of the proof given for (23), we have to establish the legality of the use of formula (I). This use will indeed be legitimate if the rank equations (I) hold for $B_i(\Sigma_3 - \Sigma_2)$ and $B_i(\Sigma_2 - \Sigma_1)$, since then all ranks in (I) for $\Sigma = \Sigma_3$ are finite, by remark (a) of page 838 and equation (19), and the employment of (I) is thus justified for all the three Betti groups appearing in (23). This condition will in fact be satisfied whenever (23) is used in the sequel.

Equation (23), however, is true without restrictions, as we show by the use of a formula of a more general type, namely, (4′) of page 46 in M, TG. By this formula, (18) implies the isomorphism $\Sigma_3 - \Sigma_2 \approx (\Sigma_3 - \Sigma_1) - (\Sigma_2 - \Sigma_1)$ which generalizes a well known group-isomorphism to group-systems. Using (I′) for the replacement of Σ and Σ_1 by $\Sigma_3 - \Sigma_1$ and $\Sigma_2 - \Sigma_1$ respectively we again arrive at formula (23), but now without any restriction.

After this remark we extend the proof of formula (3). Let the space Σ be subdivided by a countable set of subspaces Σ_k, $k = -1, 0, 1, \cdots$, satisfying the inclusion relation

$$\Sigma_{-1} \subset \Sigma_0 \subset \Sigma_1 \subset \cdots \subset \Sigma_m \subset \cdots \subset \Sigma \tag{24}$$

with Σ_{-1} empty and $\sum_{m=0}^{\infty}(\Sigma_m) = \Sigma$ (we write $\Sigma = \Sigma_\infty$). Then formula (3) holds for an infinite but bounded sum

$$M_i = \sum_{k=1}^{\infty} rB_i(\Sigma_k - \Sigma_{k-1}) \tag{25}$$

if only the partition satisfies an additional assumption on its topological nature, namely:

(C) *Any compact point set of* Σ *lies in some* Σ_k $(k\neq\infty)$.

As a matter of fact, (C) is satisfied if the partition (24) is defined by the level surfaces of a continuous real function $\mathfrak{J}$ unbounded above, the subspace Σ_k being the set of all points P such that $\mathfrak{J}(P)\leqq\sigma_k$ and $\sigma_{-1}<\sigma_0<\sigma_1<\cdots$ being an unbounded sequence, with $\sigma_{-1}<\mathfrak{J}(P)$ for all points P of Σ.

Since by assumption M_i is finite, the left side of (3) will be finite, and thus (3) holds for any m. Thus each sum

$$\sum_{k=1}^{m} rD_i(\Sigma_{k-1},\Sigma_k),\qquad \sum_{k=1}^{m} rD_{i-1}(\Sigma_{k-1},\Sigma_k) \tag{26}$$

in (3), being not larger than M_i, converges as $m\rightarrow\infty$. As a further consequence, $rB_i(\Sigma_m)$ also converges as $m\rightarrow\infty$:

$$\lim_{m\rightarrow\infty} rB_i(\Sigma_m) = R_i'. \tag{27}$$

If only we can show that $R_i'=rB_i(\Sigma)$, we shall have proved formula (3) for $m=\infty$, that is,

$$M_i = R_i + \sum_{k=1}^{\infty} rD_i(\Sigma_{k-1},\Sigma_k) + \sum_{k=1}^{\infty} rD_{i-1}(\Sigma_{k-1},\Sigma_k), \tag{28}$$

writing $R_i=rB_i(\Sigma)$.

Due to the convergence of the sequences (26) and (27), all of whose terms—as ranks—are positive integers, or zeros, there exists an integer N such that for $k\geqq N$

$$\begin{aligned} rB_i(\Sigma_k-\Sigma_{k-1}) &= 0, & rD_i(\Sigma_{k-1},\Sigma_k) &= 0,\\ rD_{i-1}(\Sigma_{k-1},\Sigma_k) &= 0, & rB_i(\Sigma_k)-R_i' &= 0. \end{aligned} \tag{29}$$

Consequently, for $h>k>N$ the right-hand terms of the inequality

$$rB_i(\Sigma_h-\Sigma_k)\leqq\sum_{j=k+1}^{h} rB_i(\Sigma_j-\Sigma_{j-1}) \tag{30}$$

are zero, so that

$$rB_i(\Sigma_h-\Sigma_k) = 0. \tag{31}$$

From (31) and the assumption (C) we easily conclude that for $k>N$

$$rB_i(\Sigma-\Sigma_k) = 0. \tag{32}$$

In fact, (32) is proved if, for any C^i representing an element of $B_i(\Sigma-\Sigma_k)$, we can show that $C^i\sim 0$ in $\Sigma-\Sigma_k$. As a compact set on Σ, ($C^i\subset\Sigma$ with $R(C^i)\subset\Sigma_k$), C^i lies in some $\Sigma_h\supset\Sigma_k$, and thus represents an element of $B_i(\Sigma_h-\Sigma_k)$. But since this group has the rank zero, by (31), $C^i\sim 0$ in $\Sigma_h-\Sigma_k$. Therefore there exists a K^{i+1} and a K^i satisfying

$$K^{i+1}\subset\Sigma_h,\qquad K^i\subset\Sigma_k, \tag{33}$$

such that

$$R(K^{i+1}) = C^i + K^i; \tag{34}$$

this shows that $C^i\sim 0$ in $\Sigma-\Sigma_k$ and proves (32).

Using (32) in connection with the main formula (I) we have for $k>N$

$$0 = rB_i(\Sigma) - rB_i(\Sigma_k) + rD_i(\Sigma_k,\Sigma) + rD_{i-1}(\Sigma_k,\Sigma). \tag{35}$$

Remark. The use of (I) is again justified, since all ranks involved are finite. (For $rB_i(\Sigma)$ and $rD_{i-1}(\Sigma_k,\Sigma)$ see remark (b) on page 838; remember also that $rB_i(\Sigma_k)=R_i^i$ and $rD_i(\Sigma_k,\Sigma)\leqq rB_i(\Sigma_k)$.) According to our remark (c) on page 838 we have $rD_{i-1}(\Sigma_k,\Sigma)=0$ in formula (35). *We shall prove that in addition* $rD_i(\Sigma_k,\Sigma)=0$.

This again is shown if for any C^i representing an element of $D_i(\Sigma_k,\Sigma)$ we can prove that $C^i\sim 0$ in Σ_k. Let C^i be such an element; then there exists a K^{i+1} such that

$$C^i=R(K^{i+1}),\qquad C^i\subset\Sigma_k. \tag{36}$$

But, again, the compact set K^{i+1} lies in some $\Sigma_h\supset\Sigma_k$, and thus C^i represents an element of $D_i(\Sigma_k,\Sigma_h)$, whose rank is not larger than

$$\sum_{j=k}^{h-1} rD_i(\Sigma_j,\Sigma_{j+1}) = 0.$$

Therefore $rD_i(\Sigma_k,\Sigma_h)=0$, so that C^i is homologous to zero in Σ_k, which was to be proved.

This shown, (35) reduces to

$$rB_i(\Sigma_k) = rB_i(\Sigma),\qquad k > N, \tag{37}$$

thus proving (28).

If for the concept of variational calculus in the large ($f(x,x')$ positive definite and of class C^2) we restrict ourselves to the case where only a finite number of stationary points (extremals) are below any $\mathfrak{J}$-level ($\mathfrak{J}=\int_P^Q f(x,x')dt>0$) and if the sequence σ_k, $k=0,1,\cdots$, unbounded above, includes all the stationary values, then $\mathfrak{J}$-deformations exist (S.T., p. 66), and can be used to prove homomorphisms between the systems (α_k) and (β_k) on page 841 in exactly the same way as before.

Hence (9) will hold, and again the type numbers

$$m_i(\sigma_k) = rB_i(\Sigma_k - \dot{\Sigma}_k) \tag{38}$$

are finite (S.T., §14, Theorem II and §17, Theorem II), so that (3) is true for any m.

In taking the sum $M_i = \Sigma m_i(\sigma_k)$, only critical values σ_k (by their definition) add nonzero terms, and if noncritical values σ were used in the construction, they can be omitted in writing M_i. Hence for this case too the Morse equations hold for a finite M_i.

Remark. If, in the case just considered, for any two subspaces Σ', Σ'' of Σ the ranks of $D_i(\Sigma', \Sigma'')$ and $D_{i-1}(\Sigma', \Sigma'')$ are zero $(\Sigma' \subset \Sigma'')$, then $M_i = R_i$. For finite M_i, of course, this follows from (28), but for infinite M_i we have to prove the statement. First, from (I)—or more exactly from the proof leading to (I)—we have, for any[1] k

$$rB_i(\Sigma) \geqq rB_i(\Sigma_k); \tag{39}$$

then for any m we have

$$\sum_{k=0}^{m} rB_i(\Sigma_k - \Sigma_{k-1}) = rB_i(\Sigma_m). \tag{40}$$

Since the left side of (40) diverges for $m \to \infty$, so does $rB_i(\Sigma_m)$ diverge, and hence, by (39), $rB_i(\Sigma)$ cannot be finite.

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[1] In fact, we have $B_i(\Sigma - \Sigma_k) \approx B_i(\Sigma) - F_i(\Sigma, \Sigma_k)$, $F_i(\Sigma, \Sigma_k) \approx B_i(\Sigma_1)$ (M, TG, pp. 55–57).

A THEOREM CONCERNING CLOSED AND COMPACT POINT SETS WHICH LIE IN CONNECTED DOMAINS[1]

HARLAN C. MILLER

The purpose of this paper is to show that the following theorem holds in any space which satisfies Axioms 0, 1, and 2 of R. L. Moore's *Foundations of Point Set Theory.*[2]

If g denotes a point set, $\bar{g}$ will be used to denote the set g together with all its limit points. For each positive integer n, G_n will denote the collection G_n of Axiom 1.

THEOREM. *If M is a closed and compact subset of a connected domain D, then there exists a compact continuum containing M and lying in D.*

PROOF. For each point P of D, there exists a region g_P of G_1 containing P such that $\bar{g}_P$ is a subset of D. By Axiom 2, there exists a connected domain d_P containing P which is a subset of g_P. Let U_1 denote the collection of all domains d_P for each point P of D. The point set M is closed and compact, and hence, by Theorem 22 of Chapter I, it is covered by a finite subcollection W_1 of U_1. By Theorem 77 of Chapter I, for each pair of domains x and y of W_1 there exists a simple chain xy whose links are domains of U_1 and whose first and last links are x and y respectively. Let V_1 denote the collection of all domains v such that for some two domains x and y of W_1, v is a link of the chain xy. The sum of all the domains of the finite collection V_1 is a connected domain D_1. Similarly, there exists a finite collection V_2 of connected domains such that if v is any domain of V_2, then $\bar{v}$ is a subset of some region of G_2 and of some domain of V_1, and such that the sum of the domains of V_2 is a connected domain D_2. This process can be continued. Thus there exists an infinite sequence $V_1, V_2, V_3, \cdots$ such that, for each n, (1) V_{n+1} is a finite collection of connected domains such that if v is any one of them then $\bar{v}$ is a subset of some region of G_{n+1} and of some domain of V_n and of D, and (2) the sum of all the domains of V_n is a connected domain D_n containing M. By Theorems 79 and 80 of Chapter I, the set of all points common to all the sets of the sequence $D_1, D_2, D_3, \cdots$ is a compact continuum, and it contains M and lies in D.

A modification of this argument proves this theorem for a space which satisfies Axioms 0 and 1 and is locally arcwise connected.

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[1] Presented to the Society, February 24, 1940.

[2] American Mathematical Society Colloquium Publications, vol. 13, New York, 1932. All references are to this book.

ON THE MEAN VALUES OF AN ANALYTIC FUNCTION[1]

RAPHAEL M. ROBINSON

This note contains improvements on the results in two recent papers by Nehari.[2]

The first paper shows that if $f(z)$ is regular for $|z|<1$, and if the mean of $|f(z)|$ on the circle $|z|=r$ is less than or equal to 1 for each $r<1$, then the mean of $|f(z)|^2$ on $|z|=r$ is less than or equal to 1 for $r\leqq 6^{-1/2}$. We shall show that the conclusion is true for $r\leqq 2^{-1/2}$, but not always for a larger value of r. *More generally, we shall show that the mean of* $|f(z)|^p$ *on* $|z|=r$ *is less than or equal to* 1 *for* $r\leqq p^{-1/2}$ (*where* $p>1$ *is an integer*), *and that this result is the best possible.*

It will be sufficient to prove that if $g(z)$ is a function which is regular for $|z|\leqq 1$ and different from 0 for $|z|<1$, and such that the mean of $|g(z)|$ on $|z|=1$ is less than or equal to 1, then the mean of $|g(z)|^p$ on $|z|=r$ is less than or equal to 1 for $r\leqq p^{-1/2}$. For suppose $0<R<1$, and put

$$g(z) = f(Rz) : \prod_{\nu=1}^{n} \frac{z-\alpha_\nu}{1-\bar{\alpha}_\nu z},$$

where $\alpha_1, \alpha_2, \cdots, \alpha_n$ are the zeros of $f(Rz)$ in $|z|<1$. We note that $|g(z)|=|f(Rz)|$ for $|z|=1$, while $|g(z)|>|f(Rz)|$ for $|z|<1$. The function $g(z)$ evidently satisfies the above conditions. From the conclusion that the mean of $|g(z)|^p$ on $|z|=r$ is less than or equal to 1 for $r\leqq p^{-1/2}$, we see that the mean of $|f(Rz)|^p$ on $|z|=r$ is not greater than 1 for $r\leqq p^{-1/2}$, or that the mean of $|f(z)|^p$ on $|z|=r$ is not greater than 1 for $r\leqq Rp^{-1/2}$. The desired result follows by letting $R\to 1$.

We have to show that from the hypothesis $(1/2\pi)\int_0^{2\pi}|g(e^{i\theta})|d\theta\leqq 1$ the conclusion

$$\frac{1}{2\pi}\int_0^{2\pi} |g(re^{i\theta})|^p d\theta \leqq 1, \qquad \text{for } r \leqq p^{-1/2},$$

follows. Now since $g(z)\neq 0$ for $|z|<1$, we may put $g(z)=h(z)^2$, where $h(z)$ is regular for $|z|<1$. If we put

$$h(z) = \sum_{n=0}^{\infty} a_n z^n, \qquad h(z)^p = \sum_{n=0}^{\infty} c_n z^n,$$

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[1] Presented to the Society December 2, 1939.

[2] Comptes Rendus de l'Académie des Sciences, Paris, vol. 206 (1938), pp. 1943–1945; vol. 208 (1939), pp. 1785–1787. My results were obtained during a summer (1939) spent at Stanford University. The two papers mentioned were called to my attention by Professor Szegö.

and use the well known formula

$$\frac{1}{2\pi}\int_0^{2\pi} | h(re^{i\theta}) |^2 d\theta = \sum_{n=0}^{\infty} | a_n |^2 r^{2n},$$

we see that the hypothesis becomes $\sum_{n=0}^{\infty} | a_n |^2 \leqq 1$, while in a similar manner the desired conclusion becomes

$$\sum_{n=0}^{\infty} | c_n |^2 r^{2n} \leqq 1, \qquad \text{for } r \leqq p^{-1/2}.$$

Now $c_n = \sum a_{n_1} a_{n_2} \cdots a_{n_p}$, where the sum extends over all sets $(n_1, n_2, \cdots, n_p)$ of integers not less than 0 whose sum is n. Hence by the Cauchy-Schwarz inequality, we have

$$| c_n |^2 \leqq \sum 1 \cdot \sum | a_{n_1}^2 a_{n_2}^2 \cdots a_{n_p}^2 |,$$

where the sums have the same range as before. Now $\sum 1$ is the number of ways of distributing n units among p terms, and hence is not greater than p^n, which is the number of ways of distributing n *different* things among p sets. Hence for $r \leqq p^{-1/2}$ we have

$$| c_n |^2 r^{2n} \leqq \sum | a_{n_1}^2 a_{n_2}^2 \cdots a_{n_p}^2 |,$$

and therefore

$$\sum_{n=0}^{\infty} | c_n |^2 r^{2n} \leqq \left(\sum_{n=0}^{\infty} | a_n |^2 \right)^p \leqq 1.$$

The theorem is not true for $r > p^{-1/2}$. For if $\epsilon > 0$ and we put $f(z) = (1 + \epsilon z)^2/(1 + \epsilon^2)$, then the hypothesis of the theorem is satisfied. On the other hand, $f(z)^p = (1 + p\epsilon z + \cdots)^2/(1 + p\epsilon^2 + \cdots)$, so that the mean of $|f(z)|^p$ on $|z| = r$ is $(1 + p^2\epsilon^2 r^2 + \cdots)/(1 + p\epsilon^2 + \cdots)$, which is greater than 1 if $pr^2 > 1$ and ϵ is sufficiently small. This negative conclusion is true also for non-integral values of p; but we have been able to prove the positive statement only for integral values of p.

We turn now to the second paper mentioned. In this, it is proved that if $f(0) = 0$ and if the mean of $|f(z)|$ along each radius of the unit circle is not greater than 1, then the mean of $|f(z)|$ along $|z| = r$ is less than or equal to 1 for $r \leqq \frac{1}{2}$, but not always for a larger value of r. The negative part of the statement is immediate, the counter-example being $f(z) = 2z$. *We shall show that the hypothesis* $f(0) = 0$ *is unnecessary, and that the stronger statement that the mean of* $|f(z)|^2$ *along* $|z| = r$ *is less than or equal to* 1 *for* $r \leqq \frac{1}{2}$ *is also true.*

We prove first the following result. *If the mean of* $| F(z) |^2$ *on* $|z| = r$

is not greater than 1 *for* $r<1$, *then the mean of* $|F'(z)|^2$ *on* $|z|=r$ *is less than or equal to* 1 *for* $r\leqq\frac{1}{2}$. If we put $F(z)=\sum_{n=0}^{\infty}b_nz^n$, we have only to prove that

$$\sum_{n=1}^{\infty} n^2|b_n|^2r^{2(n-1)} \leqq \sum_{n=0}^{\infty}|b_n|^2 \qquad \text{for } r \leqq \tfrac{1}{2}.$$

Since $n\leqq 2^{n-1}$ for all positive integral values of n, we see that $nr^{n-1}\leqq 1$, so that the inequality is true. (The result is not correct for $r>\frac{1}{2}$; counterexample, $F(z)=z^2$.)

We suppose now that the mean of $|f(z)|$ along each radius of the unit circle is not greater than 1, and put

$$F(z) = \int_0^z f(\zeta)d\zeta.$$

Since the integral may be taken along a radius, we see that

$$|F(z)| \leqq 1, \qquad \text{for } |z| < 1.$$

Hence the mean of $|F(z)|^2$ on $|z|=r$ is certainly not greater than 1 for any $r<1$. Therefore the mean of $|F'(z)|^2=|f(z)|^2$ on $|z|=r$ is not greater than 1 for $r\leqq\frac{1}{2}$.

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REMARKS ON A NOTE OF MR. R. WILSON AND ON RELATED SUBJECTS[1]

G. SZEGÖ

Introduction. *Let* $w(x)$ *be a nonnegative weight function on the interval* $-1 \leqq x \leqq +1$, *and let the integral*

$$\int_{-1}^{+1} \log w(x)\cdot(1-x^2)^{-1/2}dx = \int_0^\pi \log w(\cos\theta)d\theta \tag{1}$$

exist in the sense of Lebesgue.

If $\{p_n(x) = k_n x^n + \cdots\}$ *denotes the orthonormal set of polynomials associated with* $w(x)$, *we have*

$$\lim_{n\to\infty} \max_{-1\leqq x\leqq +1} |p_n(x)|^{1/n} = 1, \tag{2}$$

and[2]

$$\lim_{n\to\infty} k_n^{1/n} = 2. \tag{3}$$

In 1921 I found[3] the following asymptotic formula for the orthogonal polynomials $p_n(x)$ for $n\to\infty$, holding for x not on the segment $[-1, +1]$:

$$\lim_{n\to\infty} z^n p_n(x) = \Delta(z) \tag{4}$$

where $2x = z + z^{-1}$, $|z| < 1$, and $\Delta(z)$ is a certain analytic function regular and nonzero for $|z| < 1$. Of course, $\Delta(z)$ depends on the weight function $w(x)$. The formula (4) holds uniformly for

$$|z| \leqq r,\ r < 1.$$

From this result the formulas (3) and, by an additional elementary remark (cf. below (9)), (2) follow immediately. Also it furnishes (cf. OP, p. 302, Theorem 12.7.1):

$$\lim_{n\to\infty} 2^{-n}k_n = \pi^{-1/2}\exp\left\{-\frac{1}{2\pi}\int_0^\pi \log w(\cos\theta)d\theta\right\}. \tag{5}$$

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[1] Presented to the Society, February 24, 1940.

[2] Concerning the notation see my book *Orthogonal Polynomials* (American Mathematical Society Colloquium Publications, vol. 23, 1939). Hereafter this book will be referred to as OP.

[3] G. Szegö, *Über die Entwickelung einer analytischen Funktion nach den Polynomen eines Orthogonalsystems*, Mathematische Annalen, vol. 82 (1921), pp. 188–212; p. 191. Cf. also OP, p. 290, Theorem 12.1.2.

Mr. R. Wilson devoted a recent note[4] to the proof of the relation (2), or rather of the following relation:

$$\lim_{n\to\infty} \max_{-1\leqq x\leqq 1} \left| k_n^{-1} p_n(x) \right|^{1/n} = \tfrac{1}{2}, \tag{6}$$

which is, on account of (3), equivalent to (2). His argument is based on certain results of Mr. Shohat,[5] which are incidentally consequences of the asymptotic formula (4). The conditions used by Shohat are more restrictive than the existence of (1).

By applying a classical theorem of Poincaré on recurrence formulas, Shohat proves that

$$\lim_{n\to\infty} \left| p_n(x) \right|^{1/n} = \left| z \right|^{-1}, \tag{7}$$

where x and z have the same meaning as before and $|z| < 1$. (From this, (3) follows for $x = \infty$ or $z = 0$.) Based on (7), a proof of (2) or (6) can easily be arranged. Wilson (loc. cit., p. 191) prefers, however, to use another theorem on recurrence formulas due to Perron.[6]

In the present note, I give first a very simple direct approach to (2), (3), and (6), assuming the existence of (1). Naturally the deeper result (5) requires more refined methods.

Further, we deal with the following related result of Shohat (loc. cit., pp. 34–36): *Let* $w(x) \geqq 0$ *be an arbitrary weight function on the interval* $-1 \leqq x \leqq +1$. *Employing the former notation, the relations*

$$2^{-n}k_n = O(1), \qquad \lim_{n\to\infty} 2^{-n}k_n \textit{ exists} \tag{8}$$

are equivalent. Of course, this means that the second relation follows from the first one. The proof, given below, is essentially Shohat's argument; we found it, however, convenient and possible to eliminate every reference to the theory of continued fractions used by Shohat.

Finally, by means of the deeper result (5) we show that the existence of the integrals (1) is not only *sufficient* but also *necessary* for the relations (8). More precisely: *Let* $w(x)$ *be a nonnegative weight*

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[4] R. Wilson, *A note on the asymptotic properties of orthogonal polynomials*, this Bulletin, vol. 45 (1939), pp. 190–192.

[5] See J. Chokhatte (Shohat), *Sur le développement de l'intégrale* $\int_a^b [p(y)/(x-y)]dy$ *en fraction continue et sur les polynômes de Tchebycheff*, Rendiconti del Circolo Matematico di Palermo, vol. 47 (1923), pp. 25–46; cf. in particular pp. 43–44.

[6] The following objection can be made to his argument. In the present case the coefficients of the recurrence formula contain x as a parameter. However, max $|p_n(x)|$, $-1 \leqq x \leqq +1$, will be in general attained for an $x = x(n)$ which varies with n. Therefore, a generalization of Perron's theorem is needed here stating the *uniform* existence of the limit involved.

function on the interval $-1 \leqq x \leqq +1$, *and let* $p_n(x) = k_n x^n + \cdots$ *have the same meaning as before. If* $2^{-n} k_n = O(1)$, *the integrals* (1) *exist.*

Proof of (2), (3), (6). It is well known that

$$\max_{-1 \leqq x \leqq +1} \left| k_n^{-1} p_n(x) \right| \geqq 2^{1-n}, \tag{9}$$

so that for (6) it suffices to show that in $-1 \leqq x \leqq +1$

$$k_n^{-1} \left| p_n(x) \right| < A \cdot 2^{-n}(1+\epsilon)^n; \tag{10}$$

here $\epsilon > 0$ is arbitrary and A depends only on $w(x)$ and ϵ.

Let $z = re^{i\phi}$, $0 \leqq r < 1$. By use of the inequality between the arithmetic and geometric mean of a function,[7] we find that

$$\begin{aligned}
\pi^{-1} &= \frac{1}{2\pi} \int_{-\pi}^{+\pi} w(\cos\theta) \left| \sin\theta \right| \left\{ p_n(\cos\theta) \right\}^2 d\theta \\
&\geqq \frac{1-r}{1+r} \frac{1}{2\pi} \int_{-\pi}^{+\pi} w(\cos\theta) \left| \sin\theta \right| \left\{ p_n(\cos\theta) \right\}^2 \\
&\qquad \cdot \frac{1-r^2}{1-2r\cos(\theta-\phi)+r^2} d\theta \\
&\geqq \frac{1-r}{1+r} \cdot \exp \left\{ \frac{1}{2\pi} \int_{-\pi}^{+\pi} \log \left[w(\cos\theta) \left| \sin\theta \right| \right] \right. \\
&\qquad \left. \cdot \frac{1-r^2}{1-2r\cos(\theta-\phi)+r^2} d\theta \right\} \\
&\qquad \cdot \exp \left\{ \frac{1}{2\pi} \int_{-\pi}^{+\pi} \log \left\{ p_n(\cos\theta) \right\}^2 \right. \\
&\qquad \left. \cdot \frac{1-r^2}{1-2r\cos(\theta-\phi)+r^2} d\theta \right\}.
\end{aligned} \tag{11}$$

The harmonic function

$$\log \left| z^n p_n(\tfrac{1}{2}(z+z^{-1})) \right|^2 = 2\Re \log \left\{ z^n p_n(\tfrac{1}{2}(z+z^{-1})) \right\} \tag{12}$$

is regular for $|z| < 1$, and it has logarithmic singularities at $z = e^{\pm i\alpha_\nu}$, if $\cos\alpha_\nu$, $\nu = 1, 2, \cdots, n$, denote the roots of $p_n(x)$. Therefore, the second exponential expression in (11) becomes

$$\exp \left\{ \log \left| z^n p_n(\tfrac{1}{2}(z+z^{-1})) \right|^2 \right\} = \left| z \right|^{2n} \left| p_n(\tfrac{1}{2}(z+z^{-1})) \right|^2.$$

Consequently,

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[7] See, for instance, G. Pólya and G. Szegö, *Aufgaben und Lehrsätze aus der Analysis*, vol. 1, Berlin, 1925, pp. 54, 210, Problem 79.

$$(13)\qquad \begin{aligned} |z|^n|p_n(\tfrac{1}{2}(z+z^{-1}))| &< \pi^{-1/2}\left(\frac{1+r}{1-r}\right)^{1/2} \\ &\quad\cdot\exp\left\{\frac{1}{2\pi}\int_0^\pi |\log[w(\cos\theta)\sin\theta]|\,d\theta\cdot\frac{1+r}{1-r}\right\}. \end{aligned}$$

For $z=0$ this results in the inequality

$$2^{-n}k_n < \pi^{-1/2}\exp\left\{\frac{1}{2\pi}\int_0^\pi |\log[w(\cos\theta)\sin\theta]|\,d\theta\right\}.$$

On the other hand, for $x=\cos\theta$,

$$\begin{aligned} k_n^{-2} &= \min\int_{-1}^{+1} w(x)(x^n+\cdots)^2dx \\ &\leqq \int_{-1}^{+1} w(x)(2^{1-n}\cos n\theta)^2dx < 2^{2-2n}\int_{-1}^{+1} w(x)dx, \end{aligned}$$

so that $2^{-n}k_n$ remains between two positive bounds. From here (3) follows. Choosing $|z|=1-\delta$ in (13) where $\delta>0$ is sufficiently small, and applying the maximum-principle in the corresponding ellipse of the x-plane, we obtain (10) which implies (6) and also (2).

Proof of the equivalence of the relations (8) **(Shohat's theorem).** Let (see OP, p. 41, Theorem 3.2.1)

$$(14)\quad p_n(x) = k_n\bar{p}_n(x) = k_n(x^n - S_nx^{n-1}+\cdots),$$

$$(15)\quad \bar{p}_n(x) = (x-c_n)\bar{p}_{n-1}(x) - \left(\frac{k_{n-2}}{k_{n-1}}\right)^2\bar{p}_{n-2}(x),$$

$$(16)\qquad b_{2n} = -\frac{k_{n-1}}{k_n}\,\frac{p_n(-1)}{p_{n-1}(-1)} > 0,\quad b_{2n+1} = -\frac{k_{n-1}}{k_n}\,\frac{p_{n-1}(-1)}{p_n(-1)} > 0,$$
$$n = 1, 2, 3, \cdots;\ \bar{p}_{-1}(x) = 0.$$

Obviously,

$$(17)\qquad k_0^{-2}k_n^2 = (b_2b_3\cdots b_{2n}b_{2n+1})^{-1},$$

$$(18)\quad c_n+1 = b_{2n}+b_{2n-1},\ S_n = \sum_{\nu=1}^{2n} b_\nu - n,\ \ n = 1, 2, 3, \cdots;\ b_1 = 0.$$

Since

$$(19)\qquad c_n\int_{-1}^{+1}\{p_{n-1}(x)\}^2w(x)dx = \int_{-1}^{+1} x\{p_{n-1}(x)\}^2w(x)dx,$$

we have $|c_n|<1$, so that from (18) $0<b_n<2$ follows (except $b_1=0$).

If $T_n(x)$ denotes the polynomial of Tchebichef, we write

$$T_n(x) = h_0p_0(x) + h_1p_1(x) + \cdots + h_np_n(x); \tag{20}$$

then

$$\begin{gathered} h_\nu = \int_{-1}^{+1} T_n(x)p_\nu(x)w(x)dx, \\ | h_\nu | \leqq \int_{-1}^{+1} | p_\nu(x) | \, w(x)dx \leqq \left\{ \int_{-1}^{+1} w(x)dx \right\}^{1/2}, \\ \nu = 0, 1, 2, \cdots, n. \end{gathered} \tag{21}$$

Comparing the coefficients of x^n and x^{n-1} in (20), we find $2^{n-1} = h_nk_n$, $0 = -h_nk_nS_n + h_{n-1}k_{n-1}$, so that

$$S_n = 2^{1-n}h_{n-1}k_{n-1}, \qquad | S_n | \leqq 2^{1-n}k_{n-1}\left\{ \int_{-1}^{+1} w(x)dx \right\}^{1/2}.$$

Now, we define $U_n = 2b_n - 1$, $-1 < U_n < 3$ (except $U_1 = -1$). Then, as $n \to \infty$,

$$\begin{gathered} \sum_{\nu=1}^{2n} U_\nu = 2S_n = 2^{1-n}k_{n-1}\cdot O(1), \\ \prod_{\nu=2}^{2n+1} (1 + U_\nu)^{-1} = \prod_{\nu=2}^{2n+1} (2b_\nu)^{-1} = 2^{-2n}k_0^{-2}k_n^2 = (2^{-n}k_n)^2\cdot O(1). \end{gathered} \tag{22}$$

Assuming $2^{-n}k_n = O(1)$, both expressions (22) remain bounded.

Let c be a positive constant such that

$$u^{-2}\{u - \log (1 + u)\} > c, \qquad -1 < u < 3. \tag{23}$$

Thus $cU_n^2 < U_n - \log (1 + U_n)$, so that $\sum_{\nu=1}^{2n} U_\nu^2 = O(1)$, that is, $\sum U_n^2$ is convergent. The same holds for

$$\begin{aligned} \sum_{n=1}^{\infty} U_{2n}U_{2n+1} &= \sum_{n=1}^{\infty} \{4b_{2n}b_{2n+1} - 2(b_{2n} + b_{2n+1}) + 1\} \\ &= \sum_{n=1}^{\infty} \{4b_{2n}b_{2n+1} - 2(b_{2n} + b_{2n-1}) + 1\} \\ &\quad + \sum_{n=1}^{\infty} 2(b_{2n-1} - b_{2n+1}) \\ &= \sum_{n=1}^{\infty}\left\{4\left(\frac{k_{n-1}}{k_n}\right)^2 - 2(S_n - S_{n-1}) - 1\right\} \\ &\quad + \sum_{n=1}^{\infty} 2(b_{2n-1} - b_{2n+1}). \end{aligned} \tag{24}$$

The last series is convergent since $U_n \to 0$ or $b_n \to \frac{1}{2}$. Therefore,

$$(25)\qquad \sum_{n=1}^{\infty}\left\{4\left(\frac{k_{n-1}}{k_n}\right)^2 - 2(S_n - S_{n-1}) - 1\right\}$$

is convergent. Applying this result to the polynomials $(-1)^n p_n(-x) = k_n(x^n + S_n x^{n-1} + \cdots)$ associated with the weight function $w(-x)$ on $[-1, +1]$, we obtain the convergence of

$$(26)\qquad \sum_{n=1}^{\infty}\left\{4\left(\frac{k_{n-1}}{k_n}\right)^2 - 2(-S_n + S_{n-1}) - 1\right\},$$

or that of $\sum_{n=1}^{\infty}(S_n - S_{n-1})$. This is equivalent to the existence of $\lim_{n\to\infty} S_n$, or to the convergence of $\sum U_n$. This, together with the convergence of $\sum U_n^2$, implies the convergence of the product $\prod(1+U_n)^{-1}$ or the existence of $\lim_{n\to\infty} 2^{-n}k_n$.

Proof of the equivalence of the conditions (8) **to the existence of the integrals** (1). The relations (8) are equivalent to the fact that for every polynomial $q(x) = x^n + \cdots$ of the nth degree with the highest term x^n

$$(27)\qquad \int_{-1}^{+1}\{q(x)\}^2 w(x)dx > c\cdot 2^{-2n}$$

holds, where $c > 0$ is independent of n. Let $\epsilon > 0$; we have

$$(28)\qquad \int_{-1}^{+1}\{q(x)\}^2 (w(x) + \epsilon)dx > c\cdot 2^{-2n}.$$

The minimum of the left-hand side is $\{k_n(\epsilon)\}^{-2}$, where $k_n(\epsilon)$ denotes the highest coefficient of the orthonormal polynomial of the nth degree associated with $w(x)+\epsilon$. For this weight function, the integral condition (1) is satisfied so that according to (5)

$$(29)\qquad \lim_{n\to\infty} 2^{-n}k_n(\epsilon) = \pi^{-1/2}\exp\left\{-\frac{1}{2\pi}\int_0^{\pi}\log\left[w(\cos\theta) + \epsilon\right]d\theta\right\}.$$

Therefore,

$$\pi \exp\left\{\pi^{-1}\int_0^{\pi}\log\left[w(\cos\theta) + \epsilon\right]d\theta\right\} \geqq c,$$

$$\int_0^{\pi}\log\left[w(\cos\theta) + \epsilon\right]d\theta \geqq c'$$

where c' is independent of ϵ. Now, $\log(\alpha+\beta) < \log\beta + 1$ for $0 < \alpha < \frac{1}{2} < \beta$; so if $0 < \epsilon < \frac{1}{2}$,

$$\int_{w(x)>1/2} \log\,[w(\cos\theta)+\epsilon]d\theta < \int_{w(x)>1/2} \{\log w(\cos\theta)+1\}d\theta,$$

hence

$$\int_{w(x)\leqq 1/2} \log\,[w(\cos\theta)+\epsilon]d\theta > c''$$

where c'' is independent of ϵ. The same inequality holds if the integration is extended over the set $0 < \eta \leqq w(x) \leqq \frac{1}{2}$. But for a decreasing sequence of bounded (negative) functions the operations of integration and passing to the limit as $\epsilon \to +0$ are interchangeable; consequently,

$$\int_{\eta \leqq w(x) \leqq 1/2} \log w(\cos\theta)d\theta \geqq c'',$$

and since this is true for all $\eta > 0$, the integral (1) exists.

STANFORD UNIVERSITY

4

THE SUMMER MEETING IN HANOVER

The forty-sixth Summer Meeting of the Society and the twenty-second Colloquium were held at Dartmouth College, Hanover, New Hampshire, from Tuesday to Thursday, September 10–12, 1940. The Mathematical Association of America met on Monday morning and afternoon and the Institute of Mathematical Statistics on Tuesday morning and afternoon and Wednesday and Thursday mornings. About four hundred fifty persons attended the meetings, among whom were the following two hundred ninety-eight members of the Society:

C. R. Adams, R. B. Adams, R. P. Agnew, Leon Alaoglu, Warren Ambrose, R. C. Archibald, Herbert E. Arnold, H. T. R. Aude, R. P. Bailey, J. L. Barnes, R. C. F. Bartels, Felix Bernstein, R. A. Beth, E. E. Betz, Harry Birchenough, Garrett Birkhoff, G. D. Birkhoff, R. P. Boas, Salomon Bochner, C. B. Boyer, A. T. Brauer, R. W. Brink, B. H. Brown, B. L. Brown, R. E. Bruce, G. S. Bruton, J. A. Bullard, C. T. Bumer, R. S. Burington, F. J. H. Burkett, J. H. Bushey, Jewell H. Bushey, W. D. Cairns, B. H. Camp, M. E. Carlen, Leonard Carlitz, W. B. Carver, J. A. Clarkson, R. F. Clippinger, L. W. Cohen, Nancy Cole, Esther Comegys, G. M. Conwell, T. F. Cope, A. H. Copeland, L. P. Copeland, Richard Courant, A. P. Cowgill, A. T. Craig, A. R. Crathorne, H. B. Curry, D. R. Curtiss, J. H. Curtiss, Tobias Dantzig, D. R. Davis, F. F. Decker, L. S. Dederick, M. R. Demers, A. H. Diamond, C. E. Dimick, C. H. Dix, W. J. Dixon, H. A. DoBell, J. L. Doob, T. L. Downs, Arnold Dresden, John Dyer-Bennet, Samuel Eilenberg, Churchill Eisenhart, L. P. Eisenhart, M. L. Elveback, Benjamin Epstein, Paul Erdös, G. C. Evans, G. W. Evans, W. K. Feller, W. B. Fite, H. T. Fleddermann, M. M. Flood, L. R. Ford, G. E. Forsythe, Tomlinson Fort, R. M. Foster, A. H. Fox, J. S. Frame, Philip Franklin, K. O. Friedrichs, Orrin Frink, T. C. Fry, R. E. Fullerton, M. G. Gaba, H. L. Garabedian, H. J. Gay, H. P. Geiringer, B. P. Gill, R. E. Gilman, Michael Goldberg, Michael Golomb, C. H. Graves, L. J. Green, E. H. Hadlock, D. W. Hall, Marshall Hall, P. R. Halmos, Israel Halperin, J. A. Hamilton, D. C. Harkin, O. G. Harrold, O. C. Hazlett, E. R. Hedrick, M. H. Heins, Robert Henderson, J. G. Herriot, E. H. C. Hildebrandt, T. H. Hildebrandt, Einar Hille, M. P. Hollcroft, T. R. Hollcroft, G. M. Hopper, Harold Hotelling, S. E. Hotelling, E. M. Hull, E. V. Huntington, W. A. Hurwitz, M. H. Ingraham, Dunham Jackson, S. B. Jackson, Fritz John, R. A. Johnson, F. E. Johnston, A. W. Jones, B. W. Jones, H. A. Jordan, Mark Kac, M. L. Kales, E. R. van Kampen, Wilfred Kaplan, E. S. Kennedy, L. S. Kennison, S. C. Kleene, J. R. Kline, J. C. Knipp, E. R. Kolchin, W. D. Lambert, O. E. Lancaster, A. E. Landry, R. E. Langer, R. A. Leibler, B. A. Lengyel, C. A. Lester, Norman Levinson, D. C. Lewis, M. I. Logsdon, W. R. Longley, L. H. Loomis, C. I. Lubin, N. H. McCoy, E. J. McShane, L. A. MacColl, C. C. MacDuffee, Saunders MacLane, H. M. MacNeille, W. G. Madow, Dorothy Manning, L. C. Mathewson, Walther Mayer, W. E. Milne, Richard von Mises, E. B. Mode, Deane Montgomery, T. W. Moore, F. M. Morgan, W. K. Morrill, Max Morris, Marston Morse, E. J. Moulton, G. W. Mullins, C. W. Munshower, O. E. Neugebauer, John von Neumann, C. V. Newsom, A. V. Newton, Jerzy Neyman, C. O. Oakley, R. E. O'Connor, P. S. Olmstead, J. M. H. Olmsted, Isaac Opatowski, Oystein Ore, F. W. Owens, H. B. Owens, Gordon Pall, E. W. Paxson, F. W. Perkins, O. J. Peterson, C. G. Phipps, Everett Pitcher, Hillel

Poritsky, G. B. Price, E. J. Purcell, H. A. Rademacher, E. D. Rainville, J. F. Randolph, M. S. Rees, R. W. Rempfer, Harris Rice, R. G. D. Richardson, D. E. Richmond, R. F. Rinehart, J. F. Ritt, H. E. Robbins, Robin Robinson, S. L. Robinson, Benjamin Rosenbaum, Joseph Rosenbaum, R. A. Rosenbaum, Arthur Rosenthal, J. B. Rosser, H. G. Russell, N. E. Rutt, S. T. Sanders, Peter Scherk, E. R. Schneckenburger, I. J. Schoenberg, G. E. Schweigert, C. H. W. Sedgewick, R. W. Sedgewick, W. E. Sewell, I. M. Sheffer, L. W. Sheridan, L. L. Silverman, R. G. Simond, M. E. Sinclair, Abraham Sinkov, H. L. Slobin, M. M. Slotnick, M. F. Smiley, C. E. Smith, P. A. Smith, T. L. Smith, W. M. Smith, Virgil Snyder, M. R. Solt, Joseph Spear, D. C. Spencer, V. E. Spencer, Abraham Spitzbart, G. R. Stibitz, C. N. Stokes, M. H. Stone, E. C. Strayhorn, R. E. Street, D. J. Struik, Otto Szász, Gabor Szegö, J. D. Tamarkin, J. M. Thomas, E. W. Titt, M. M. Torrey, J. I. Tracey, W. J. Trjitzinsky, A. W. Tucker, Bryant Tuckerman, J. W. Tukey, S. M. Ulam, G. B. Van Schaack, J. H. Van Vleck, H. E. Vaughan, Oswald Veblen, Abraham Wald, G. L. Walker, G. W. Walker, R. J. Walker, H. S. Wall, Charles Walmsley, J. L. Walsh, Warren Weaver, W. D. A. Westfall, A. P. Wheeler, H. S. White, A. L. Whiteman, P. M. Whitman, Hassler Whitney, J. K. Whittemore, G. T. Whyburn, W. M. Whyburn, D. V. Widder, Norbert Wiener, L. R. Wilcox, C. E. Wilder, S. S. Wilks, W. L. G. Williams, E. W. Wilson, Aurel Wintner, Audrey Wishard, E. E. Witmer, Jacob Wolfowitz, W. D. Wray.

The Colloquium Lectures on the subject *Analytic topology* were delivered by Professor G. T. Whyburn of the University of Virginia on Tuesday morning and afternoon and Wednesday and Thursday mornings. The presiding officers at the four lectures were, in order, President G. C. Evans, Professor J. R. Kline, Vice President C. R. Adams, and Vice President T. C. Fry.

On Thursday afternoon, by invitation of the Program Committee, Professor Leonard Carlitz of Duke University gave an address entitled *Arithmetic of polynomials in a Galois field.*

President G. C. Evans presided at the business meeting and Professor J. L. Walsh at the general session on Thursday morning; Professor Gordon Pall at the general session on Thursday afternoon. The presiding officers of the sectional sessions were as follows: Analysis, Professors L. W. Cohen, J. F. Randolph, I. M. Sheffer, Gabor Szegö; Algebra, Topology and Foundations, Dr. Marshall Hall; Algebra and Theory of Numbers, Professor B. W. Jones; Geometry, Professor M. G. Gaba; Mathematical Statistics (a joint session with the Institute of Mathematical Statistics), Professor B. H. Camp.

Headquarters for the meeting were in Thayer Hall. The dormitories of the college were available to the members of the three organizations and their guests.

Those attending the meetings were entertained Monday evening by a music recital, by moving pictures showing winter sports at Dartmouth, and by an exhibition of string models. On Monday afternoon, tea was served at the Graduate Club by the ladies of the Dartmouth faculty.

A dinner for the members of the Society, the Association and the Institute was held Tuesday evening in Thayer Hall. Professor B. H. Brown was toastmaster. The speakers were Dean E. G. Bill, Dean R. G. D. Richardson, Professors P. A. Smith, Hassler Whitney, and S. S. Wilks. The attendance was 324.

Wednesday afternoon was devoted to an excursion to Franconia Notch.

The Bell Telephone Laboratories exhibited a machine for computing with complex numbers. The recording instrument at Hanover was connected by telegraph with the computing mechanism in New York. This machine was available to members from 11 A.M. to 2 P.M. each day of the meetings.

A resolution by Professor E. J. Purcell thanking the President, officers and members of the Mathematics Department of Dartmouth College for their competent arrangements and cordial hospitality, was adopted unanimously at the general session Thursday morning.

The Board of Trustees met on Monday at 6:00 P.M. in Thayer Hall. It reviewed the financial affairs of the Society with a view to a report to the Council.

There were three meetings of the Council, one being a joint session with the Board of Governors of the Mathematical Association of America. This joint session to hear the report of the War Preparedness Committee was held in Dartmouth Hall at 9:00 P.M. on Monday. The other meetings of the Council were in Thayer Hall at 4:30 P.M. on Tuesday and at 9:00 P.M. on Wednesday.

The Secretary announced the election of the following sixteen persons to membership in the Society:

Miss Helen C. Brodie, New York, N. Y.;
Professor D. van Dantzig, Technische Hoogeschool, Delft, Netherlands;
Professor Joseph J. Dennis, Clark University, Atlanta, Ga.;
Mr. Albert B. Farnell, University of Oklahoma;
Professor James Norman Goodier, Cornell University;
Dr. L. Louise Johnson, Reed College, Portland, Ore.;
Professor Eugene Leimanis, University of Latvia, Riga, Latvia;
Dr. Beppo Levi, The Mathematical Institute, Rosario, Argentina;
Dr. Harry Miller Lydenberg, New York, N. Y.;
Mr. (Albert) Wayne McGaughey, University of Cincinnati;
Sister Teresa M. Madden, College of Our Lady of the Elms, Chicopee, Mass.;
Mr. James R. Newman, New York, N. Y.;
Dr. Maxwell Reade, Ohio State University;
Dr. Charles V. Robinson, University of Missouri;

Professor Arthur Rosenthal, University of Michigan;
Dr. Walter Stephen Snyder, Ohio State University.

The following appointments by President Evans were reported: as representative of the Society at the dedication of buildings at the University of Colorado on June 8–9, Professor A. J. Kempner; as chairman of the Committee on Publicity (to replace Professor C. N. Moore, who resigned), Mr. R. M. Foster; as an additional member of this Committee, Professor J. B. Rosser (the newly constituted committee is as follows: Messrs. R. M. Foster (chairman), Saunders MacLane, G. B. Price, J. B. Rosser, and J. M. Thompson); as representative of the Society at the Conference on Science, Philosophy, and Religion and Their Relation to the Democratic Way of Life, held at the Jewish Theological Seminary of America in New York City on September 9–11, 1940, Professor Hermann Weyl; as representative of the Society at the inauguration of George N. Shuster as fifth president of Hunter College on October 8–11, Professor H. S. White.

Meetings of the Society during 1941 were set as follows: in New York City on February 22; in Stanford University, California on April 5; in Washington, D. C. on May 2–3; at the University of Chicago on September 2–6; in New York City on October 25. Summer Meetings were arranged for Cornell University in 1942 and for the University of Colorado in 1943.

An invitation to meet at the University of New Hampshire on June 23–27, 1941, in connection with the Congress of Sciences and Humanities and the American Association for the Advancement of Science was received.

The Committee on War Preparedness presented an extended report, which is summarized in an article by the Chairman, Professor Marston Morse, in the September number of this Bulletin. A Roster of Scientific Personnel is being compiled by the National Resources Planning Board of the U. S. Government, and Dean L. P. Eisenhart, Professor G. C. Evans, and Dean R. G. D. Richardson represent mathematics on the committee.

The Committee on Scientific Aids to Learning (supported by funds from the Carnegie Corporation) has appropriated a sum not to exceed $10,000 for microfilm reading machines, the distribution of which is to be for the benefit of Mathematical Reviews and to be arranged by the Executive Committee of that journal.

Reports were sent in July to the Carnegie Corporation and Rockefeller Foundation regarding their grants to Mathematical Reviews.

Certain invitations to give hour addresses were announced: Professors J. B. Rosser and H. S. Wall for the November 1940 meeting in Detroit, Professors Saunders MacLane and Leo Zippin for the 1940 Annual Meeting, and Professor M. R. Hestenes for the 1941 April meeting in Chicago.

The sixteenth Josiah Willard Gibbs Lecture is to be given by Professor Sewall Wright at the Summer Meeting of 1941 in Chicago. The committee to select Gibbs lecturers for 1942 and 1943 consists of Professors H. P. Robertson (chairman), N. H. McCoy, and J. L. Synge.

Professors H. E. Buchanan and H. J. Ettlinger were appointed representatives on the Council of the American Association for the Advancement of Science for 1941 and Professor F. D. Murnaghan was appointed representative on the National Research Council for a period of three years beginning July, 1941.

The report of the Committee to Consider Possible Freer Methods of Nominating and Electing Council Members as adopted by the Council provides for a decrease in the number of ex-officio members of the Council, the opportunity for choice on the ballot in the election of vice presidents and elected members of the Council, and the chance on the ballot to suggest names for the various offices for the year following. A committee consisting of Professors W. M. Whyburn (chairman), Alonzo Church, and Philip Franklin was appointed to make appropriate revisions of the By-Laws with a view to presenting them for adoption by the Society. Provision is to be made on the Council for a representative of Mathematical Reviews.

A Committee on the Award of the Cole Prize in the Theory of Numbers for the period 1936–1940 was appointed, consisting of Professors H. H. Mitchell (chairman), A. J. Kempner, and Hermann Weyl.

A committee consisting of Professor Richard Courant (chairman), Mr. R. M. Foster, and Professor Harold Hotelling was named to consider the problem of holding informal discussion groups in regard to important papers read before the Society which touch on applications.

Titles and cross references to the abstracts of the papers read are given below. The papers were read as follows: papers 1–8 in the section for Analysis on Tuesday morning; papers 9–13 in the section for Geometry on Tuesday morning; papers 14–19 in the section for Mathematical Statistics on Tuesday afternoon; papers 20–23 in the section for Analysis on Tuesday afternoon; papers 24–31 in the section for Analysis on Wednesday morning; papers 32–38 in the section for Algebra, Topology and Foundations on Wednesday morning; papers 39–43 in the General Session on Thursday morning; papers 44–

50 in the section for Analysis on Thursday afternoon; papers 51–55 in the section for Algebra and Theory of Numbers on Thursday afternoon; and papers 56–96, whose abstract numbers are followed by the letter *t*, were read by title. Dr. Wundheiler was introduced by Professor D. J. Struik, Dr. Brown by Professor S. S. Wilks, Professor Zygmund by Professor T. R. Hollcroft, Dr. Hutchinson by Professor D. J. Struik, Professor Alexandroff by Professor P. A. Smith, Dr. Bergman by Dr. D. C. Spencer, Professor Hua by Professor T. R. Hollcroft, Dr. Manel by Professor Richard Courant, Professor Pólya by Professor Einar Hille, and Mr. Rosenthall by Professor E. T. Bell. Paper 1 was presented by Dr. Stephan Bergman, paper 4 by Mr. Epstein, paper 8 by Dr. Poritsky, paper 12 by Professor Struik, paper 21 by Dr. Boas, paper 22 by Dr. Halmos, paper 30 by Dr. Spencer, paper 31 by Professor Wiener, paper 38 by Dr. Hall, paper 39 by Professor Bochner, and paper 46 by Dr. Alaoglu.

1. Abe Gelbart: *On approximations of functions of two complex variables regular in multiply-connected four-dimensional domains.* (Abstract 46-9-418.)

2. K. O. Friedrichs: *On hyperbolic differential operators.* (Abstract 46-9-416.)

3. Orrin Frink: *Series expansions in linear vector space.* (Abstract 46-9-417.)

4. D. G. Bourgin and Benjamin Epstein: *A class of kernels generated by a Laplace-Mellin transformation.* (Abstract 46-11-448.)

5. Isaac Opatowski: *On the motion of an electric particle.* (Abstract 46-11-468.)

6. D. C. Harkin: *Fourier series as limits of Gaussian sums.* (Abstract 46-11-457.)

7. Einar Hille: *A class of differential operators of infinite order.* (Abstract 46-11-491.)

8. Hillel Poritsky and H. D. Snively: *Doubly periodic networks.* (Abstract 46-11-471.)

9. Alexander Wundheiler: *Some basic notions in meta-geometry.* (Abstract 46-9-445.)

10. Marshall Hall: *Projective planes.* (Abstract 46-9-421.)

11. W. L. G. Williams: *The application of hyperbolic complex numbers to the geometry of the triangle.* (Abstract 46-11-482.)

12. D. J. Struik and Alexander Wundheiler: *Affine representation of systems of forces.* (Abstract 46-9-439.)

13. E. J. Purcell: *Space Cremona transformations of order $m+n-1$.* (Abstract 46-9-433.)

14. Hilda P. Geiringer: *A generalization of the law of large numbers.* (Abstract 46-11-455.)

15. S. S. Wilks: *On the problem of two samples from normal populations with unequal variances.* (Abstract 46-11-481.)

16. Harold Hotelling: *Experimental determination of the maximum of an empirical function.* (Abstract 46-9-424.)

17. Abraham Wald: *Asymptotically shortest confidence intervals.* (Abstract 46-11-479.)

18. G. W. Brown: *Reduction of certain composite statistical hypotheses.* (Abstract 46-11-449.)

19. Jerzy Neyman: *Conception of equivalence in the limit of statistical tests and its application to certain forms of* χ^2 *and to the tests based on the* λ *principle.* (Abstract 46-11-467.)

20. J. F. Randolph: *Some properties of sets of the Cantor type.* (Abstract 46-11-472.)

21. R. P. Boas and D. V. Widder: *Functions with positive differences.* (Abstract 46-9-404.)

22. P. R. Halmos and R. A. Leibler: *Square roots of measure preserving transformations.* (Abstract 46-9-422.)

23. Antoni Zygmund: *Power series of the class* H^λ *on the circle of convergence.* (Abstract 46-11-483.)

24. J. G. Herriot: *Nörlund summability of double Fourier series.* Preliminary report. (Abstract 46-9-423.)

25. M. H. Heins: *A proof of Picard's theorem by the fundamental theorem on the conformal mapping of simply-connected Riemann surfaces.* Abstract 46–11-459.)

26. I. M. Sheffer: *On the singularities of functions under certain linear transformations.* (Abstract 46-11-475.)

27. I. J. Schoenberg: *On positive definite functions on spheres.* (Abstract 46-11-474.)

28. W. J. Trjitzinsky: *Properties of growth for solutions of differential equations of dynamical type.* (Abstract 46-11-478.)

29. J. M. H. Olmsted: *Lebesgue theory on a Boolean algebra.* Preliminary report. (Abstract 46-9-430.)

30. Stefan Bergman and D. C. Spencer: *On distortion in pseudo-conformal mapping.* (Abstract 46-9-402.)

31. Norbert Wiener and Aurel Wintner: *Discrete chaos.* (Abstract 46-11-480.)

32. P. M. Whitman: *Splittings of a lattice.* (Abstract 46-9-442.)

33. L. R. Wilcox: *A topology for semi-modular lattices.* (Abstract 46-9-443.)

34. O. G. Harrold (National Research Fellow): *Characterizations of some continua by continuous functions.* Preliminary report. (Abstract 46-11-458.)

35. Everett Pitcher: *The cap heights of a sum function on a product space.* (Abstract 46-11-470.)

36. S. C. Kleene: *Recursive predicates and quantifiers.* Preliminary report. (Abstract 46-11-464.)

37. Samuel Eilenberg: *On homotopy groups.* (Abstract 46-11-453.)

38. D. W. Hall and W. T. Puckett: *Strongly arcwise connected sets.* (Abstract 46-11-456.)

39. Salomon Bochner and I. J. Schoenberg: *On positive definite functions on compact spaces.* (Abstract 46-11-447.)

40. Gordon Pall: *On the arithmetic of ternary quadratic forms.* (Abstract 46-11-469.)

41. R. E. Langer: *On the theory of irregular differential boundary problems.* Preliminary report. (Abstract 46-11-465.)

42. G. C. Evans: *Surfaces of minimum capacity.* (Abstract 46-9-413.)

43. H. B. Curry: *The inconsistency of certain formal logics.* (Abstract 46-11-451.)

44. Warren Ambrose: *Change of velocities in a continuous ergodic flow.* Preliminary report. (Abstract 46-11-446.)

45. Benjamin Epstein: *A contribution to the theory of two complex variables in certain domains.* (Abstract 46-9-412.)

46. Leon Alaoglu, Garrett Birkhoff, and B. J. Pettis: *General ergodic theorems.* II. Preliminary report. (Abstract 46-9-394.)

47. Fritz John: *Discontinuous convex solutions of difference equations.* (Abstract 46-9-425.)

48. J. H. Curtiss: *Necessary conditions in the theory of interpolation in the complex domain.* (Abstract 46-7-378.)

49. Otto Szász: *On the jump of almost periodic functions and Fourier integrals.* (Abstract 46-9-399.)

50. H. T. Fleddermann: *The equality of measure functions.* (Abstract 46-9-414.)

51. L. C. Hutchinson: *On the classification of the trivector.* (Abstract 46-11-460.)

52. Mark Kac: *On a question of Ramanujan.* (Abstract 46-9-426.)

53. J. S. Frame: *The double cosets of a finite group.* (Abstract 46-9-415.)

54. A. T. Brauer: *On a property of k consecutive integers.* (Abstract 46-9-405.)

55. John Dyer-Bennet: *A note on partitions of the set of positive integers.* (Abstract 46-9-411.)

56. Paul Alexandroff: *General combinatorial topology.* (Abstract 46-7-374-*t*.)

57. Stefan Bergman: *Boundary values of functions satisfying partial differential equations of elliptic type.* (Abstract 46-9-401-*t*.)

58. R. P. Boas: *A general moment problem.* (Abstract 46-9-403-*t*.)

59. Herbert Busemann and Walther Mayer: *On the foundations of calculus of variations.* (Absract 46-9-406-*t*.)

60. Leonard Carlitz: *On the Staudt-Clausen theorem.* (Abstract 46-11-450-*t*.)

61. R. H. Cole: *The expansion problem associated with an ordinary linear differential equation and boundary conditions applying at a set of collinear points.* (Abstract 46-9-407-*t*.)

62. H. V. Craig: *On extensors.* (Abstract 46-9-408-*t*.)

63. H. B. Curry: *A revision of the fundamental rules of combinatory logic.* (Abstract 46-9-409-*t*.)

64. H. B. Curry: *Consistency and completeness of the theory of combinators.* (Abstract 46-9-410-*t*.)

65. M. M. Day: *Ordered sets and closures.* (Abstract 46-9-395-*t*.)

66. Samuel Eilenberg: *Local connectedness in higher dimensions.* (Abstract 46-11-452-*t*.)

67. C. H. Forsyth: *Statistical interpretations of the first auxiliaries of the Gaussian symmetrical method.* (Abstract 46-11-454-*t*.)

68. H. L. Garabedian, H. S. Wall, and Einar Hille: *Hausdorff means and integral equations.* (Abstract 46-9-397-*t*.)

69. L. J. Green: *Twisted cubics associated with a space curve.* II. (Abstract 46-9-419-*t*.)

70. D. W. Hall and W. T. Puckett: *Arc-preserving transformations of a certain class of spaces.* (Abstract 46-9-420-*t*.)

71. O. G. Harrold (National Research Fellow): *A note on strongly irreducible maps of an interval.* (Abstract 46-7-381-*t*.)

72. Loo-Keng Hua: *Some "Anzahl" theorems in the theory of groups of prime-power order.* (Abstract 46-11-492-*t*.)

73. Loo-Keng Hua: *The lattice points in a circle.* (Abstract 46-11-493-*t*.)

74. W. H. Ingram: *A generalization of Erhard Schmidt's solution of the nonhomogeneous integral equation.* Preliminary report. (Abstract 46-11-494-*t*.)

75. Nathan Jacobson: *Restricted Lie algebras of characteristic p.* (Abstract 46-11-461-*t*.)

76. R. L. Jeffery: *Integration in abstract space.* (Abstract 46-11-462-*t*.)

77. F. B. Jones: *Quasi-continuous curves and certain boundary problems.* (Abstract 46-11-463-*t*.)

78. Edward Kasner and J. J. DeCicco: *Equilong and conformal transformations of period two.* (Abstract 46-9-398-*t*.)

79. W. H. McEwen: *On the degree of convergence of a series associated with an integro-differential system.* (Abstract 46-9-427-*t*.)

80. W. G. Madow: *Contributions to the theory of the representative method of sampling.* (Abstract 46-11-466-*t*.)

81. Bella Manel: *Conformal mapping of multiply connected domains on the basis of Plateau's problem.* (Abstract 46-7-384-*t*.)

82. H. E. Newell: *On the asymptotic forms of the solutions of an ordinary linear matric differential equation in the complex domain.* Preliminary report. (Abstract 46-9-428-*t*.)

83. Rufus Oldenburger: *La teoria de los polinomios de orden superior.* (Abstract 46-9-429-*t*.)

84. F. W. Perkins: *Polygonal and polyhedral means of functions.* Preliminary report. (Abstract 46-9-431-*t*.)

85. G. Pólya: *Sur l'existence de fonctions entières satisfaisant à certaines conditions linéaires.* (Abstract 46-9-432-*t*.)

86. W. T. Reid: *A new class of self-adjoint boundary value problems.* (Abstact 46-11-473-*t*.)

87. J. F. Ritt: *On a type of algebraic differential manifold.* (Abstract 46-9-434-*t*.)

88. L. B. Robinson: *Singular solutions of nonlinear functional equations.* (Abstract 46-9-435-*t*.)

89. R. M. Robinson: *Stencils for solving* $x^2 \equiv a \pmod{m}$. (Abstract 46-9-436-*t*.)

90. E. Rosenthall: *On the diophantine equation* $x^3+y^3=z^3+w^3$. (Abstract 46-9-437-*t*.)

91. A. R. Schweitzer: *Concerning general abstract relational spaces.* II. (Abstract 46-9-438-*t*.)

92. D. C. Spencer: *On an inequality of Grunsky.* (Abstract 46-11-476-*t*.)

93. Gabor Szegö: *Power series with multiply monotonic sequences of coefficients.* (Abstract 46-11-477-*t*.)

94. J. L. Walsh and W. E. Sewell: *Degree of polynomial approximation to analytic functions. Problem* β. (Abstract 45-7-369-*t*.)

95. J. H. C. Whitehead: *On incidence matrices,' nuclei and homotopy types.* (Abstract 46-9-441-*t*.)

96. Orla V. Wood: *On relations between solutions of the differential equation of the second order with four regular singular points.* (Abstract 46-9-444-*t*.)

T. R. HOLLCROFT,
Associate Secretary

BOOK REVIEWS

La Théorie des Spineurs. By Élie Cartan. (Actualités Scientifiques et Industrielles, nos. 643 and 701.) Paris, Hermann, 1938. 95 pp. and 91 pp.

The theory of spinors has been treated in the literature from three points of view: the infinitesimal, the algebraic and the geometric. Cartan originated the theory in its infinitesimal aspect in 1913 and he has now written a full length account in which it has been his intention "de développer systematiquement la théorie des spineurs en donnant de ces êtres mathématiques une définition purement géométrique."

Cartan's definition of a spinor (§§93 and 109) is based upon the one-to-two correspondence between the ν-vectors which are coordinates of linear spaces of ν dimensions on the fundamental quadric cone in 2ν (or $2\nu+1$) dimensions and certain vectors with 2^ν components, called "simple" spinors. Despite this geometric introduction to the concept of a spinor, knowledge of the algebraic theory of the representation of linear groups will be of more use to the reader than any acquaintance with the classical ideas of geometry.

The first two chapters of the first volume do not contain the word "spinor." Instead, they include an elementary discussion of the following topics: euclidean space (including the complex and the real definite and indefinite cases), rotations and reflections, multivectors, definition of tensors, tensor algebra, irreducible and reducible tensors, matrices (their algebra and a brief account of unitary, orthogonal and hermitian matrices), and the irreducibility of multivectors. Any reader already acquainted with the material in these forty-seven pages will delight in the concise elegance of the exposition while others will benefit from its lucidity and accuracy.

In the third and fourth chapters the spinor theory is developed in detail for a three dimensional space. The components of a spinor are first introduced as specific functions of the components of an isotropic vector, affected however by an ambiguous sign. After showing that a spinor is a "euclidean tensor," the author introduces in a natural fashion the relation $(x_1)^2+(x_2)^2+(x_3)^2=XX$, where X is a matrix with linear forms in x_1, x_2 and x_3 as elements and the left member is a scalar matrix. This association of vectors and matrices is used to obtain the double-valued representation of the rotation group by linear transformations on spinors. Chapter III concludes with a discussion of reality conditions in the euclidean and pseudo-euclidean cases. In Chapter IV several theorems on the linear representations of the various rotation groups are proved by making use of the infinitesimal generators of the groups.

The first chapter of the second volume extends the spinor theory to spaces of any odd number of dimensions while the following chapter does the same for spaces of an even number of dimensions. In each of these cases the chapter closes with a brief discussion of reality conditions.

The last three chapters (totalling thirty pages) are entitled: Spinors in the Space of Special Relativity; Linear Representations of the Lorentz Group; and Spinors and Dirac Equations in Riemannian Geometry.

Comparison of this exposition of the spinor theory with the work of Veblen and his students[1] reveals a surprisingly small amount of common material. Thus Cartan does not employ Veblen's useful concept of a linear family of geometric transformations, which is a generalization of the well known linear family of involutions on the projective line. The avoidance of any use of the index notation is also interesting since the use of dotted indices has sometimes been regarded (quite erroneously) as a characteristic feature of the theory. These monographs do contain a full and excellent account of the group representation aspect of the theory and are an invaluable contribution to the literature of the subject.

Aside from a few misprints, the only error noted was on page 18 of the first volume, where the first lemma is false in case the fundamental quadratic form in n variables is of signature $\pm(n-2)$. Even in this case a weakened form of the lemma suffices for the following proofs.

WALLACE GIVENS

Gesammelte Werke. By Johannes Kepler. Vol. 1. *Mysterium Cosmographicum. De Stella Nova.* Edited by Max Caspar. 1938. 15+493 pp. Vol. 2. *Astronomiae Pars Optica.* Edited by F. Hammer. 1939. 467 pp. Vol. 3. *Astronomia Nova.* Edited by Max Caspar. 1937. 488 pp. Munich, Beck.

In 1936 Max Caspar published the *Bibliographia Kepleriana* as a part of this series.

The three imposing volumes of Kepler's collected works represent in a way the first conclusive steps in an established mathematically harmonious universe. The desire for simplicity in the theory of the motions of the heavenly bodies activated both Copernicus and Kepler. The former attained some simplification but retained the difficult systems of epicycles by which Greek mathematicians had achieved an explanation of celestial phenomena sufficient for some eighteen hundred years of observation. The new path in theory taken by Copernicus was the result not so much of phenomena not explainable by the old theories but of the desire for a more evident mathematical simplicity.

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[1] In his introduction Cartan refers to Veblen's work as unpublished, although in his bibliography he lists Veblen's paper in the Comptes Rendus Congrès Oslo and the reviewer's dissertation, which was done under Veblen's direction. The most complete publication of the work at Princeton was in the form of mimeographed notes on lectures by Veblen and Givens under the title "Geometry of Complex Domains."

Man's interest in the observed movements of the heavenly bodies is doubtless the most persistent intellectual occupation of record. The ancient Babylonians derived from these observations and records material for mathematical calculations. With the Greeks comes not only the use of the older observations and records but, further, the study of associated mathematical and physical problems, connected with light and theories of vision.

In India also the heavenly bodies secured intelligent attention, with simplified methods of computation, and with algebraic devices and the use of the half-chord and even the shadow function in astronomical problems.

All of these developments contributed to the Moslem interest in astronomy, geography, and trigonometry, and the material was communicated through their works and instruments to Europe in the twelfth to fifteenth centuries. Kepler, more than any other of the European group which includes also Copernicus, Galilei, Tycho Brahe and Newton, appreciated most explicitly the achievements of the past centuries and their manifold contributions towards new theories.

Kepler began his search for a harmonious universe with an inspiration which today would be termed a delusion. However, Kepler kept faith with the observations as recorded and the continuance of this faith led him to observations not reconcilable with his inspiration nor with the theories of Ptolemy or Copernicus. The conclusive observations were furnished by Tycho Brahe, with a few by Kepler himself, and the mathematical universe involved in Kepler's laws of planetary motions gradually evolved as a result of enormous computations, treated with profound mathematical insight, establishing definitely the orbits of the planets as ellipses.

Kepler began his interest in astronomy at the University of Tübingen. The struggle for an existence led him to the casting of horoscopes and to the publication of almanacs, which were largely for such uses. Despite this occupation the scientist Kepler sought a mathematically harmonious universe. It seemed to him to come as an inspired thought "on July 19, 1595," as he records in a letter, that there were six planets and five regular solids, and that, in consequence, the latter could be circumscribed with the sun as center about the orbits of the planets and thus harmoniously, possibly, fill in the gaps between the six planetary orbits. In the *Mysterium Cosmographicum* of 1596 Kepler elaborates this thesis, arriving at only partially successful conclusions.

In 1594 when Kepler began to teach mathematics the Copernican theory was not yet widely accepted. In his study of astronomy under an ardent Copernican, M. Mästlin, at Tübingen, Kepler had acquired an insight into the problems, leading him to desire further information concerning the size and the orbits of the planets. As the varying speeds of the planets were known and their approximate relative distances from the sun, Kepler was able to attain preliminary success in circumscribing around the recorded orbits the five regular solids, "limited by the unchangeable laws of geometry." With these five solids the sphere as a sixth solid was employed for the outermost sphere of the fixed stars.

The haste with which Kepler composed his *Mysterium Cosmographicum* is explained by the author's impending marriage and the necessity for impressing the University officials of Graz where he was employed. Doubtless had Kepler lingered, as he did with later projects, the *Mysterium* might, indeed, have never seen the light. Kepler spent some seven months in Tübingen, on vacation from Graz, completing the manuscript and arranging for the supervision of the printing by his former teacher, Mästlin.

A friendly reception was accorded the *Mysterium Cosmographicum* despite the assumption of the Copernican theory and the peculiarities inherent in the Keplerian hypothesis.

In placing the sun at the center of the universe, Kepler arrived at an appreciation, often reiterated in his later works, of the sun as the center of force of our universe. Copernicus also refers to the sun as the center of power, but he actually used in his epicycles the centers of the orbits as centers; Kepler made his measurements as far as possible directly from the sun. In Chapter XX (p. 70) Kepler refers to the sun as not only the fountain head of light, but the source of life, movement and active force (*anima*); in a later edition Kepler uses the word *"vis."*

Naturally the completed *Mysterium Cosmographicum* of 1595 was sent to the most distinguished astronomer then living, Tycho Brahe, and the later association of Kepler with Brahe hinged upon this connection. The scholarly documentation of an untenable theory established Kepler's fame as an astronomer of great ability. For more correct planetary orbits Tycho Brahe had the data, and it was with an eye to these that Kepler made contact with the great Danish observer of the heavens.

With the *Mysterium Cosmographicum* Kepler included the *Narratio Prima* which had been published in 1540 by G. J. Rheticus,[1] as a preliminary statement concerning the Copernican system. To this was added an appendix by M. Mästlin, *De dimensione orbium et sphaerarum coelestium.*

The modern editors have included in this volume with the *Mysterium* four of Kepler's minor astronomical treatises, published under the title, *De Stella Nova* . . . , 1606; separate title pages are included for the other three treatises. These were publications designed in large part to take advantage, financially, of somewhat wide popular interest in astronomical occurrences.

Late in 1599 Tycho Brahe invited Kepler to visit him in Prague. In February of 1600 Kepler began somewhat tentatively employment at the Observatory in Prague.

When Kepler first approached Tycho Brahe he rather naively supposed that Tycho would be able to hand to him complete data of the orbits of the planets, so that within two or three weeks Kepler himself could check the results with his theory. In fact it took Kepler years to compute the orbit of Mars, the first one to be completed.

Tycho Brahe was having his difficulties in 1599 when he moved to Prague with the mathematical computations in connection with his observations and particularly with the orbit of Mars. Shortly thereafter Kepler was practically without a job, and as Tycho Brahe needed a mathematical astronomer, Kepler found himself in 1600 in Prague as an associate with Tycho Brahe, replacing the astronomer Longomontanus, who returned to Denmark. The observations of Mars and the problems of their reduction were placed in the hands of Kepler.

While Kepler was occupied with these computations, Tycho Brahe died October 24, 1601. Kepler was for a time prevented from free use of the observations of Tycho Brahe. Partly because of these hindrances the complete construction of the orbit of Mars was not effected until some years after the death of Tycho and then even further delayed in publication.

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[1] See Edward Rosen, *Three Copernican Treatises*, New York, Columbia University Press, 1939; the English translation of the Rheticus treatise is given.

In the interval Kepler attacked a number of problems in optics and in observational astronomy which were essential to any correct interpretation of the observations. Kepler's treatise on optics started with the work of Witelo as a basis. Kepler established a modern point of view for the science and even touched some of the problems shortly to be created by the introduction of the telescope.

The *Astronomiae Pars Optica* includes a wide range of discussions on the nature and properties of light, the functioning of the eye, the optical image, reflection and refraction of light rays, and astronomical optics including the lights and shadows of heavenly bodies as well as problems of parallax. Through the work of Witelo written about 1270 A.D. Kepler takes account of the work of the Arabs, notably Al-Hazan (Ibn al-Haitam) and the optics of Euclid and Ptolemy; the whole science was modernized and innumerable mistaken points of view eliminated.

Notably in connection with the observation of eclipses Kepler introduced quite new methods and hurried the book through the press in order to make his methods available to astronomers for an eclipse to occur on October 2 (12) of 1605.

Within six years after this work by Kepler appeared, observational astronomy was revolutionized by the telescope. Kepler was able to make more obvious contributions to the new theory, particularly of lenses, in his *Dioptrice* of 1611. The *Optics* of 1604 revived the problems of refraction and stimulated the definitive solution by Snell and Descartes.

The orbit of Mars had resisted the efforts of Tycho Brahe and Longomontanus. To this problem Kepler gave himself with utmost devotion over a period of five years. About Easter of 1605 Kepler abandoned the eccentric circles and the ovals to make an attempt at the ellipse as the orbit. In view of the attention given in the *Pars Optica* to the focal properties of the conics and their possible applications, one is justified in the conclusion that this study paved the way to Kepler's Laws of Planetary Motion.

While in 1601 Kepler had arrived at the orbit of Mars as in a plane inclined to the ecliptic, not until October 11 of 1605 did Kepler announce the path of Mars as an ellipse; this was done in a letter to the astronomer, Fabricius. This achievement more than any other marks the beginning of the modern theory of the movements of the heavenly bodies.

In a measure the crucial step taken by Kepler was to take the sun itself and not the center of the orbit of a planet about the sun as the center from which computations were made. In the Copernican theory the center of the earth's orbit about the sun is taken as the center of the universe. Kepler's innovation moved the center (focus) to the sun itself and gave a mathematical law for the varying velocity of each planet in its orbit. Vital in the computations was new light upon the earth's orbit. In fact, the second law of Kepler that equal areas are swept out in equal times was first demonstrated by Kepler for the earth; the formulation gave the velocity as inversely proportional to the distance from the sun, the "radius-theorem." The final establishment for the orbit of Mars was dependent upon the corrections made possible by the revisions made in the earth's orbit.

In preparing for publication of Kepler's collected works translations proved to be out of the question; the editors have introduced the device of admirable summaries of contents of each volume. In addition there are notes, often giving an indication of the method of work of the author, and the genesis of the ideas; in fact, Kepler himself gives much information along this line, including even material finally rejected. The editors have also included in each volume numerous notes and for the periods concerned available lists of Kepler's correspondence, largely with brief statements of contents of the letters.

In every way the series, *Johannes Kepler Gesammelte Werke*, can be commended as worthy of the great genius of Kepler.

LOUIS C. KARPINSKI

An Introduction to the Theory of Functions of a Real Variable. By S. Verblunsky. Oxford, Clarendon Press, 1939. 11+169 pp. $4.25.

This text for students and teachers was written for the special purpose of furnishing a more rigorous and accurate treatment of the elements of the theory of functions of a real variable. It is based on notes of lectures delivered by Verblunsky to students in their first year at the University of Manchester. The subject matter is entirely standard, but the treatment involves much that is new and original—ingenious and elegant proofs for certain theorems and new approaches to some parts of the subject. The dominant feature of the book is the presentation of the subject as a body of deductions from specified hypotheses.

The material treated will be sufficiently indicated by the following list of chapter headings: Chapter I, Number; II, Sets and Functions; III, Convergence; IV, Continuity and the Derivative; V, The Elementary Functions; VI, Primitives; VII, Limits and Higher Derivatives; VIII, Integrals; IX, Series. The elementary character of the book should be noted. There is no treatment of the properties of sets of points, of measure, of Lebesgue integration, or of other modern topics.

The treatment is different or unusual in a number of respects; some of them are the following. (a) In Chapter I there is a set of postulates from which all the results in the book are derived. These postulates do not form a mutually independent set; they were chosen in an effort to make the initial progress of the student rapid and easy. (b) In Chapter I also the principle of induction is stated as a postulate; it is then used constantly, and whenever occasion demands, in proofs throughout the book. (c) Both Heine's and Cauchy's definitions of continuity are given, but the treatment is based largely on the former. Although there is a proof that the two definitions are equivalent, attention is not called to the use in it of the axiom of choice. Two definitions of a continuous function $f(x)$ are given, one for a closed interval (Chapter IV, p. 51) and another for an open interval (Chapter VI, p. 83). Finally, the treatment of continuity is separated from the treatment of limits of functions of a continuous variable, the former being placed first in Chapter IV and the latter afterward in Chapter VII. (d) The treatment of the derivative, placed as it is in Chapter IV before the study of limits in Chapter VII, obviously involves new features. The derivative of $f(x)$ at $x=\xi$ is shown to be the value $\phi(\xi)$ of an auxiliary function $\phi(x)$ which is continuous at $x=\xi$. There follow elegant derivations of all the standard properties of the derivative. (e) The treatment of the elementary functions (exponential, logarithmic, trigonometric, hyperbolic) in the short space of fourteen pages in Chapter V is a gem. (f) The introduction of an equivalence relation indicates the care used in the treatment of primitives in Chapter VI. The familiar theorem now reads: a primitive of the sum of two functions is equivalent (not *equal*!) to the sum of primitives of these functions. (g) The integral treated in Chapter VIII is essentially a special case of the Perron integral. It is quite general and integrates with equal ease bounded functions and certain functions which become infinite at a finite number of points. It is shown that for a continuous function it is possible to approximate to the integral by the Riemann sums; in general, however, there is no treatment of integrals as the limits of sums. (h) The book contains a large number of examples. Some illustrate definitions and theorems, and others are additional propositions which are essential for later developments. Most of the examples are accompanied by proofs.

To this reviewer it seems that Verblunsky has succeeded admirably in his effort to write a rigorous and accurate introduction to the theory of functions of a real variable. Furthermore, the literature has been enriched permanently by original treatments of certain topics and the elegant proofs of numerous theorems which he has contributed. At the same time, the book has certain characteristics which this reviewer finds unfortunate in a text for beginners. They are these. (a) The subject has been completely divorced from its intuitive background and also from its historical development. The book contains no figures, and there is no indication of the origin of any of the ideas involved or of their applications. There are no references to the history of the subject. Even the names of mathematicians have been omitted; those of Rolle and MacLaurin are the only ones this reviewer could find. (b) There is an authoritative air of finality about the book. There is no suggestion that other treatments can be given for many of the topics, or that interesting extensions and generalizations are possible in some cases. A book with these characteristics is not likely to offer much *inspiration* to a beginning student. Nevertheless, it is a valuable text for superior students who are reading independently and for classes of an inspiring lecturer.

The printing has been done with an accuracy worthy of the author's success in producing a book of high mathematical quality. The reviewer noticed only three typographical errors, all of which were small and obvious; no one would cause confusion even to the beginning student. The author and printer are to be congratulated on producing an excellent mathematical text.

G. Baley Price

NOTES

The University of Pennsylvania celebrated the two hundredth anniversary of its origin in September 1940. Professor Hermann Weyl of the Institute for Advanced Study gave an address on *The mathematical way of thinking*. Dr. T. C. Fry of the Bell Telephone Laboratories was chairman of the symposium on statistical methods in engineering. Associate Professor S. S. Wilks of Princeton University and Dr. W. A. Shewhart of the Bell Telephone Laboratories participated in this symposium.

The following members of the American Mathematical Society are in residence at the Institute for Advanced Study for all or part of this academic year: Dr. Warren Ambrose, Dr. Valentin Bargmann, Assistant Professor D. G. Bourgin of the University of Illinois, Dr. A. T. Brauer, Professor Orrin Frink of Pennsylvania State College, Dr. Guido Fubini, Dr. Kurt Gödel, Dr. P. R. Halmos, Dr. M. H. Heins, Dr. Dorothy Maharam, Assistant Professor Gordon Pall of McGill University, Dr. Abraham Schwartz, Dr. C. E. Shannon, Dr. Seymour Sherman, Assistant Professor A. H. Taub of the University of Washington, Dr. R. M. Thrall, and Professor W. J. Trjitzinsky of the University of Illinois.

The following persons are also to be at the Institute for Advanced Study during a part or all of the current academic year: Dr. C. L. Critchfield, Professor Shizuo Kakutani of Osaka Imperial University, Professor Wolfgang Pauli of Zürich Polytechnicum, and Professor C. L. Siegel of the University of Göttingen.

Assistant Professor R. G. Sanger of the University of Chicago has been awarded one of three $1000 awards given at the University of Chicago to selected teachers for excellence in undergraduate teaching.

Professor H. H. Dalaker of the University of Minnesota has been given the title emeritus.

Professor Hermann Weyl of the Institute for Advanced Study has been awarded an honorary doctorate of science by the University of Pennsylvania.

Dr. Frances E. Baker of Mount Holyoke College has been promoted to an assistant professorship.

Dr. E. F. Beckenbach of Rice Institute has been appointed to an assistant professorship at the University of Michigan.

Dr. J. H. Curtiss of Cornell University has been promoted to an assistant professorship.

Dr. Churchill Eisenhart of the University of Wisconsin has been promoted to an assistant professorship.

Professor W. B. Ford of the University of Michigan has retired.

Professor E. G. Harrell of the North Dakota State Teachers College, has been appointed to a professorship at the State Teachers College, Platteville, Wisconsin.

Mr. L. B. Hedge of Brown University has been appointed to an assistant professorship at The Citadel.

Dr. C. C. Hurd of Michigan State College has been promoted to an assistant professorship.

Dr. J. F. Kubis of Fordham University has been promoted to an assistant professorship.

Dr. S. B. Littauer of the U. S. Naval Academy at Annapolis has been promoted to an assistant professorship.

Professor C. C. MacDuffee of the University of Wisconsin has been appointed to a professorship at Hunter College.

Dr. W. H. Myers of the University of Utah has been appointed to an assistant professorship at San Jose Junior College, San Jose, California.

Assistant Professor N. N. Royall of The Citadel has been appointed to an associate professorship at Winthrop College, Rock Hill, South Carolina.

Dr. S. S. Smith of the University of Utah has been promoted to an assistant professorship.

Associate Professor R. C. Stephens of Knox College has been promoted to a professorship.

Dr. J. A. Ward of the Tennessee Polytechnic Institute has been appointed to an associate professorhip at Delta State Teachers College, Cleveland, Mississippi.

Dr. L. R. Wilcox of the University of Wisconsin has been appointed to an assistant professorship at the Illinois Institute of Technology.

Associate Professor P. A. De Vore of Central Missouri State Teachers College has been granted a leave of absence of this academic year.

Dr. R. H. Fox of the Institute for Advanced Study has been made an associate at the University of Illinois.

The following appointments to instructorships are announced: Brooklyn College: Mr. Saul Gorn, Dr. D. C. Harkin; College of the City of New York: Dr. G. N. Garrison; Cooper Union: Dr. Alvin Sugar; Duke University: Miss Mary E. Layne; Eastern Washington College of Education, Cheney, Washington: Mr. R. F. Bell; Gogebic Junior College, Ironwood, Michigan: Mr. K. F. McLaughlin; Illinois Institute of Technology: Dr. Herbert Busemann, Dr. J. J. DeCicco; Iowa State College: Dr. A. T. Lonseth; Johns Hopkins University: Dr. R. B. Kershner; University of Michigan: Dr. G. E. Hay, Dr. Wilfred Kaplan; University of Minnesota: Dr. J. M. H. Olmsted; University of Missouri: Dr. M. E. Shanks, Dr. J. V. Wehausen; University of Nevada: Mr. E. M. Beesley; North Carolina State College: Dr. Abe Gelbart; Oklahoma Agricultural and Mechanical College: Dr. P. E. Lewis; Purdue University: Mr. C. D. Olds; Southern Methodist University: Dr. D. W. Starr; Texas Technological College: Dr. P. W. Gilbert; Vanderbilt University: Dr. John Dyer-Bennet; University of Wisconsin: Dr. S. M. Ulam.

Mr. Jonathan Hodgkinson of Jesus College, Oxford University, died June 21, 1940, at the age of fifty-four years.

The death of Professor A. E. H. Love of Oxford University has been reported.

Dr. Frederic Marty lost his life June 14, 1940 while in the service of France.

Dr. E. G. C. Poole of Oxford University died June 28, 1940, at the age of forty-nine years.

Dean F. G. Wren of Tufts College died July 17, 1940, at the age of sixty-six years He had been a member of the Society since 1922.

ABSTRACTS OF PAPERS

SUBMITTED FOR PRESENTATION TO THE SOCIETY

The following papers have been submitted to the Secretary and the Associate Secretaries of the Society for presentation at meetings of the Society. They are numbered serially throughout this volume. Cross references to them in the reports of the meetings will give the number of this volume, the number of this issue, and the serial number of the abstract.

446. Warren Ambrose: *Change of velocities in a continuous ergodic flow.* Preliminary report.

A flow is a 1-parameter group T_t, $-\infty < t < \infty$, of measure preserving transformations of a space into itself. It is measurable (continuous) if T_tP is a measurable (continuous) (P, t)-function If S is any measure preserving transformation of a space Ω into itself, then a flow can be "built on S" as follows: consider the product space of Ω with $0 \leqq x < 1$, with measure defined multiplicatively. Then define the flow by $T_t(P, x) = (S^nP, t+x-n)$, where n is equal to the integral part of $t+x$. It is shown that for any continuous ergodic flow on a separable metric space of finite measure the velocities along the streamlines can be altered to obtain a flow built on a measure preserving transformation, that is, a subspace of the original space can be found whose product with $0 \leqq x < 1$ is, with respect to measure properties, the original space (where measure on that original space is now the measure invariant under the altered flow) and such that the altered flow is built on a measure preserving transformation on this subspace. It is intended to use this theorem in a study of spectral properties which are invariant under a change of velocities. (Received August 5, 1940.)

447. Salomon Bochner and I. J. Schoenberg: *On positive definite functions on compact spaces.*

The expansion theorem for positive definite functions on finite-dimensional euclidean spheres, as stated in *On positive definite functions on spheres* by I. J. Schoenberg (abstract 46-11-474), is contained in a general theorem concerning such functions on compact spaces on which a transitive group of transformations is defined. In the general case, as in the special case, the function is given as a function of two independent points which in addition to being positive definite is invariant under simultaneous transformations of both points by the same group element. (Received August 5, 1940.)

448. D. G. Bourgin and Benjamin Epstein: *A class of kernels generated by a Laplace-Mellin transformation.*

The kernels discussed include those of the type of the generalized zeta function. The authors treat the inversion in L_2 of integrals with these kernels. (Received August 5, 1940.)

449. G. W. Brown: *Reduction of certain composite statistical hypotheses.*

The results obtained make it possible to reduce a large class of "composite" statistical hypotheses to equivalent "simple" hypotheses. The fundamental theorem established states essentially that if two distributions give rise, in sampling, to the same distribution of the set of differences between observations, then one distribution must be a translation of the other, subject to a condition requiring that the characteristic function of one of the distributions be such that any interior intervals of zeros be not too large. The result is established by means of the functional equation $\phi(t_1)\phi(t_2)\phi(-t_1-t_2)=\psi(t_1)\psi(t_2)\psi(-t_1-t_2)$ relating the characteristic functions. Similar results are obtained for scale, and combination of location and scale, and the corresponding situations in multivariate distributions. This type of uniqueness theorem permits one to reduce a composite statistical hypothesis involving an unknown location parameter (or scale, or both) to an equivalent simple hypothesis. (Received August 2, 1940.)

450. Leonard Carlitz: *On the Staudt-Clausen theorem.*

The principal object of this note is to simplify both the proof and formulation of the "polynomial" analogue of the Staudt-Clausen theorem given some time ago (Duke Mathematical Journal, vol. 3 (1937), pp. 503–517). (Received August 6, 1940.)

451. H. B. Curry: *The inconsistency of certain formal logics.*

In 1935 Kleene and Rosser published a proof that certain systems of formal logic were inconsistent in the sense that every formula which could be formulated in them was also demonstrable. This proof has been revised and simplified by the present author, who showed that the inconsistency depends, essentially, on the conjunction of two incompatible properties, called combinatorial and deductive completeness respectively (see abstract 43-3-118). The argument, in both the original proof and the revision, was a refinement of the Richard paradox. In the present paper it is shown that a contradiction can be derived by a much simpler argument based on the paradox of Epimenides. (Received August 6, 1940.)

452. Samuel Eilenberg: *Local connectedness in higher dimensions.*

A metric space Y which is locally p-connected in the homotopy sense for $k\leqq p\leqq n$ (notation: Y is LC_k^n) has the following property (P): Given any metric separable space X and a closed subset X_1 which is LC_0^{k-1} such that $\dim X-X_1\leqq n+1$, every continuous mapping $f(X_1)\subset Y$ has an extension $f(U)<Y$ where U is some open set containing X_1. If Y is locally 0-connected then the property (P) is equivalent with Y being LC_k^n. If the property (P) holds without the restriction $\dim X-X_1\leqq n+1$ then Y is said to be an absolute neighborhood retract starting from the dimension k (notation: Y is ANR_k). If Y is compact then Y is an absolute neighborhood retract if and only if Y is LC_0^{k-1} and ANR_k. A compact space Y which is ANR_1 may not be locally p-connected for any $p\geqq 0$; however, if Y is locally 0-connected then it is LC_0^n for every n. (Received August 9, 1940.)

453. Samuel Eilenberg: *On homotopy groups.*

Let Y be an arcwise connected topological space. A singular simplex in Y is a couple (G^p, f) where G^p is a closed euclidean p-simplex and f is a mapping $f(G^p)\subset Y$. After a proper equivalence relation has been introduced, the totality of all the singular cells in Y was shown by Lefschetz to form a closure-finite complex $S(Y)$. The nth homology group $H^n(S(Y))$ coincides with the homology group $H^n(Y)$ obtained using singular chains and cycles. Given $y_0\varepsilon Y$ and $m\geqq 0$ let $S_m(Y)$ be the subcomplex of $S(Y)$ consisting of all singular simplexes (G^p, f) such that every face of G^p of dimension less than

m is mapped by f into y_0. For every n the groups $H^{n,m}(Y)=H^n(S_m(Y))$ are considered where $0\leq m\leq n$. The groups $H^{n,0}(Y)$, $H^{n,1}(Y)$ and $H^n(Y)$ are isomorphic. If $n>1$ and the coefficients are integers, then $H^{n,n}(Y)$ is the nth homotopy group $\pi_n(Y)$ of Y. (Received August 9, 1940.)

454. C. H. Forsyth: *Statistical interpretations of the first auxiliaries of the Gaussian symmetrical method.*

While the Gaussian symmetrical method of applying least squares is well known and fully treated in mathematical literature, the fact that the first auxiliaries of that method form a veritable treasure house for most of the fundamental formulas of statistical theory (including those of means, dispersions and even of correlation coefficients) has not, apparently, been duly appreciated. The author applies the notation peculiar to statistical theory to discover all these formulas. (Received August 7, 1940.)

455. Hilda P. Geiringer: *A generalization of the law of large numbers.*

Let $V_1(x)$, $V_2(x)$, $\cdots$, $V_n(x)$ be n probability distributions which are not supposed to be independent and let $F(x_1, \cdots, x_n)$ be a "statistical function" of n observations in the sense of von Mises, $V_i(x)$, $i=1, \cdots, n$, indicating as usual the probability of getting a result equal to or less than x at the ith observation. Then it can be proved that under fairly general conditions $F(x_1, \cdots, x_n)$ converges stochastically to its theoretical value; or, in other words, that under these general conditions a great class of statistics $F(x_1, \cdots, x_n)$ are consistent in the sense of R. A. Fisher. Well known particular cases of this theorem result (a) if $F(x_1, \cdots, x_n)$ is taken for the average $(x_1+x_2+\cdots+x_n)/n$ of the n observations, (b) if it is assumed that the $V_i(x)$ are independent distributions. (Received August 5, 1940.)

456. D. W. Hall and W. T. Puckett: *Strongly arcwise connected sets.*

In this paper two characterizations of strongly arcwise connected (see W. T. Puckett, this Bulletin, abstract 46-5-316) sets are obtained. The first characterization states that a locally connected continuum A is strongly arcwise connected if and only if every infinite collection of open sets in A contains an infinite subcollection intersecting some arc of A. Strong arcwise connectivity of a set at a point p is defined and a necessary and sufficient condition that the locally connected continuum M fail to be strongly arcwise connected at a point p is obtained. From this criterion it is deduced that an arbitrary locally connected continuum M is strongly arcwise connected at all save possibly a countable number of its points. If no two points separate M, then M is strongly arcwise connected. (Received September 11, 1940.)

457. D. C. Harkin: *Fourier series as limits of Gaussian sums.*

By means of the cyclotomic algebra which Gauss introduces in the seventh section of the *Disquisitiones Arithmeticae*, Fourier series can be had as limiting cases of sums similar to those there considered. (Received August 15, 1940.)

458. O. G. Harrold (National Research Fellow): *Characterizations of some continua by continuous functions.* Preliminary report.

In this paper a study is made of mappings of a given continuum into the interval (0, 1) which are characteristic. This method has been used by Mazurkiewicz, Whyburn and Eilenberg. New characterizations are given for the continua of finite degree and the continua having no continuum of condensation. For the last mentioned class of continua, a subclass of all continuous mappings is found with respect to which this type of continuum has much the same properties as the class of all continuous mappings with respect to the class of all Peano continua. If a mapping of A into B is called of *almost all finite sections* provided that $f^{-1}(y)$ is finite, $y \varepsilon B$, except for a countable set of points each of which has a countable inverse (A.F.S.), then a necessary and sufficient condition that a Peano continuum admit an A.F.S. map into $(0, 1)$ is that each dendrite in the continuum have this property. This is related to a problem proposed by Cech (Fundamenta Mathematicae, vol. 18). (Received August 6, 1940.)

459. M. H. Heins: *A proof of Picard's theorem by the fundamental theorem on the conformal mapping of simpy-connected Riemann surfaces.*

Let G_w^∞ denote the universal covering surface of G_w, the extended w-plane with three distinct points a, b, c deleted. G_w^∞ is simply-connected, and, as the universal covering surface of the Riemann surface G_w, G_w^∞ may be taken to be a Riemann surface. Hence G_w^∞ can be mapped one-to-one and conformally onto one and only one of the following canonical domains: (1) the extended z-plane, (2) the finite z-plane, (3) the interior of the unit circle in the z-plane. The first possibility is readily excluded. As for the second, let $w=w(z)$ denote a mapping function which allegedly maps the finite z-plane one-to-one and conformally onto G_w^∞. $w=w(z)$ would be automorphic under a group of linear transformations mapping the finite z-plane onto itself. From this one infers readily that $w(z)$ must be doubly-periodic, simply-periodic, or a constant. In each case a contradiction is manifest. Hence G_w^∞ must be of hyperbolic type. Picard's theorem that a function, not a constant, meromorphic in the finite plane, can omit at most two values follows immediately. (Received August 6, 1940.)

460. L. C. Hutchinson: *On the classification of the trivector.*

In this paper the author finds by a new method the known canonical forms for trivectors, or alternating tensors of the third valency, in eight dimensions, as well as new forms which do not exist as canonical forms except in the real domain, and examines some of their geometrical properties. This paper is based on the author's work for the doctorate at the Massachusetts Institute of Technology. (Received August 6, 1940.)

461. Nathan Jacobson: *Restricted Lie algebras of characteristic p.*

In this paper certain identities previously noted connecting pth powers, $a+b$, $a\alpha$, α in the field and $[a, b]=ab-ba$ in an associative algebra of characteristic $p\neq 0$, are proved to be characteristic of these operations. This is shown by defining the concept of an abstract restricted Lie algebra and proving that any such algebra L may be obtained as a subset of an associative algebra closed under the above operations. Among the minimal associative algebras determined by L in this way, there is one which is universal in a certain sense. The derivation algebra of any algebra of characteristic p is an instance of a restricted Lie algebra. A notion of a restricted derivation of a restricted Lie algebra is defined. These form a restricted Lie algebra and may be used to define a restricted holomorph. If L has a finite basis it has a (1-1) representation by finite matrices. An analogue of Engel's theorem holds in this case. (Received August 9, 1940.)

462. R. L. Jeffrey: *Integration in abstract space.*

A theory of integration in a complete normed vector space is given which is shown to be equivalent to that given by G. Birkhoff (Transactions of this Society, vol. 38 (1935), pp. 357–378). This theory is in the spirit of the classical theories for real and complex variables. In particular it makes no use of convex sets, and, as a consequence of this, the developments are as simple as those of the classical theories. (Received August 3, 1940.)

463. F. B. Jones: *Quasi-continuous curves and certain boundary problems.*

A continuum M is said to be *quasi-connected at* P if P belongs to M and for each point X of $M-P$ there exists a continuum which lies in $M-X$ and contains an open subset of M containing P. A continuum which is quasi-connected at each of its points is said to be a *quasi-continuous curve.* The class of quasi-continuous curves includes as proper subclasses both the class of continuous curves and the class of semi-locally connected continua, whereas neither of the latter classes includes the other. The first part of the paper deals largely with the similarity between quasi-continuous curves and continuous curves in a Hausdorff space. The second part of the paper deals with boundary problems in a space whose only two-dimensional feature is the Jordan curve theorem. If P is a point of a continuum K in such a space so that (1) K is locally compact at P and (2) K is quasiconnected at P, then K together with all but a finite number of its complementary domains is connected im kleinen at P. As an interesting application of this theorem, it is shown that a proposition analogous to Whyburn's generalization of the Torhorst theorem for the plane holds true in these abstract spaces. (Received August 10, 1940.)

464. S. C. Kleene: *Recursive predicates and quantifiers.* Preliminary report.

Consider the following forms of expression: $R(n)$, $(Ex)R(n, x)$, $(x)R(n, x)$, $(x)(Ey)R(n, x, y)$, $(Ex)(y)R(n, x, y)$, $\cdots$. The variables $n, x, y, \cdots$ range over non-negative integers. The quantifiers $(E.)$ and $(.)$ express "there exists an . such that" and "for .," respectively. Choose one of the forms, and let R be a predicate (that is, propositional function) of the variables used in that form. Then the expression represents a predicate P of the variable n. As defined by Herbrand and Gödel, a general recursive predicate has a certain kind of algorithm, which, for given values of the variables, leads to a decision as to the truth or falsity in a finite number of steps. According to Church, this kind of algorithm is the most general possible. To each of the above forms, a general recursive R can be chosen, such that the resulting P is not given by any of the other forms with the same or fewer quantifiers, for any general recursive R whatsoever. This includes, for the first of the above forms, Church's result on an unsolvable problem of elementary number theory, and for the second, a part of Gödel's results on the incompleteness of formal deductive systems. The R chosen for each form is primitive recursive. (Received August 7, 1940.)

465. R. E. Langer: *On the theory of irregular differential boundary problems.* Preliminary report.

A matrix differential boundary problem $Y'=(\lambda R+B)Y$, $W_aY(a)+W_bY(b)=0$, is familiarly classifiable as regular, mildly irregular, or highly irregular, according as the characteristic equation possesses or does not possess certain structural peculiarities. In any event, if an infinity of characteristic values exists, an "arbitrary" function may be formally associated with an "expansion" in characteristic solutions of the problem. Under appropriate conditions this expansion is known to be convergent or summable to the given function if the problem is respectively regular or mildly irregular. There is, however, little in the way of general theory for the highly irregular cases. This paper seeks to obtain such theory by the study of a boundary problem which depends upon an additional parameter ν, in such a way that it reduces to the given (irregular) problem for some specific value of ν, while it remains regular for adjacent values. It is shown that in a specifiable sense the expansions associated with irregular cases of certain categories may be regarded as summable to the given function. (Received August 2, 1940.)

466. W. G. Madow: *Contributions to the theory of the representative method of sampling.*

The theory of representative sampling may be regarded as a dual process, the first consisting in sampling different random variables and the second in repeating several times the experiments associated with each of the different random variables. It follows that while the theory of sampling from finite populations without replacement may be required for the first process, the second leads directly into the theory of sampling from infinite populations. There is, however, one difference. The usual theory is concerned with the evaluation of fiducial or confidence limits for, say, the mean of a sample of N when n, $(N \geqq n)$, of the values are known. It is thus possible to use the usual theories of estimation in obtaining estimates of the parameters and to allow the effects of the subsampling process to show themselves in the different values of the fiducial limits. It is shown that the limits obtained are almost identical with those obtained by the theory of sampling from a finite population. Distributions of the statistics used in these limits are derived. Besides these results, the theory is extended to that of sampling vectors and the conditions are stated under which the "best" allocation of the number in a sample among several strata is proportional to the kth roots of the generalized variance of a random vector having k components. (Received August 2, 1940.)

467. Jerzy Neyman: *Conception of equivalence in the limit of statistical tests and its application to certain forms of χ^2 and to the tests based on the λ principle.*

Denote by T_1 and T_2 two different tests of the same hypothesis H and by N the number of observations to be used to test H. Definition: if the probability of T_1 and T_2 contradicting themselves tends to zero as N is increased, then T_1 and T_2 are called equivalent in the limit. Consider an experiment which will produce one of the results $E_1, E_2, \cdots, E_s$, the probability of E_1 being p_i. This experiment is to be repeated n times and n_i will denote the number of occurrences of E_i. Denote by H the hypothesis that $p_i = f_i(\theta_1, \theta_2, \cdots, \theta_h)$, where f_i is a function of h parameters θ_j, with continuous derivatives of second order such that $\partial(f_1, f_2, \cdots, f_h)/\partial(\theta_1, \theta_2, \cdots, \theta_h) \neq 0$. Suppose that H is tested against a set of alternatives ascribing to the p's any non-negative values, such that $\sum_i p_i = 1$. Under those conditions each of the following χ^2 tests is equivalent in the limit to the test based on the λ principle: (1) test T_1 consists in the rule of rejecting H whenever $\sum_i (n_i - np_i')^2/np_i' > \chi^2_\alpha$, where p_i' denotes the maximum likelihood estimate of p_i; (2) test T_2 consists in rejecting H whenever

$\sum_i(n_i-n\bar{p}_i)^2/n\bar{p}_i>\chi^2_\alpha$, where $\bar{p}_i$ denotes the value of p_i minimizing $\sum_i(n_i-np_i)^2/np_i$; (3) test T_3 consists in rejecting H whenever $\sum_i(n_i-np_i'')^2/n_i>\chi^2_\alpha$, where p_i'' denotes the value of p_i minimizing $\sum_i(n_i-np_i)^2/n_i$. (Received September 11, 1940.)

468. Isaac Opatowski: *On the motion of an electric particle.*

The motion of an electrically charged point mass under the action of a magnetic force $\mathfrak{H}=-\text{grad } U(P)$ and an electric force $\mathfrak{E}=-\text{grad } V(P)$ is studied in the case in which U and V are harmonic functions satisfying the conditions: (1) grad $W(P)=\mathfrak{u}\times\mathfrak{H}$, $C=\mathfrak{u}\cdot\mathfrak{E}$ where W is a certain function (generalization of the ordinary stream function), C is a scalar constant, $\mathfrak{u}=\mathfrak{a}+\mathfrak{b}\times\mathfrak{r}$, where $\mathfrak{a}$, $\mathfrak{b}$ are two constant vectors and $\mathfrak{r}$ is the radius vector. The most important fields of mathematical physics, that is, general plane fields, all fields symmetric about an axis of revolution and many others are included in (1). The integral of momentum is explicitly given in terms of W and C. Regions of space are determined within which, for a given energy and a given constant of momentum integral, no motion is possible (generalization of Stoermer's "forbidden regions"). Equations of certain trajectories are explicitly given. The paper deals with both constant and relativistic mass. (Received August 2, 1940.)

469. Gordon Pall: *On the arithmetic of ternary quadratic forms.*

The fundamentals of the arithmetic of ternary quadratics is presented with modifications of the usual invariants, leading to very considerable simplifications. Among the topics treated are: order invariants, form-residues, generic characters, five equivalent definitions of genus, the number of existing genera, simultaneous representation, simultaneous characters, the numbers represented by a genus, zero and universal forms, rational equivalence, reduction formulas for the representations of pn or p^2n, and so on. (Received August 5, 1940.)

470. Everett Pitcher: *The cap heights of a sum function on a product space.*

This paper contains a solution under some restrictions of a problem in critical point theory arising in the work of M. Morse and C. Tompkins (see Annals of Mathematics, (2), vol. 38 (1937), pp. 386–449 and vol. 40 (1939), pp. 443–472, and forthcoming papers) on the problem of Plateau. M and N denote metric spaces and $f(x)$ and $g(y)$ positive functions defined upon them. It is assumed that (i) the sets $f\leqq c<\infty$ are compact, and (ii) the cap heights of f form an increasing sequence, finite or with $+\infty$ as limit. It is assumed that g satisfies (i) and (ii) also. On $M\times N$, $h[x\times y]=f(x)+g(y)$ and satisfies (i) and (ii). Vietoris topology with coefficients from a field is used. An arbitrary maximal group of k-caps of h at height e is shown to be isomorphic to a direct sum of homology groups $H^k(c, d)$. The summation extends over all pairs c, d of cap heights of f and g respectively with sum e. The group $H^k(c, d)$ is the homology group of classes of k-cycles on the relative product space $[f\leqq c]\times[g\leqq d]d-\text{mod } [f\leqq c]\times[g<d]\cup[f<c]\times[g\leqq d]$. Some consideration is given to homology groups of a product space. (Received August 6, 1940.)

471. Hillel Poritsky and H. D. Snively: *Doubly periodic networks.*

A solution is obtained for the doubly infinite number of linear equations of a doubly periodic network. Expressions are obtained for the self and mutual impedances as double integrals which are then reduced to elliptic integrals (of all three kinds).

These impedances are also evaluated by direct numerical methods of successive approximation. (Received August 21, 1940.)

472. J. F. Randolph: *Some properties of sets of the Cantor type.*

Given $a_1 \geqq a_2 \geqq \cdots > 0$ where $\sum_n a_n = 1$, let S consist of the numbers x for which there exists a sequence $\{\epsilon_n\}$, where each ϵ_n is 0 or 1, such that $x = \sum_n \epsilon_n a_n$. (If $a_n = 2/3^n$ then S is the Cantor set.) Theorem: $a_n \geqq 2\sum_{i=n+1}^{\infty} a_i$ is a necessary and sufficient condition that corresponding to $0 \leqq z \leqq 2$ there exists an $x \varepsilon S$ and $y \varepsilon S$ such that $x+y=z$. Also some results are given on the Carathéodory and Gillespie linear measures of cartesian products of sets of the Cantor type. (Received August 6, 1940.)

473. W. T. Reid: *A new class of self-adjoint boundary value problems.*

This paper treats a differential system consisting of the vector differential equation $\mathcal{L}[y] \equiv y' - A(x)y - \lambda B(x)y$ and two-point boundary conditions $s[y] \equiv My(a) + Ny(b) = 0$ which is self-adjoint under the nonsingular transformation $z = T(x)y$ and satisfies the further conditions: (1) the matrix $S(x) = \tilde{T}(x)B(x)$ is symmetric on ab; (2) $H[y] = \int_a^b y\tilde{T}\mathcal{L}[y]dx > 0$ for all vectors y satisfying $s[y] = 0$, $B(x)y(x) \not\equiv 0$, and for which there exists a corresponding vector $g(x)$ such that $\mathcal{L}[y] = Bg$ on ab; (3) $y \equiv 0$ is the only vector satisfying on ab the conditions $\mathcal{L}[y] = 0$, $By = 0$, $s[y] = 0$. The chief distinction between these conditions and those defining a definitely self-adjoint boundary problem (Bliss, Transactions of this Society, vol. 44 (1938), pp. 413–428) is that the above hypothesis (2) on $H[y]$ replaces a corresponding definiteness assumption on the matrix $S(x)$. Such systems are shown to possess fundamental properties similar to those previously established for definitely self-adjoint problems. In particular, certain important types of boundary problems associated with the calculus of variations which are not definitely self-adjoint do belong to this new class of problems. (Received August 6, 1940.)

474. I. J. Schoenberg: *On positive definite functions on spheres.*

The positive definite functions on m-dimensional spheres and on spheres in Hilbert space are explicitly determined. In ordinary terminology the results are as follows: I. Let $f(x)$ be real and continuous in the interval $-1 \leqq x \leqq 1$. Let $A = \|a_{ik}\|$ denote an arbitrary n-rowed symmetric matrix of a positive quadratic form whose rank does not exceed the fixed number $m+1$ $(m \geqq 1)$. Moreover $a_{11} = a_{22} = \cdots = a_{nn} = 1$. The quadratic form corresponding to the transformed matrix $f(A) = \|f(a_{ik})\|$ is also always positive if and only if $f(x)$ admits throughout $[-1, 1]$ an expansion in ultraspherical polynomials $f(x) = \sum_{\nu=0}^{\infty} p_\nu P_\nu^{((m-1)/2)}(x)$ with nonnegative coefficients $p_\nu \geqq 0$. II. The assumptions on $f(x)$ and the matrix A are as above except that there is now no restriction whatever on the rank of A. The quadratic form of $f(A) = \|f(a_{ik})\|$ is also always positive if and only if $f(x) = p_0 + p_1 x + p_2 x^2 + \cdots$, $(-1 \leqq x \leqq 1;\ p_\nu \geqq 0)$. (See Pólya and Szegö, vol. 2, p. 107, where the sufficiency of the power series expansion for $f(x)$ is stated as Problem 37.) The second theorem is derived from the first by letting $m \to \infty$. This passage to the limit is carried through by means of the following new estimate: $|P_n^{(\lambda)}(\cos\theta)| \leqq P_n^{(\lambda)}(1)\{((1+\cos^2\theta)/2)^{n/2} + (4/n)\cdot(1+\cos^2\theta)/\sin\ (2\theta)\}$ if $0 < \theta < \pi/2$, $\lambda \geqq 5$, and $n = 1, 2, 3, \cdots$. (Received August 5, 1940.)

475. I. M. Sheffer: *On the singularities of functions under certain linear transformations.*

The linear transformations under consideration are all of the form $L[y(x)]$

$\equiv \sum_0^\infty L_n(x)y^{(n)}(x)$. When $L[y(x)]=F(x)$, the relation between the singularities of $F(x)$ and those of $y(x)$ is examined. The theorems obtained are analogous to the well known theorems of Hadamard and Hurwitz on the singularities of power series compounded from given power series; indeed, these latter theorems are but particular instances of the results of the present work. (Received August 6, 1940.)

476. D. C. Spencer: *On an inequality of Grunsky.*

Let W be a Riemann domain which, for simplicity, is assumed to be the map of $|z|<1$ by a function $f(z)$ regular in $|z|\leqq 1$, and let $n(w)$ be the number of times the point w is covered by W. Suppose that $n(w)\leqq n(0)$. Then Grunsky (Dissertation, Schriften des mathematischen Seminars der Universität Berlin, vol. 1 (1932), pp. 113–118) proves the following inequality (a case of the theorem of the geometric and arithmetic means in which the weight function is not necessarily of constant sign): $\exp \{(1/2\pi n(0))\int_B \lg R^2 d\Phi\} \leqq (1/2\pi n(0))\int_B R^2 d\Phi$, where B is the boundary of W. In this note a new and simple proof of the inequality is given for more general domains W which are only mean $n(0)$-valent (that is, under conditions less restrictive than $n(w)\leqq n(0)$). The inequality is applied to obtain a more precise version of a theorem of Bermant (Comptes Rendus de l'Académie des Sciences, Paris, vol. 207 (1938), pp. 31–33). (Received August 6, 1940.)

477. Gabor Szegö: *Power series with multiply monotonic sequences of coefficients.*

A sequence $\{a_0, a_1, \cdots\}$ is called monotonic of order k if the differences $\Delta^{(v)}a_n$ defined by $\Delta^{(0)}a_n=a_n$, $\Delta^{(1)}a_n=a_n-a_{n+1}$, $\Delta^{(v)}=\Delta^{(1)}\Delta^{(v-1)}$ are non-negative for $0\leqq v\leqq k$, $n\geqq 0$. Let $a_n\neq 0$. Fejér proved (Transactions of this Society, vol. 39 (1936), p. 57) that $\sum a_n z^n$ is regular and univalent for $|z|<1$ provided $\{a_n\}$ is monotonic of order $k=4$. He also showed that this is not true for $k=1$. In the present paper it is proved that the assertion in question remains true for $k=3$, but not for $k=2$. The latter fact has been established by a different method by Mr. S. Szidon (Acta Szeged, 1940). (Received September 11, 1940.)

478. W. J. Trjitzinsky: *Properties of growth for solutions of differential equations of dynamical type.*

In the background of this work are, in part, certain memoirs of A. Liapounoff, P. Bohl, E. Cotton, and O. Perron. The developments for linear systems are based on characteristic numbers and product integration. The nonlinear systems are investigated, in part, with the aid of the theory of linear systems. In the study of the nonlinear problem, use is made of characteristic numbers and of successive approximations. The latter method is associated with some of the present author's earlier work in the field of differential equations. (Received August 5, 1940.)

479. Abraham Wald: *Asymptotically shortest confidence intervals.*

Let $f(x, \theta)$ be the probability density function of a variate x involving an unknown parameter θ. Let $x_1, x_2, \cdots, x_n$ be n independent observations on x and let $C_n(\theta)$ be a positive function of θ such that the probability that $F(x, \theta)=|n^{-1/2}\partial\sum_{\alpha=1}^n \log f(x_\alpha, \theta)/\partial\theta|\leqq C_n(\theta)$ is a constant β when θ is the true value of the parameter. Denote by $\underline{\theta}(x_1, \cdots, x_n)$ the root in θ of the equation $F(x, \theta)=C_n(\theta)$ and by $\overline{\theta}(x_1, \cdots, x_n)$ the root of $F(x, \theta)=-C_n(\theta)$. Under some weak assumptions on $f(x, \theta)$ the limiting interval $\delta_n(x_1, \cdots, x_n) = [\underline{\theta}(x_1, \cdots, x_n), \overline{\theta}(x_1, \cdots, x_n)]$ for $n \to \infty$ is a shortest unbiased confidence interval (for the definition of a shortest unbiased confidence interval, see J. Neyman, Philosophical Transactions of the Royal Society of London, vol. 236 (1937), pp. 333–380) of θ corresponding to the confidence coefficient β. This confidence interval is identical with that given by S. S. Wilks (Annals of Mathematical Statistics, vol. 9 (1938)). Wilks has shown that $\delta_n(x_1, \cdots, x_n)$ is asymptotically shortest in the average compared with all confidence intervals computed on the basis of statistics belonging to a certain class C. In the present paper it has been proved that the confidence interval in question is asymptotically shortest compared with any arbitrary unbiased confidence interval, without any restriction to a certain class of functions. (Received August 5, 1940.)

480. Norbert Wiener and Aurel Wintner: *Discrete chaos.*

The ordinary method of introducing the Lebesque integration needed in statistical mechanics in terms of a probability density referred to a product space works in systems of a finite number of degrees of freedom, but breaks down in the case of an infinity of degrees of freedom. An alternative method is developed in this latter case, depending on an infinity of probability densities of finite sets of particles. (Received September 11, 1940.)

481. S. S. Wilks: *On the problem of two samples from normal populations with unequal variances.*

Suppose O_{n_1} and O_{n_2} are samples of n_1 and n_2 elements from normal populations Π_1 and Π_2 respectively. Let a_1, σ_1^2 and a_2, σ_2^2 be the means and variances of Π_1 and Π_2 and let O_{n_1} and O_{n_2} have means $\bar{x}_1$ and $\bar{x}_2$ and variances s_1^2 and s_2^2 (unbiased estimates of σ_1^2, σ_2^2) respectively. It is shown that there exists no function (Borel measurable) of $\bar{x}_1$, $\bar{x}_2$, s_1^2, s_2^2, $a_1 - a_2$ independent of σ_1^2 and σ_2^2, having its probability law independent of the four population parameters. It is therefore impossible to obtain exact confidence limits for $a_1 - a_2$ corresponding to a given confidence coefficient. Functions of the four parameters and four statistics are devised from which one can set up confidence limits for $a_1 - a_2$ with associated confidence coefficient inequalities. (Received August 2, 1940.)

482. W. L. G. Williams: *The application of hyperbolic complex numbers to the geometry of the triangle.*

A complex number $a + bj$, in which a and b are real numbers and $j^2 = 1$, is called a hyperbolic complex number. If such a number is designated by z and is represented as a two-dimensional vector, the vector zj is said to be pseudo-perpendicular to z. When the concept pseudo-perpendicular takes the place of perpendicular, new points, related to a triangle, are found, analogous to orthocentre, circumcentre and nine-point centre. These points lie on a straight line with the centroid of the triangle. There exist rectangular hyperbolas analogous to the circumcircle and nine-point circle. There is an analogue of Feuerbach's theorem. (Received August 6, 1940.)

483. Antoni Zygmund: *Power series of the class H^λ on the circle of convergence.*

If the function $f(z) = \sum c_n z^n$ belongs to the class H^λ, where $0 < \lambda < 1$, then the series $\sum c_n e^{in\theta}$ is summable $(C, 1/\lambda - 1)$ for almost every θ. (Received August 19, 1940.)

484. A. T. Brauer: *On a problem of partitions.*

Let $a_1, a_2, \cdots, a_k$ be relatively prime positive integers. The purpose of this paper is to determine bounds $F(a_1, a_2, \cdots, a_k)$ such that the Diophantine equation $a_1x_1+a_2x_2+\cdots+a_kx_k=n$ always has solutions in positive integers $x_1, x_2, \cdots, x_k$ for $n>F(a_1, a_2, \cdots, a_k)$. (Received September 27, 1940.)

485. A. B. Brown: *On the number of independent parameters.*

Let $f(x_1, \cdots, x_n, \alpha_1, \cdots, \alpha_m)$ be a function of class $C^{(q)}$ (q specified in the paper). To form matrix (M), take $\partial f/\partial\alpha_1, \cdots, \partial f/\partial\alpha_m$ as first row. As further trial rows, take the derivatives to $\alpha_1, \alpha_2, \cdots, \alpha_m$ of derivatives of f to the x's, in the following order: $\partial f/\partial x_n$, $\partial f/\partial x_{n-1}, \cdots, \partial f/\partial x_1$, $\partial f/\partial x_n^2, \cdots$. If the set of derivatives to $\alpha_1, \cdots, \alpha_m$ increases the rank of the matrix, the derivative, say $\partial z/\partial x_n$, is *accepted*, otherwise *rejected*. The algorithm continues until the accepted derivatives are the only derivatives of z which are not rejected derivatives or derivatives of rejected derivatives. If k is the number of rows of the resulting matrix, then *all functions f are given by varying k of the parameters, keeping the other $n-k$ constant; furthermore, no function is given twice in this way.* Certain "singular points," forming a nowhere dense closed set, must be avoided. It is proved that not all functions can be obtained by varying less than k parameters. Further, given $g(x_1, \cdots, x_n, \beta_1, \cdots, \beta_l)$ where $f(x, \alpha)$ and $g(x, \beta)$ are the same set of functions of (x), but nothing is assumed as to relationship between (α) and (β), the same number k of parameters is needed for $g(x, \beta)$. (Received September 25, 1940.)

486. M. M. Day: *Some more uniformly convex spaces.*

For a sequence of Banach spaces $[B_i]$, $i=1, 2, \cdots$, define $B=\prod^p\{B_i\}$ to be the space of sequences $b=\{b_i\}$ with $b_i\in B_i$ and $\|b\|=(\sum\|b_i\|^p)^{1/p}<\infty$. Complementing an earlier result of this writer (abstract 46-9-396) it is shown that $\prod^p\{B_i\}$ is uniformly convex if $B_i=l^{p_i}$ or L^{p_i} and if $1<m\leqq M<\infty$ exist with $m\leqq p_i\leqq M$ for all i. A conjecture of Boas (this Bulletin, vol. 46 (1940), p. 304) that the spaces $\prod^p\{B_i\}$ with $B_i=B_0$ for all i, as well as the spaces $L^p(B_0)$ (consisting of all Bochner integrable functions f on, say, $(0, 1)$ with values in B_0 and $\|f\|=(\int_0^1|f(t)|^p dt)^{1/p}<\infty$), are uniformly convex if B_0 is shown to be true. A sequence of spaces $\{B_i\}$ is said to have a common modulus of convexity if for each ϵ, $0<\epsilon\leqq 2$, there is a $\delta(\epsilon)>0$ such that for each i and each pair of points $b_i, b_i'\in B_i$ with $\|b_i\|=\|b_i'\|=1$ and $\|b_i-b_i'\|>\epsilon$, the relation $\|b_i+b_i'\|<2(-\delta(\epsilon))$ holds. The results mentioned above are corollaries of the following theorem: $\prod^p\{B_i\}$ is uniformly convex if and only if the B_i have a common modulus of convexity. (Received September 21, 1940.)

487. Aaron Fialkow: *The conformal theory of a hypersurface.*

For a hypersurface V_{n-1} in any Riemann space V_n $(n>2)$, the author shows that it is possible to define three quadratic differential forms which remain unchanged under any conformal mapping of V_n, not necessarily on itself. These forms are the *conformal fundamental forms* of the hypersurface V_{n-1}. If $n>3$, the coefficients of the third conformal fundamental form may be expressed in terms of the coefficients of the first and second conformal fundamental forms. If V_{n-1} and $\bar{V}_{n-1}$ are hypersurfaces of V_n and $\bar{V}_n$ respectively and $V_n\leftrightarrow\bar{V}_n$, $V_{n-1}\leftrightarrow\bar{V}_{n-1}$ by a conformal map, then the conformal fundamental forms of V_{n-1} and $\bar{V}_{n-1}$ are equal. Conversely, if V_n and $\bar{V}_n$ are conformally euclidean spaces and the conformal fundamental forms of V_{n-1} and $\bar{V}_{n-1}$ are equal, then a conformal transformation exists so that $V_n\leftrightarrow\bar{V}_n$, V_{n-1}

$\leftrightarrow \bar{V}_{i-1}$. In any V_n, a hypersurface exists whose conformal fundamental forms are any preassigned quadratic differential forms whose coefficients satisfy certain partial differential equations analogous to the classical Gauss-Codazzi equations. The Weyl conformal curvature tensor of V_n plays a role analogous to that of the Riemann curvature tensor in classical differential geometry. The case $n=3$ occupies a special position in this theory. (Received September 23, 1940.)

488. G. E. Forsythe: *On Riesz summability methods of order r, for $\Re(r)<0$.* Preliminary report.

Let A_r and B_r represent M. Riesz's (discrete *resp.* continuous) summability methods for sequences, as defined by Agnew (Transactions of this Society, vol. 35 (1933), pp. 532–548). It is known that A_r and B_r and Cesàro C_r are equivalent for $-1<r\leqq 1$, while A_r and B_r are not equivalent for the real part of r, $\Re(r)<-1$, nor for certain values of $r>1$. By aid of Theorem 5.1 (op. cit.) the present paper obtains a criterion for the equivalence of A_r and B_r for $\Re(r)<0$ in terms of the reciprocal of $\phi_r(x)=\sum_1^\infty n^r x^n$, and applies it to prove that A_{-1+ik} and B_{-1+ik} $(-\infty<h<\infty)$ are equivalent if and only if $h=0$. (Received September 25, 1940.)

489. Saul Gorn: *Homomorphisms and modular functionals.*

Using CA to represent the complements of elements of A, an ideal in a complemented modular L is called a C-ideal if $C\mathfrak{A}$ is an ideal; $C\mathfrak{A}$ is a π-ideal if $\mathfrak{A}$ is a σ-ideal and dually. If $\mathfrak{A}=CC\mathfrak{A}$, $\mathfrak{A}$ is called neutral (identical with Garrett Birkhoff's definition). An $\mathfrak{A}$ fulfilling both conditions is called C-neutral. The general homomorphism theorem for complemented modular lattices is given by the C-neutral ideals. Letting $\mathfrak{A}'$ be the elements orthogonal to all elements of $\mathfrak{A}$, a C-neutral ideal is normal if and only if it is normal in Stone's sense: $\mathfrak{A}''=\mathfrak{A}$. The homomorphism $L\rightarrow L/\mathfrak{A}$ takes normal ideals containing or equal to $\mathfrak{A}$ into normal ideals; conversely if $\mathfrak{A}$ is normal. L is called ideally irreducible if it contains no non-trivial C-neutral normal ideal. $L/\mathfrak{A}$ is ideally irreducible if and only if $\mathfrak{A}$ is a maximal (or prime for π-ideals) C-neutral normal ideal. Using the Wilcox-Smiley continuity conditions, a quasi-norm is uniquely determined by its generating normal C-neutral $\mathfrak{A}$ if and only if $L/\mathfrak{A}$ is ideally irreducible (that is, $\mathfrak{A}$ maximal if non-trivial). This generalizes results mentioned in abstract 45-1-16. (Received September 12, 1940.)

490. N. A. Hall: *The solution of the quadrinomial equation in infinite series by the method of iteration.*

The roots of the quadrinomial equation $z^{m+n+k}-z^{m+n}+bz^m+a=0$ may be expressed as double power series in the two complex variables a and b. These solutions have been previously developed by use of the Lagrange expansion and also by use of differential resolvents. In this paper, these results are reproduced and extended by the direct method of solution of the equation by iteration as previously applied by the author to the trinomial equation (this Bulletin, vol. 44 (1938), p. 337). The solutions are exhibited as generalized hypergeometric series in the two variables a and b. These series give the set of $m+n+k$ roots covering the quadrant of absolute values of a and b except for the small region covered by the branch points of the roots. The relation between these singularities and the domains of convergence for the double series is shown. (Received September 27, 1940.)

491. Einar Hille: *A class of differential operators of infinite order.*

The author studies the differential operator $G(\delta_z)$ where $G(w)$ is an entire function and $\delta_z = z^2 - d^2/dz^2$ is the operator of the Hermite-Weber equation. Necessary and sufficient conditions are found in order that $G(\delta_z)$ shall apply to various classes of analytic functions, in particular entire functions. The conditions differ in some respects fundamentally from corresponding conditions for the operator $G(d/dz)$. Relations are found between the orders and types of $G(w)$, $f(z)$, and $G(\delta_z) \cdot f(z)$. The equation $G(\delta_z) \cdot W = F(z)$ is studied with the aid of summable Fourier-Hermite series. If $F(z) \sim \sum F_n h_n(z)$, then formally $W(z) \sim \sum W_n h_n(z)$, where $G(2n+1) W_n = F_n$. Sufficient conditions are found under which this formal procedure leads to an actual solution and a uniqueness theorem is proved. Some examples are discussed which show that if the operator δ_z is replaced by other second order differential operators D_z then the applicability of $G(D_z)$ to entire functions may be governed by completely different conditions. (Received August 30, 1940.)

492. Loo-Keng Hua: *Some "Anzahl" theorems in the theory of groups of prime-power order.*

A group G of order p^n (p being a prime) is said to be of rank δ if the maximum of the order of elements of G is equal to $p^{n-\delta}$. The author selects the following theorem to be announced: If G is a group of order p^n ($p \geqq 3$, $u \geqq 2\alpha$) of rank α, then (i) G contains one and only one subgroup of order p^n ($\alpha \leqq m \leqq n$) of rank α; (ii) G contains p^α cyclic subgroups of order p^m ($\alpha < m < n - \alpha + 1$) and (iii) G contains $p^{m+\alpha}$ ($\alpha \leqq m \leqq n - \alpha$) elements which satisfy $x^{p^m} = 1$. The second and the third statement solve the "Anzahl" theorems of Miller's and Kalakoff's type. (Received August 31, 1940.)

493. Loo-Keng Hua: *The lattice points in a circle.*

Let $R(x)$ denote the number of lattice points in the circle $m^2 + n^2 = x$. The object of the paper is to prove that $R(x) = \pi x + O(x^{\alpha+\epsilon})$, $\alpha = 13/40$, which is better than Titchmarsh's result (Proceedings of the London Mathematical Society, (2), vol. 38 (1933), pp. 96–115) with an error term $O(x^{15/46+\epsilon})$. Notice that Vinogradow's proof of the error term $O(x^{17/53+\epsilon})$ is incorrect (Bulletin de l'Académie des Sciences de l'URSS, vol. 7 (1932), pp. 313–336). (Received August 31, 1940.)

494. W. H. Ingram: *A generalization of Erhard Schmidt's solution of the nonhomogeneous integral equation.* Preliminary report.

It is first proved that the Hilbert transform of an arbitrary step-wise continuous vector matrix has, in terms of the characteristic solutions of the equation $\phi_r(x) = \lambda_r \cdot \int_b^a \mathfrak{K}(x, \alpha)\phi_r(\alpha)d\alpha$, an absolutely and uniformly convergent development which represents the transform in the sense of least squares when the scalar coefficients are computed by means of the associated characteristic functions ψ_r. As in abstract 46-5-282, $\mathfrak{K}$ is a non-symmetric square matrix, and ϕ_r and ψ_r, respectively, column and line matrices which can be biorthonormalized. In this sense of equality and on the basis of this expansion theorem, the generalization of Schmidt's result previously announced is independently derived. (Received August 27, 1940.)

495. L. H. Loomis: *The decomposition of meromorphic functions into rational functions of univalent functions.*

The general question discussed is this: given $f(z)$ meromorphic in $|z| < 1$, when does there exist a decomposition $f(z) \equiv f_2(f_1(z))$ where $\zeta = f_1(z)$ is univalent in $|z| < 1$, and $f_2(\zeta)$ is a rational function? A simple but typical result is that if $f(z)$ is meromorphic in $|z| \leqq 1$, then the decomposition is possible. Necessary and sufficient conditions are derived for the general case. Stronger theorems are proved for situations in which $\zeta=f_1(x)$ is the mapping function of a simply-connected plane region which is not of the most general type, for instance, a Jordan region. Analogous theorems are developed for $f(z)$ regular and bounded in $|z|<1$ and $f_1(\zeta)$ restricted to special types of rational functions, say polynomials. In proof, the author considers the possibility of imbedding the Riemann surface which is the map of $|z|<1$ under $w=f(z)$ in the Riemann surface of the inverse of a rational function. (Received September 23, 1940.)

496. P. T. Maker: *A topological characterization of monotone functions.* Preliminary report.

Let $f(s)$, a function from S to X, both topological spaces, be in $M(S, X)$ if the complete inverse image of every connected set in X is connected. It is shown that $M(R_1, R_1)$ is the class of monotone functions, and that for other spaces M includes the functions "monotone" by previous generalizations. Conditions on S and X, in order that $f(s)$ have certain continuity properties, or that M be complete or compact, are determined. (Received September 28, 1940.)

497. E. R. Ott: *A locus determined by an algebraic correspondence.*

Let P and Q be two variable vertices of a hexagon inscribed in a conic and let the other four vertices be fixed. If P and Q have parameters $t'=F(t)$ and $t'=G(t)$, where $F(t)$ and $G(t)$ are functions which establish an (m, n) algebraic correspondence between P and Q, then as t varies the Pascal line of the hexagon envelopes a curve of class $(m+n)$ and of order $2(m+n-1)$. Each of the two fixed sides of the hexagon which are adjacent to the variable side is a multiple tangent of the locus. All of the Pluecker characteristics of the curve are obtained and are independent of the order of the vertices of the hexagon. (Received September 14, 1940.)

498. Francisco Perez: *A generalization of the theory of invariant factors and similar matrices.*

E. H. Moore's general identity matrix $\delta_{\mathfrak{M}}$ (E. H. Moore, *General Analysis*, Part I) for a right linear space $\mathfrak{M}$ of vectors on a finite range $\mathfrak{P}$ to $\mathfrak{A}$ is shown to exist even when $\mathfrak{A}$ is a number system of *type B*, provided $\mathfrak{M}$ has the property of being *perfect*. In a perfect space $\mathfrak{M}$ of rank r, to every matrix ϕ of type $\mathfrak{M}\overline{\mathfrak{M}}$ and of rank r corresponds uniquely a matrix ϕ^{-1} of type $\mathfrak{M}\overline{\mathfrak{M}}$ such that $S\phi^{-1}\phi=S\phi\phi^{-1}=\delta_{\mathfrak{M}}$. When multiplication is commutative, it is shown (without using determinants) that every matrix κ of type $\mathfrak{M}\overline{\mathfrak{M}}$ determines uniquely a set of polynomials $P_i(z)$ $(i=1, 2, \cdots, t)$ called the *invariant factors of* κ *in* $\mathfrak{M}$. The sum of the degrees of these polynomials is equal to r, and P_{i+1} divides P_i. It is shown that $P_1(\kappa)=O(\mathfrak{P}\mathfrak{P})$, where $\kappa^0=O(\mathfrak{P}\mathfrak{P})$. Two matrices κ_1 and κ_2 of type $\mathfrak{M}\overline{\mathfrak{M}}$ are said to be *similar in* $\mathfrak{M}$ if there exists a matrix ϕ of type $\mathfrak{M}\overline{\mathfrak{M}}$ and of rank r such that $\kappa_2=SS\phi^{-1}\kappa_1\phi$. Two matrices of type $\mathfrak{M}\overline{\mathfrak{M}}$ are similar in $\mathfrak{M}$ if and only if they have the same invariant factors in $\mathfrak{M}$. (Received September 14, 1940.)

499. L. B. Robinson: *A functional equation with a singular line.*

The equation (I) $u'(x)=a(x)u(x^2)$ can be transformed into (II) $w'(z)=e^{z}a(e^{z})w(2z)$ which in general has a solution with $z=0$ as singular point. The exceptional case is given by the vanishing of a determinant. In general the circumference of the unit circle is a singular line of $u(x)$. The necessary and sufficient condition for the failure of this rule has been found. (Received September 14, 1940.)

500. L. B. Robinson: *A functional equation with negative exponent.*

Consider the system $u_x'(x)=(\lambda/(1+x^2))[P_n(x)/Q_n(x)]u(x^{-2})$, $u_x'(x^{-1})=(-\lambda/(1+x^2))[P_n(x^{-1})/Q_n(x^{-1})]u(x^2)$. The solution of this system is $u(x)=u_0\{1+r\lambda\phi_1(x)+\lambda^2\phi_2(x)+r\lambda^3\phi_3(x)+\cdots\}$, $u(x^{-1})=u_0\{r+\lambda\psi_1(x)+r\lambda^2\psi_2(x)+\lambda^3\psi_3(x)+\cdots\}$ where $\phi_n(x)=\int_0^x(1/(1+\xi^2))[P_n(\xi)/Q_n(\xi)]\psi_{n-1}(\xi^{-2})d\xi$, $\psi_n(x)=\int_0^x(-1/(1+\xi^2))[P_n(\xi^{-1})/Q_n(\xi^{-1})]\cdot\phi_{n-1}(x^2)d\xi$. If the value of r is rightly determined, the two series $u(x)$ and $u(x^{-1})$ converge and are consistent. (Received September 14, 1940.)

501. Jenny E. Rosenthal: *Generating functions and properties of certain orthogonal polynomials.* II.

The nonhomogeneous differential equation satisfied by generating functions for orthogonal polynomials reduces in the simplest possible case to an algebraic equation. Its solution is the generating function of a certain type of Szegö-Bernstein polynomials. A method is given for obtaining the weight factor from the generating function and for finding some additional properties of these polynomials. The second order differential equation satisfied by the polynomials is derived and is shown to reduce in a special case to an equation which was obtained by Shohat. (Received September 27, 1940.)

502. Peter Scherk: *On real closed curves of the order $n+1$ in the projective n-space.* Preliminary report.

The author discusses closed curves K^{n+1} in the real projective n-space which have osculating spaces of all the dimensions less than n and which have the real order $n+1$; that is, the maximum number of points of intersection of a K^{n+1} with an $(n-1)$-space shall be $n+1$. These curves are a generalization of the algebraic curves of order $n+1$ with one real branch. Analogous to the algebraic case, singular points can be defined and provided with multiplicities. The sum of the multiplicities of the singular points of a K^{n+1} is less than or equal to $n+1$ and congruent to $n+1$ (mod 2). The types for which this sum equals $n+1$ are characterized. Furthermore, the pairs of curves in the projective n-space are described which have no more than $n+1$ points with any $(n-1)$-space in common. These pairs generalize the algebraic curves of order $n+1$ with two real branches. Some special qualities of the K^{n+1} are indicated. The main tool of the discussion is the continuous transformation of the K^{n+1} into itself which correlates to each point of the K^{n+1} the point at which its osculating $(n-1)$-space intersects the K^{n+1} again. (Received September 27, 1940.)

503. W. T. Scott and H. S. Wall: *A geometrical method in the theory of continued fractions.*

In this paper the authors consider continued fractions of the form $1/1+a_2/1+a_3/1+a_4/1+\cdots$, where the a_n's are complex numbers. By regarding the continued fraction as a succession of linear fractional transformations they determine conditions under which the nth approximant A_n/B_n of the continued fraction will lie in a region V of the complex plane whenever the a_n's lie in a region U. Under these conditions any convergent continued fraction whose elements lie in U must have a value in V. As a supplement to their recent "parabola" theorem (Transactions of this Society, vol. 47 (1940), p. 166) they show that if the a_n's lie in the parabola $|z|-R(z)\leq\frac{1}{2}$ every approximant A_n/B_n lies in the circle $|z|^2-2R(z)\leq 0$, and that this is the "best" circular region having the above property. (Received August 19, 1940.)

504. W. T. Scott and H. S. Wall: *Linear manifolds of Hausdorff means.*

A set M of moment sequences is a manifold if when $\{c_n\}$, $\{d_n\}$ are in M, then $\{c_n+d_n\}$ and $\{rc_n\}$ (r real) are in M. A manifold M is regular if when $\{c_n\}\in M$, $c_0\neq 0$ then $\{c_n/c_0\}$ is a regular Hausdorff sequence. The authors determine conditions under which a sequence $\{\beta_n(u)\}$ of real functions on the interval $(0, 1)$ constitutes a basis for a manifold $M(\beta)$, in the sense that M is made up of the sequences $c_n=\int_0^1\beta_n(u)\cdot d\phi(u)$, $(n=0, 1, 2, \cdots)$, for $\phi(u)\in BV[0,1]$. They also determine conditions upon the basis such that $M(\beta)$ shall be regular, and conditions such that every regular Hausdorff mean determined by the sequences of $M(\beta)$ shall include a given Hausdorff mean. The general theory is applied to a number of special examples. Manifolds exist which include any given Hausdorff mean, and, in particular, which include (C, α) but not $(C, \alpha+\epsilon)$, $(\alpha>0, \epsilon>0)$. For example, if $\beta_n(u)=(u+1)/(u+n+1)$, then $M(\beta)\supset(C, 1)$ but not $(C, 1+\epsilon)$, $(\epsilon>0)$. (Received August 19, 1940.)

505. I. E. Segal: *The space of Besicovitch almost periodic functions.*

Let B^p $(1\leqq p<\infty)$ be the space of all Besicovitch almost periodic functions of order p on the infinite interval $(-\infty, \infty)$ with the usual norm. It is shown that there is a bicompact space S and a completely additive measure μ over the field of Borel subsets of S such that B^p is isometric with the space $L_p(S, \mu)$ of all complex-valued functions which are measurable and whose pth power is summable with respect to μ; the norm is the usual one. This result follows readily from a recent theorem of Gelfand on the representation of the class of Bohr almost periodic functions. It is also connected with some recent work of Bochner and of Wecken. A similar result holds for the spaces of almost periodic functions on any locally bicompact group. (Received September 28, 1940.)

506. William Wernick: *Distributive properties of set operators.*

A point set operator a which maps a set A of a given space S uniquely into a set $aA\in S$ may have certain distributive properties defined by expressions of the form $a(Af_1B)\cdot R:aA\cdot f_2\cdot aB$; where f_1, f_2 are set operators $+$ or $\cdot$; and where R is a set relation, either $=$, $\subset$, or $\supset$. There are twelve such properties, but *monotonicity* and *inverse monotonicity* are included to obtain a list of fourteen "distributive" properties. These fourteen properties (α_i) and their negations $(\bar{\alpha}_i)$ form a collection of 28 properties which may be assumed or deduced for set operators. Their interrelations are investigated. The distributive character of a is determined with respect to α_i if it is known that a has property α_i $(a:\alpha_i)$; or that $a:\alpha_i$. An operator a is "completely determined" if its distributive character is determined with respect to every α_i. In view of the interrelations among the α_i, there are relatively few "completely determined" operators. A list of 25 is given, with examples. (Received September 13, 1940.)

507. William Wernick: *Functional dependence in the calculus of propositions.*

In the two-valued calculus of propositions a function F of n variables is completely determined by 2^n constants, independently 0, or 1 (call their sum S). F is independent of a particular variable x_i if no change in F results from a change in x_i alone. A function A_i is defined whose vanishing is a necessary and sufficient (NS) condition that F be independent of x_i. In terms of the A_i a function $A(F)$ is defined whose nonvanishing is a NS condition that F be really a function of all its arguments, that is, not independent of any of them. Another function B_i is defined, and it is shown that $S=2B_i$ is another NS condition that F be independent of x_i. From this are obtained very simply the following two theorems: "A sufficient condition that F be really a function of all its arguments is that S be odd." "A necessary condition that F be independent of some or all of its arguments is that S be even." (Received September 20, 1940.)

508. Hassler Whitney: *The mappings of a 3-complex into a space with vanishing fundamental group.*

Extending methods used formerly for classifying the mappings of a 3-complex into a 2-sphere S^2 (see this Bulletin, vol. 42 (1936), p. 338), S^2 is now replaced by any space S without fundamental group. The 2-dimensional homotopy group $\pi^2(S)$ is a direct sum of cyclic groups, with generators $\rho_1, \cdots, \rho_n$. There is a natural function $\theta(\rho) \in \pi^3(S)$ and a multiplication $\rho \times \rho \in \pi^3(S)$; $2\theta(\rho) = \rho \times \rho$. If $a\rho = 0$, then $2a\theta(\rho) = 0$. If f is a mapping of the 2-dimensional part K^2 of K into S, a 2-π^2-chain X^2 is defined (compare Duke Mathematical Journal, vol. 3 (1937), pp. 51–55); f may be extended through K^3 and K^4 if and only if $\delta X^2 = 0$ and $X^2 \cup^* X^2 \sim 0$. Here, if $\sigma^r \cup \sigma^s = \sigma^{r+s}$, then $\sum \alpha_i \rho_i \sigma^r \cup^* \sum \beta_i \rho_i \sigma^s = [\sum \alpha_i \theta(\rho_i) + \sum_{i<j} \alpha_i \beta_j (\rho_i \times \rho_j)] \sigma^{r+s}$ if no ρ_i are of finite order. Two mappings f and f' are homotopic if and only if their difference is $\sim 2(Y^1 \cup^* X^2)$ for some 1-π^2-cocycle Y^1. (The factor 2 was omitted in the abstract cited above.) (Received September 28, 1940.)

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ARITHMETICS OF RATIONAL GENERALIZED QUATERNION ALGEBRAS

D. M. BROWN

1. **Introduction.** A rational generalized quaternion algebra is a linear associative algebra of order four and rank two, having a principal modulus I_0, ranging over the field of rational numbers. A basis (I_0, I_1, I_2, I_1I_2) may be found[1] such that

$$(1) \quad I_1^2 = \alpha, \quad I_2^2 = \beta, \quad I_1I_2 = -I_2I_1, \qquad \alpha \text{ and } \beta \text{ rational integers.}$$

An algebra in such form will be denoted by $Q(\alpha, \beta)$, a number in it by

$$(2) \qquad Q = a_0I_0 + a_1I_1 + a_2I_2 + a_{12}I_1I_2, \qquad a\text{'s rational,}$$

and its norm by

$$(3) \qquad N(Q) = a_0^2 - \alpha a_1^2 - \beta a_2^2 + \alpha\beta a_{12}^2.$$

Q satisfies its rank equation

$$(4) \qquad X^2 - 2a_0X + N(Q) = 0.$$

a_0 is called the real part of Q.

An arithmetic S of $Q(\alpha, \beta)$ is a set of numbers having the following properties:

C_a: S is closed with respect to algebraic addition.

C_m: S is closed with respect to multiplication.

R: For every number of S, (4) has integral coefficients.

U: S contains I_0, I_1 and I_2 (and hence I_1I_2 by C_m).

M: S is maximal; that is, S is contained in no larger set having Properties C_a, C_m, R and U.

It is the purpose of this paper to determine a set of bases for the arithmetics of those algebras for which α and β contain no squared prime factors.[2] In any case, it has been proved[3] that for a given arithmetic S in the algebra, a basis $(I_0', I_1', I_2', I_1'I_2')$ can be found such that $I_1'^2=\alpha'$, $I_2'^2=\beta'$, $I_1'I_2'=-I_2'I_1'$, α' and β' containing no squared prime factors, and such that the given arithmetic is an arithmetic of $Q(\alpha', \beta')$. The problem of determining bases for the arithmetics of such algebras has been solved by Latimer[4] when $\alpha\equiv\beta\equiv 1 \pmod 2$, and by Darkow[5] when $\alpha\equiv\beta\equiv 0 \pmod 2$. Darkow's results can be extended to apply to the remaining cases (when $\alpha+\beta\equiv 1 \pmod 2$).[6] All possible cases are treated in this paper, with greatly simplified results.[7]

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[1] Dickson, L. E., *Algebren und ihre Zahlentheorie*, Zurich, 1927, pp. 43–44. The definition of an arithmetic is made there also.

[2] If α and β contain squared prime factors, the number of arithmetics varies with the form of those factors. See M. Eichler, *Untersuchungen in der Zahlentheorie der rationalen Quaternionalgebren*, Journal für die reine und angewandte Mathematik, vol. 174 (1936), p. 149, Theorem 12.

[3] Latimer, C. G., *The classes of integral sets in a quaternion algebra*, Duke Mathematical Journal, vol. 3 (1937), pp. 246–247, §7. On pages 237–238, §2, of this reference is stated a theorem giving necessary and sufficient conditions that a basis of $Q(\alpha, \beta)$ be a basis of an *arithmetic* of $Q(\alpha', \beta')$.

Albert[8] has shown that every rational generalized quaternion *division* algebra $Q(\alpha, \beta)$ may be transformed into $Q(\tau, \sigma)$, in which τ and σ have special properties. Concerning the treatments by Latimer and Darkow (similar to that used in this paper) Albert writes:[9]

> The · · · division into special cases is certainly not desirable. Nor is it necessary. For it is obvious that at least an attempt should be made to show that transformations carrying all cases into a canonical form are possible, and it is this canonical form which should be studied. · · · In particular, it is evident that the integral sets of S should contain the integers of the realm $R(i)$.

Concerning the above quotation, a few comments seem pertinent. Albert's results are readily obtainable from the results of this paper or those of Latimer and Darkow. His canonical form is not unique, and the number (two) of arithmetics obtained by it is, in general, much less than the number of arithmetics as defined by this paper. There being no essential difference between the basal elements I_1 and I_2, or, for that matter, I_1I_2, there seems to be little reason for requiring that the integers of the realm $R(I_1)$ be contained in an integral set (arithmetic) in preference to those of realms $R(I_2)$ or $R(I_1I_2)$. Albert's results apply only to division algebras, while those of Latimer and Darkow and the writer apply to all algebras. The norm of a number of $Q(\alpha, \beta)$, given in (3), being of a form of great interest in number theory, it would seem particularly useful to have available a method of obtaining the arithmetics of an algebra expressed in terms of parameters α and β which yield that exact norm. Clearly Albert's canonical form would not suffice, so that methods of treatment as given here are conceivably not only useful but necessary.

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[4] Latimer, C. G., *Arithmetics of generalized quaternion algebras*, American Journal of Mathematics, vol. 48 (1926), pp. 57–66.

[5] Darkow, M. D., *Determination of a basis for the integral elements of certain generalized quaternion algebras*, Annals of Mathematics, (2), vol. 28 (1926), pp. 263–270.

[6] Latimer, C. G., *On the class number of a quaternion algebra with a negative fundamental number*, Transactions of this Society, vol. 40 (1936), p. 320.

[7] The writer is greatly indebted to Dr. Claiborne G. Latimer for valuable suggestions enabling much greater simplification than he had previously obtained.

[8] Albert, A. A., *Integral domains of rational generalized quaternion algebras*, this Bulletin, vol. 40 (1934), pp. 164–176.

[9] Albert, A. A., loc. cit., p. 165. The i used there is the same as the I used in this paper.

As mentioned above, the method of treatment in this paper is similar to that used by Latimer and Darkow, the work being more direct in some places.[10]

2. **Notations, transformations and definitions used.** The letters r, s and t will be used exclusively to denote permutations of 1, 2, and 3. Let $(g-1)(g+1)=0$ such that

$$g = \begin{cases} 1 \text{ if the number of inversions of } (rst) \text{ is even,} \\ -1 \text{ otherwise.} \end{cases} \tag{5}$$

Let the g.c.d. of α and β be

$$(\alpha, \beta) = \gamma_3 \text{ (positive or negative)} \tag{6}$$

and $\alpha = -\gamma_2\gamma_3$, $\beta = \gamma_3\gamma_1$. Let

$$I_1 I_2 = \gamma_3 I_3, \tag{7}$$

so that from (1), $I_r^2 = -\gamma_s\gamma_t$, and $I_r I_s = g\gamma_t I_t$. From the results of Latimer and Darkow, it can easily be shown that I_3 is in each arithmetic, as are I_1 and I_2. Hence from (6) and (7), I_1, I_2, and I_3 are formally identical, as are the γ's, the latter being formally identical to their negatives. $Q(\alpha, \beta)$ will hereafter be written in the symmetric notation

$$Q(\gamma_1, \gamma_2, \gamma_3) \equiv Q(\alpha, \beta). \tag{8}$$

Making the unitary transformation

$$\begin{gathered} i_0 = gI_0, \qquad i_1 = gI_r, \qquad i_2 = gI_s, \qquad i_3 = gI_t, \\ \gamma_r = \alpha_1, \qquad \gamma_s = \alpha_2, \qquad \gamma_t = \alpha_3, \end{gathered} \tag{9}$$

one has $i_r^2 = -\alpha_s\alpha_t$ and $i_r i_s = g\alpha_t i_t$, which is abstractly identical with (9). $Q(\gamma_1, \gamma_2, \gamma_3)$ now becomes $Q(\alpha_1, \alpha_2, \alpha_3)$. After proper change of sign of the γ's, one finds that γ_t is odd or even, $\gamma_s \equiv 1 \pmod 4$, γ_r is odd, and $\gamma_r \equiv 1 \pmod 4$ if γ_t is odd. Choosing

$$(\theta - 1)(\theta - 2) = 0, \tag{10}$$

$Q(\alpha_1, \alpha_2, \alpha_3)$ is such that

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[10] Latimer and Darkow obtain their results for $Q(\alpha, -\beta)$. Each mentions a "tentative" process for obtaining bases. See Latimer, loc. cit., footnote 3, pp. 61 and 65; Darkow, loc. cit., p. 263 and p. 267 ff.

$$\alpha_3 \equiv \theta \pmod{2\theta}, \quad \alpha_2 \equiv 1 \pmod 4, \quad \alpha_1 \equiv \alpha_2 \pmod{2(3-\theta)}. \tag{11}$$

In what follows, it will be assumed that (9) has been made so that (11) holds. All other symbols introduced hereafter will represent integers, unless they are obviously not integers. The parameters θ, m, k, n, A_r, B_r, ξ and H_j, used frequently hereafter, and most easily determined in the order given, are defined by the following, each being unique save n and H_j:

θ: $\alpha_3 \equiv \theta \pmod{2\theta}$; $(\theta-1)(\theta-2)=0$.

m: $(\alpha_1+\alpha_2+m\alpha_3) \equiv 2-\theta \pmod{4\theta}$; $(m-1)(m+1)[(\theta-1)m(m+2)+2-\theta]=0$.

k: $2(m^2+m+1)k=(m+1)(m+2)$.[11]

n: $kn^2-n=0$.

A_r, B_r: $\alpha_r = A_rB_r$, and

$$\left(\frac{-\alpha_s\alpha_t}{A_r}\right) = 1, \qquad \left(\frac{-\alpha_s\alpha_t}{B_r}\right) = -1,$$

B_r positive and a minimum.[12]

ξ: ξ is the number of prime factors (all distinct) of $A_1A_2A_3$.

H_j: $(\alpha_1\alpha_2+\alpha_2\alpha_3+\alpha_3\alpha_1)H_j^2 \equiv -1 \pmod{A_1A_2A_3}$, $(j=1, 2, \cdots, \xi)$.[13] If H_j is even, then $H_j' = H_j \pm A_1A_2A_3$ is odd. Let $f_7=(1-2m)$ and $f_8=2\theta k-2k+2$. Then if $A_2B_1H_j' \not\equiv f_7 \pmod{f_8}$, it can be shown that $H_j'' = H_j' \pm A_1A_2A_3$ is such that $A_2B_1H_j'' \equiv f_7 \pmod{f_8}$. Since H_j, H_j', and H_j'' are all in the same residue class, modulo $A_1A_2A_3$, it follows (and will be so assumed hereafter) that H_j may always be chosen such that (f_7 is odd and f_8 is even, so H_j will be odd)

$$A_2B_1H_j \equiv f_7 \pmod{f_8}. \tag{12}$$

The following symbols will also be used:

$$\begin{aligned} &f_1 = (2-\theta)kn, \quad f_2 = (\theta n - 2n + 2), \quad f_3 = k(m+\theta n-2n), \\ &f_4 = (2-\theta)(m-2n-k+1)+n, \quad f_5 = (2n-\theta n-k+2), \\ &f_6 = (\theta-1)(m-2k+2), \quad f_7 = (1-2m), \quad f_8 = 2\theta k-2k+2, \end{aligned} \tag{13}$$

---

[11] If $e=1-k$, then k may also be defined by $(\alpha_1+\alpha_2+\alpha_3)(\alpha_1+\alpha_2)(\alpha_2+\alpha_3)(\alpha_3+\alpha_1) \equiv 8e \pmod{16}$. See Latimer, C. G., *On the fundamental number of a rational generalized quaternion algebra*, Duke Mathematical Journal, vol. 1 (1935), p. 435, theorem and second footnote.

[12] In these conditions

$$\left(\frac{A}{P}\right)$$

is the Legendre symbol for quadratic residues.

[13] This congruence has exactly ξ distinct solutions. See Dickson, L. E., *Introduction to the Theory of Numbers*, University of Chicago Press, 1929, p. 12, Theorem 16.

$$(14)\quad M_j = \frac{\alpha_1 H_j i_1 + i_2}{2A_3}, \qquad N_j = \tfrac{1}{2} i_0 + \frac{-\alpha_1 H_j i_1 + \alpha_2 H_j i_2 + i_3}{2A_1A_2}.$$

3. The fundamental form of a number Q in an arithmetic. From the relations given by (2), (3), (7) and (9), a number in $Q(\alpha_1, \alpha_2, \alpha_3)$ is such that

$$(15)\qquad \begin{aligned} Q &= b_0 i_0 + b_1 i_1 + b_2 i_2 + b_3 i_3, \\ N(Q) &= b_0^2 + \alpha_2\alpha_3 b_1^2 + \alpha_3\alpha_1 b_2^2 + \alpha_1\alpha_2 b_3^2 \end{aligned}$$

(the b's rational). By Property $\mathrm{C_m}$, if Q is in S, then so are Qi_0, Qi_1, Qi_2 and Qi_3, whose real parts are b_0, $-\alpha_2\alpha_3 b_1$, $-\alpha_3\alpha_1 b_2$ and $-\alpha_1\alpha_2 b_3$, respectively, and by Property R and (4), one may write $2b_0 = X_0$, $2b_r\alpha_s\alpha_t = X_r$, so that (15) becomes

$$(16)\qquad \begin{aligned} Q &= \frac{1}{2}\left[X_0 i_0 + \frac{X_1 i_1}{\alpha_2\alpha_3} + \frac{X_2 i_2}{\alpha_3\alpha_1} + \frac{X_3 i_3}{\alpha_1\alpha_2}\right], \\ N(Q) &= \frac{1}{4}\left[X_0^2 + \frac{X_1^2}{\alpha_2\alpha_3} + \frac{X_2^2}{\alpha_3\alpha_1} + \frac{X_3^2}{\alpha_1\alpha_2}\right] \end{aligned}$$

and, by Property R,

$$(17)\qquad \alpha_1\alpha_2\alpha_3 X_0^2 + \alpha_1 X_1^2 + \alpha_2 X_2^2 + \alpha_3 X_3^2 \equiv 0 \pmod{4\alpha_1\alpha_2\alpha_3}.$$

Using (11), (17) becomes equivalent to

$$(18)\qquad \begin{aligned} \alpha_1\alpha_2\alpha_3 X_0^2 + \alpha_1 X_1^2 + \alpha_2 X_2^2 + \alpha_3 X_3^2 &\equiv 0 \pmod{4\theta} \\ \text{if also } \alpha_r X_r^2 + \alpha_s X_s^2 &\equiv 0 \pmod{\alpha_t}. \end{aligned}$$

Let δ_t be any prime factor of α_t. If δ_t divides X_r, it also divides X_s. But if $(X_r, \delta_t) = 1$, then there exists an integer p_r such that $p_r X_r \equiv 1 \pmod{\delta_t}$. Hence from (18), $p_r^2\alpha_r X_r^2 + p_r^2\alpha_s X_s^2 \equiv 0 \pmod{\delta_t}$, $\alpha_r + p_r^2\alpha_s X_s^2 \equiv 0 \pmod{\delta_t}$, or $(p_r\alpha_s X_s)^2 \equiv -\alpha_r\alpha_s \pmod{\delta_t}$. Hence

$$\left(\frac{-\alpha_r\alpha_s}{\delta_t}\right) = 1,$$

and from (13), X_r may be prime to each factor of A_t, but must be divisible by each factor of B_t, and hence by B_t itself. Therefore B_t divides X_r and X_s, and one may write

$$(19)\qquad X_0 = x_0, \qquad X_r = B_s B_t x_r,$$

and (16) and (18) become, respectively,

$$(20)\qquad \begin{aligned} Q_x &= \frac{1}{2}\left[x_0 i_0 + \frac{x_1 i_1}{A_2A_3} + \frac{x_2 i_2}{A_3A_1} + \frac{x_3 i_3}{A_1A_2}\right], \\ N(Q) &= \frac{1}{4}\left[x_0^2 + \frac{B_2B_3x_1^2}{A_2A_3} + \frac{B_3B_1x_2^2}{A_3A_1} + \frac{B_1B_2x_3^2}{A_1A_2}\right], \end{aligned}$$

and

$$(21)\qquad \begin{aligned} \alpha_1\alpha_2\alpha_3x_0^2 + \alpha_1x_1^2 + \alpha_2x_2^2 + \alpha_3x_3^2 &\equiv 0 \pmod{4\theta}, \\ (A_sB_rx_s)^2 &\equiv -\alpha_r\alpha_sx_r^2 \pmod{A_t}. \end{aligned}$$

Hereafter, the numbers x_0, x_1, x_2, x_3 will be referred to as the *coordinates of a number Q_x in S*, and (20) will be called the *fundamental form of Q in S*. Using the definition of H_j to obtain $\alpha_r\alpha_sH_j^2 \equiv -1$ (mod A_t), and multiplying both members of (21b) by H_j^2, one obtains $(A_sB_rH_jx_s)^2 \equiv -\alpha_r\alpha_sH_j^2x_r^2 \equiv x_r^2$ (mod A_t), or, reducing to linear congruences, $x_r \equiv A_sB_r(\pm H_j)x_s$ (mod A_t). Now if H_j is a solution of its congruence, then $-H_j$ is another, distinct from H_j, and one may finally write

$$(22)\qquad x_r \equiv gA_sB_rH_jx_s \pmod{A_t},$$

where g is defined as in (5). Hence from (20), (21) and (22) one concludes:

THEOREM I. *If a number Q is in an arithmetic, it is necessary that it have the form* (20), *whose coordinates satisfy* (21a) *and* (22).

4. **Analysis of the conditions imposed on the coordinates of a number Q of fundamental form lying in an arithmetic.** It can be shown that two numbers of fundamental form have Property C_a if and only if their coordinates satisfy (22) *for the same value of j*. Hereafter H_j will be replaced by H whenever the value of j is fixed in an argument. Let Q_x and Q_y be in an arithmetic, and hence satisfy (22) for the same value of j. Then by Property C_m, Q_xQ_y must also lie in S. A study of the coordinates of this new number reveals that in addition to (21a) and (22), the following conditions must hold:[14]

$$(23)\qquad \begin{aligned} \alpha_1\alpha_2\alpha_3x_0y_0 + \alpha_1x_1y_1 + \alpha_2x_2y_2 + \alpha_3x_3y_3 &\equiv 0 \pmod{2\theta}, \\ x_0y_1 + y_0x_1 + y_2x_3 + x_2y_3 &\equiv 0 \pmod{2}, \\ x_0y_2 + y_0x_2 + y_3x_1 + x_3y_1 &\equiv 0 \pmod{2}, \\ \theta(x_0y_3 + y_0x_3) + y_1x_2 - x_1y_2 &\equiv 0 \pmod{2\theta}. \end{aligned}$$

---

[14] Here begins the direct method, as opposed to the "tentative" process of Latimer and Darkow. See footnote 10.

Exhaustive analysis of (21a), (22) and (23) further reveals that the conditions that must hold in order for Q_x and Q_y *to have property M* are as follows:

$$
\begin{aligned}
x_0 &= [(2-\theta)n+(\theta-1)km]x_1 \\
&\quad + (2-\theta)(1-n)x_2+(\theta-1)x_3 \pmod{2}, \\
x_1 &\equiv 0 \pmod{\theta+k-\theta k}, \\
x_1 &\equiv (1-2n)x_2+2mx_0 \pmod{\theta k+2\theta-2k}, \text{for fixed } n, \\
x_3 &\equiv (2-\theta)[(1-n)x_1+nx_2] \pmod{3-\theta}
\end{aligned}
\tag{24}
$$

$$
x_1 \equiv B_1H(A_2x_2-A_3x_3) \pmod{A_2A_3}, \qquad x_2 \equiv A_3B_2Hx_3 \pmod{A_1}, \tag{25}
$$

which is equivalent to (22). Letting P_0, P_1, P_2, and P_3 be arbitrary integers, the f's integers as defined in (13), and M_j and N_j as in (14), then (24) and (25) can be reduced to

$$
\begin{aligned}
x_0 &= 2P_0+f_1P_1+f_3P_2+P_3, \\
x_1 &= f_2A_2A_3P_1+(f_4A_2A_3+f_5A_2\alpha_1H)P_2+(f_6A_2A_3-A_3B_1H)P_3, \\
x_2 &= f_5A_1P_2+A_3B_2HP_3, \\
x_3 &= P_3, & \text{for fixed } n.
\end{aligned}
\tag{26}
$$

Substituting these in (20a) one has

$$
\begin{aligned}
Q_{xjn} &= P_0i_0+\tfrac{1}{2}(f_1i_0+f_2i_1)P_1+[\tfrac{1}{2}(f_3i_0+f_4i_1)+f_5M_j]P_2 \\
&\quad +[\tfrac{1}{2}f_6i_1+N_j]P_3, \qquad j=1,2,\cdots,\xi.
\end{aligned}
\tag{27}
$$

Now let

$$
\begin{aligned}
U_{0jn} &= i_0, & U_{1jn} &= \tfrac{1}{2}(f_1i_0+f_2i_1), \\
U_{2jn} &= \tfrac{1}{2}(f_3i_0+f_4i_1)+f_5M_j, & U_{3jn} &= \tfrac{1}{2}f_6i_1+N_j.
\end{aligned}
\tag{28}
$$

(27) then becomes

$$
Q_{xjn} = P_0U_0+P_1U_1+P_2U_2+P_3U_3. \tag{29}
$$

All such numbers, for a fixed j and n, form an integral set of numbers with (28) as a basis. Since (28) is equivalent to (26) and (20) combined, and since (26) and (20) establish necessary and sufficient conditions that Properties C_a, C_m, R and M hold, and it is easily shown that i_1, i_2 and i_3 are of the form (29), so that Property U holds, it follows that (28) gives the bases of the algebra—one for each value of j and n. But n has $k+1$ values, and j has ξ values. By use of the inverse of the unitary transformation (9), and of (7), it is clear that the arithmetics of $Q(\alpha_1, \alpha_2, \alpha_3)$ are also the arithmetics of $Q(\gamma_1, \gamma_2, \gamma_3)$ and of $Q(\alpha, \beta)$. Hence

THEOREM II. *The numbers of* (28) *constitute the bases for the arithmetics of* $Q(\alpha_1, \alpha_2, \alpha_3)$, *of* $Q(\gamma_1, \gamma_2, \gamma_3)$, *and of* $Q(\alpha, \beta)$. *There are* $2^k\xi$ *such arithmetics.*

The number of arithmetics agrees with that determined by Latimer and Darkow, and the bases were checked by applying the theorem mentioned in footnote 3. In this theorem the fundamental number d of the algebra is used. By a known result[11]

$$d = \pm 2^{1-k} B_1 B_2 B_3,$$

the sign being positive if $\alpha_1, \alpha_2, \alpha_3$ are all of the same sign, and negative otherwise; that is, according as $N(Q)$ (see (3)) is a definite or indefinite form. It is also known that $Q(\alpha, \beta)$ is a division algebra if and only if $d \neq -1$.[15]

5. **The method of writing down the arithmetics of $Q(\alpha, \beta)$, or of $Q(\gamma_1, \gamma_2, \gamma_3)$, α and β containing no squared prime factors.** By use of (7) and (9), and a few new symbols, the arithmetics may readily be written down in terms of the original basis, (I_0, I_1, I_2, I_1I_2). Let the following definitions and order of procedure be made:

(a) $(\alpha, \beta) = \gamma_3$ (positive or negative), so that $\alpha = -\gamma_2\gamma_3$, $\beta = -\gamma_3\gamma_1$.

(b) $\gamma_r = \Gamma_r \Delta_r$, and

$$\left(\frac{-\gamma_s\gamma_t}{\Gamma_r}\right) = 1, \qquad \left(\frac{-\gamma_s\gamma_t}{\Delta_r}\right) = -1,$$

Δ_r a positive minimum and (rst) a permutation of (123).

(c) Arrange $\gamma_1, \gamma_2, \gamma_3$ (changing signs of all simultaneously if necessary) and determine θ so that $\gamma_t \equiv \theta \pmod{2\theta}$, $\gamma_s \equiv 1 \pmod 4$, $\gamma_r \equiv \gamma_s$ $(\bmod\ 2(3-\theta))$.

(d) Then determine m so that $\gamma_r + \gamma_s + m\gamma_t \equiv 2-\theta \pmod{4\theta}$, $(m-1)(m+1)[(\theta-1)m(m+2)+2-\theta] = 0$.

(e) Determine k and n such that $2(m^2+m+1)k = (m+1)(m+2)$, and $kn^2 - n = 0$.

(f) Determine $f_1, f_2, \cdots, f_8$ according to Table I.

(g) Determine the ξ solutions H_j $(j = 1, 2, \cdots, \xi)$ of $(\gamma_1\gamma_2 + \gamma_2\gamma_3 + \gamma_3\gamma_1)H_j^2 \equiv -1 \pmod{\Gamma_1\Gamma_2\Gamma_3}$, ξ being the number of prime factors of $\Gamma_1\Gamma_2\Gamma_3$. If any H_j is even, replace it by $H_j \pm \Gamma_1\Gamma_2\Gamma_3$ to make it odd.

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[15] Latimer, C. G., loc. cit., footnote 11. See §1, next to last paragraph, and the corollary to the theorem on page 435.

If then any H_j be such that $\Gamma_s\Delta_r H_j \not\equiv f_7 \pmod{f_8}$, replace it by $H_j \pm \Gamma_1\Gamma_2\Gamma_3$ and find that $\Gamma_s\Delta_r H_j \equiv f_7 \pmod{f_8}$.

(h) Then determine V_{0jn}, V_{1jn}, V_{2jn}, V_{3jn}, as the desired bases, for each value of j and n, where

$$\begin{aligned} V_{0jn} &= I_0, \qquad V_{1jn} = \frac{f_1 I_0 + f_2 I_r}{2}, \\ V_{2jn} &= \frac{f_3 I_0 + f_4 I_r}{2} + f_5 \frac{\gamma_r H_j I_r + I_s}{2\Gamma_t}, \\ V_{3jn} &= \frac{I_0 + f_6 I_r}{2} + \frac{-\gamma_r H_j I_r + \gamma_s H_j I_s + I_t}{2\Gamma_r\Gamma_s}. \end{aligned} \tag{30}$$

(i) Finally, replace $\alpha_3 I_3$ by $I_1 I_2$, giving the final forms.

Hull[16] has solved the problem of determining a set of bases for all the arithmetics of *division* algebras, by showing that every arithmetic may be obtained from some one of an infinite number of canonical generations of the algebra (merely a non-unitary transformation of units carrying α and β into α' and β' having special properties). Latimer has obtained similar results without the restriction that the algebras be division algebras.[3] However, the method of determining which canonical generations will give the arithmetics for a given un-

TABLE I

θ	1	1	1	2	2	2	2	2	2
m	-1	1	1	1	1	0	0	-1	-2
k	0	1	1	1	1	1	1	0	0
n	0	0	1	0	1	0	1	0	0
f_1	0	0	1	0	0	0	0	0	0
f_2	2	2	1	2	2	2	2	2	2
f_3	0	1	0	1	1	0	0	0	0
f_4	0	1	0	0	1	0	1	0	0
f_5	2	1	2	1	1	1	1	2	2
f_6	0	0	0	1	1	0	0	1	0
f_7	3	-1	-1	-1	-1	1	1	3	5
f_8	2	2	2	4	4	4	4	2	2

---

[16] Hull, Ralph, *The maximal orders of generalized quaternion division algebras*, Transactions of this Society, vol. 40 (1936), pp. 1–11.

altered α and β is not stated in either paper. Hull showed that the canonical forms determined by Albert[8] are special cases of his canonical generations.

One problem remains to be solved, namely a method of writing down the arithmetics of $Q(\alpha, \beta)$ when α and β contain squared prime factors.

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A NOTE ON MAXIMUM MODULUS AND THE ZEROS OF AN INTEGRAL FUNCTION

S. M. SHAH

Let $f(z)$ be an integral function of finite order $\rho \geqq 0$, let $M(r, f) = M(r) = \max_{|z|=r} |f(z)|$, and let $n(r, f) = n(r)$ be the number of zeros of $f(z)$ in $|z| \leqq r$ and on its circumference. I have discussed elsewhere[1] the behaviour of $g(r) = \log M(r)/n(r)$, as $r \to \infty$, and have proved that *for every canonical product function* $f(z)$

$$\liminf_{r=\infty} \frac{\log M(r)}{n(r)\phi(r)} = 0, \tag{1}$$

where $\phi(r)$ *is any positive L function*[2] *such that*

$$\int^{\infty} \frac{dx}{x\phi(x)} < A = \text{const.} \tag{2}$$

The question arises how large and how small $g(r)$ and $G(r) = T(r, f)/n(r)$ can be, where $T(r, f) \doteq T(r)$ is the Nevanlinna characteristic function for $f(z)$. I prove in this note the following result.

THEOREM. *Given* $\rho \geqq 0$ *and* $\psi(x)$ *any positive function such that*

$$\limsup_{x=\infty} \frac{\log \psi(x)}{\log x} \leqq \rho.$$

There exists an integral function $F(z)$ *of order* ρ *for which*

$$\limsup_{r=\infty} \frac{T(r, F)}{\psi(r)n(r, F)} = \infty \tag{3}$$

and an integral function $f(z)$ *of order* ρ *for which*

$$\liminf_{r=\infty} \frac{\psi(r)T(r, f)}{n(r, f)} = 0. \tag{4}$$

PROOF. We shall first construct an integral function $f(z)$ of order ρ for which

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[1] (i) *A theorem on integral functions of integral order*, Journal of the London Mathematical Society, vol. 15 (1940), pp. 23–31. (ii) *On integral functions of integral or zero order*, to be published. (iii) *On integral functions of perfectly regular growth*, Journal of the London Mathematical Society, vol. 14 (1939), pp. 293–302.

[2] For definition see G. H. Hardy, *Orders of Infinity*, 1924, p. 17.

$$\liminf_{r=\infty} \frac{\log M(r, f)\psi(r)}{n(r, f)} = 0 \tag{4.1}$$

from which (4) will follow. Let

$$\lambda_n = n^{n^n}, \qquad \psi(x) = e^{(\rho+\eta')\log x}$$

where we may suppose without loss of generality that $\eta_x' > 0$. We know that $\eta_x' \to 0$ as $x \to \infty$. Let η_r be the upper bound of η_t' for $t \geqq r$. Then $\eta_r \to 0$ monotonically as $r \to \infty$, and $\psi(r) \leqq e^{(\rho+\eta_r)\log r}$. Let $\epsilon_1 = \epsilon_2 = 1$, and

$$\epsilon_n = 2\eta_n + \frac{(4+\rho)\log \lambda_{n-1}}{\log \lambda_n}, \qquad n = 3, 4, 5, \cdots;$$

$$k = [\rho] + 1;$$

$$\mu_n = [\lambda_n^{\rho+\epsilon_n}], \quad \zeta_n = [\lambda_n^{\rho/k+\epsilon_n}], \quad n = 1, 2, 3, \cdots;$$

$$f(z) = \prod_1^\infty \left\{1 + \left(\frac{z}{\lambda_n}\right)^{k\mu_n}\right\}; \qquad F(z) = \prod_1^\infty \left\{1 + \frac{z^k}{\lambda_n}\right\}^{\zeta_n}.$$

It is easily seen that $f(z)$ and $F(z)$ are canonical products. Further, $n(\lambda_n, f) = k(\mu_1 + \cdots + \mu_n) \sim k\mu_n$ and hence

$$\limsup_{r=\infty} \frac{\log n(r, f)}{\log r} = \limsup_{n=\infty} \frac{\log k + \log \mu_n}{\log \lambda_n} = \rho.$$

Hence $f(z)$ is an integral function of order ρ. Let now $x = \lambda_m$. Then

$$\begin{aligned} \log f(x) &= \sum_{n=1}^{m-1} \log \left\{1 + \left(\frac{\lambda_m}{\lambda_n}\right)^{k\mu_n}\right\} \\ &\quad + \log 2 + \sum_{m+1}^{\infty} \log \left\{1 + \left(\frac{\lambda_m}{\lambda_n}\right)^{k\mu_n}\right\} \\ &= \Sigma_1 + \log 2 + \Sigma_3. \end{aligned}$$

We have

$$\Sigma_3 \leqq \sum_{m+1}^{\infty} \left(\frac{\lambda_m}{\lambda_n}\right)^{k\mu_n} \leqq \sum_{m+1}^{\infty} \left(\frac{1}{2}\right)^{k\mu_n} = O(1),$$

$$\Sigma_1 < m \log \left\{2(\lambda_m)^{k\mu_{m-1}}\right\}.$$

Hence for $m \geqq m_0$,

$$\begin{aligned} \log f(x) &< 2m\left\{\log 2 + k\mu_{m-1} \log \lambda_m\right\} \\ &\leqq \exp \left\{\log \mu_{m-1} + 2 \log \log \lambda_m\right\}, \end{aligned}$$

$$\begin{aligned}\frac{\psi(x)\log f(x)}{n(x,f)} &\leqq \exp\left\{(\rho+\eta_{\lambda_m})\log\lambda_m+\log\mu_{m-1}+2\log\log\lambda_m-\log\mu_m\right\}\\ &\leqq \exp\left\{(\rho+\eta_m)\log\lambda_m+\log\mu_{m-1}+2\log\log\lambda_m\right.\\ &\qquad \left.-(\rho+2\eta_m+(4+\rho)\log\lambda_{m-1}/\log\lambda_m)\log\lambda_m+O(1)\right\},\end{aligned}$$

and the last expression tends to zero with $1/m$. Hence

$$\liminf_{r=\infty}\frac{\psi(r)\log M(r,f)}{n(r,f)}=0$$

which proves (4.1). Further

$$n(\lambda_n^{1/k},F)=k(\zeta_1+\cdots+\zeta_n)\sim k\zeta_n.$$

Hence $F(z)$ is an integral function of order ρ. Let now $r=\frac{1}{2}\lambda_n^{1/k}$. Then

$$\begin{aligned}n(r,F)&=k(\zeta_1+\cdots+\zeta_{n-1})\leqq k\zeta_{n-1}\left(1+\frac{2\zeta_{n-2}}{\zeta_{n-1}}\right)\\ &< \exp\left\{\log k+\log\zeta_{n-1}+\frac{2\zeta_{n-2}}{\zeta_{n-1}}\right\},\end{aligned}$$

$$\begin{aligned}\log M(\tfrac{2}{3}r,F)&>\zeta_n\log\left(1+\frac{1}{3^k}\right)\\ &=A\exp\left\{\left(\frac{\rho}{k}+\epsilon_n\right)\log\lambda_n+O(\lambda_n^{-\rho/k})\right\},\end{aligned}$$

for $n\geqq n_0$. Hence[3]

$$\begin{aligned}\frac{\log M(\frac{2}{3}r,F)}{n(r,F)\psi(r)}&>A\exp\left\{\left(\frac{\rho}{k}+\epsilon_n\right)\log\lambda_n-\log\zeta_{n-1}\right.\\ &\qquad\left.-(\rho+\eta_n)\left(\frac{1}{k}\log\lambda_n-\log 2\right)+O(1)\right\}\\ &=A\exp\left\{\epsilon_n\log\lambda_n-\log\zeta_{n-1}-\frac{\eta_n}{k}\log\lambda_n+O(1)\right\}\\ &=A\exp\left\{2\eta_n\log\lambda_n+(4+\rho)\log\lambda_{n-1}\right.\\ &\qquad\left.-\left(\frac{\rho}{k}+\epsilon_{n-1}\right)\log\lambda_{n-1}-\frac{\eta_n}{k}\log\lambda_n+O(1)\right\},\end{aligned}$$

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[3] A denotes a positive constant.

and the last expression tends to infinity with n. Hence

$$\limsup_{r=\infty} \frac{\log M(\tfrac{4}{3}r, F)}{n(r, F)\psi(r)} = \infty,$$

and so

$$\limsup_{r=\infty} \frac{T(r, F)}{\psi(r)n(r, F)} = \infty.$$

Finally, I must thank Dr. T. Vijayaraghavan for helpful criticisms.

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SEPARABILITIES OF ARBITRARY ORDERS AND RELATED PROPERTIES[1]

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For the case of a metric space Gross[1a] established the equivalence of the following to each other and to other properties: (1) each point set is separable; (2) each uncountable point set contains a condensation point of itself.[2] Sierpiński has shown that for a space S Fréchet these two properties are not equivalent; also, he gives properties which are equivalent to (1) or to (2).[3] These properties involve countable collections, and, thus, are concerned with the smallest transfinite cardinal. Appert[4] generalizes by considering extensions to greater cardinals. He shows that certain of these properties imply others, but gives examples to show that they are not equivalent.[5] In a recent paper the author showed the equivalence of the following properties for the case of a space H Fréchet: (3) each uncountable point set contains a limit point of itself; (4) each uncountable point set either contains a condensation point of itself or is separable.[6] This result supplements Sierpiński's work by showing that property (3), which for the case of metric spaces is equivalent to (1) and (2), is a common basis for them in that it necessarily involves elements of either the one or of the other. In Theorem 2 property (3) is shown to be a necessary and sufficient condition for properties which involve for each point set M cardinals "as close as one pleases" to the power of M in connection with such concepts as separability of higher orders, the density in M of its limit points of higher orders, the extension of the Cantor-Bendixson theorem, and the "almost" perfect compactness of M. In Theorem 1 similar equivalences are established for orders which are not less than an arbitrary regular cardinal α, provided that the following holds: (3′) each point set of power α contains a limit point of itself. Note that if α is the smallest transfinite cardinal, then (3′) is a corollary of (3); and, if (3′) holds for a certain cardinal, it holds for greater cardinals. While our definitions differ from those of Appert, they yield a more systematic theory.

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[1] Presented to the Society, December 27, 1939, under the title *Separabilities of higher orders and related properties.*

[1a] Cf. W. Gross, *Zur Theorie der Mengen in denen ein Distanzbegriff definiert ist,* Sitzungsberichte der Kaiserlichen Akademie der Wissenschaften, part IIa, vol. 123 (1914), p. 801.

[2] References to contributions by Fréchet may be found in his *Espaces Abstraits et Leur Théorie Considérée Comme Introduction à l'Analyse Générale,* Paris, 1928. The reader is referred to this treatise for definitions.

[3] W. Sierpiński, *Sur l'équivalence de trois propriétés des ensembles abstraits,* Fundamenta Mathematicae, vol. 2 (1921), pp. 179–188. His results hold for more general spaces than the spaces S he mentions.

[4] A. Appert, *Propriétés des Espaces Abstraits les Plus Généraux,* Actualités Scientifiques et Industrielles, nos. 145 and 146, Paris, 1934, pp. 82–88. These extensions are considered in our Theorem 3.

[5] In *Über höherstufige Separabilität und Kompaktheit,* Japanese Journal of Mathematics, vol. 8 (1931), p. 114, Haratomi considers similar properties for the case of a metric space, for which they are equivalent; thus, his work is a generalization of Gross's.

[6] *Concerning limiting sets in abstract spaces* II, Transactions of this Society, vol. 43 (1938), p. 487.

Definitions. If α is a cardinal,[7] the point set M is said to be ***strongly α-separable*** provided that there exists N such that (1) $\overline{N} \supset M \supset N$, and (2) either N is countable or its power is less than[8] α. If α is the power of M and M is strongly α-separable, we say that M is ***semi-separable.*** Thus, for a point set whose power α is aleph-one, the properties separability, semi-separability, and strong α-separability are equivalent. If M is a point set, P is a point, and α is the smallest cardinal, β, such that there exists a neighborhood of P, U, for which the power of $U \cdot (M-P)$ is β, then P is a ***limit point of M of order α.*** Thus, an isolated point of M has order zero; a limit point has an order greater than zero; a condensation point has an order greater than aleph-zero; a complete limit point of M has an order equal to the power of M. In a space H Fréchet a limit point has an infinite order; at the end of this paper we give examples of spaces V Fréchet in which there exist limit points of finite orders. A point of M which has an order less than α is called an ***α-isolated point of M.*** The point set M is said to be ***almost perfectly α-compact in itself*** provided that if α and δ are cardinals, $M \supset N$, $\alpha \leqq$ power of N, and $\delta <$ power of N, then there exists in M a limit point of N of an order at least as great as δ. M is ***almost perfectly compact in itself*** provided that if $M \supset N$ and δ is a transfinite cardinal which is less than the power of N, then M contains a limit point of N of an order at least as great as δ; if, instead, we require that δ be the power of N, then M is perfectly compact in itself. Thus, a countable set is almost perfectly compact in itself. Because of the equivalence of properties (6) and (1) of Theorem 2, if M is almost perfectly compact in itself, the same property holds for each of its subsets; an analogous proposition does not hold for perfectly compact sets. Consider the properties of M: (a) it is perfectly compact in itself; it is almost perfectly α-compact in itself, (b) for α = aleph-zero, or (c) for α = aleph-one; (d) it is almost perfectly compact in itself. Then (c) and (d) are equivalent; for, if N is an uncountable subset of M and $\delta <$ power of N, there exists a transfinite cardinal δ_1 such that $\delta \leqq \delta_1 <$ power of N; the conclusion follows. Each element of the sequence (a), (b), (c) implies any that follows it, but implies none that precede it. However, for a countable set in a space H Fréchet (a) and (b) are equivalent. The following interpretation is suggestive: In (d) the power of N is the "upper bound" of the orders of the limit points of N which belong to M; in (a) this bound is *attained.*

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[7] In this paper zero and the positive integers are included among the cardinals and the ordinals. Also, according to our usage, finite or vacuous sets are countable. For the properties of transfinite numbers see F. Hausdorff, ***Grundzüge der Mengenlehre***, Leipzig, 1914.

[8] If (2) is modified as follows, M is α-separable: Either N is countable or its power is not greater than α; cf. Appert, loc. cit., p. 83.

THEOREM 1. *Let α be a regular cardinal which is greater than aleph-zero, T be a space V Fréchet in which the operation of derivation of point sets is distributive,*[9] *and the symbol M denote an arbitrary point set in T. Then the following properties are equivalent.*

(1) *Each point set of power α contains a limit point of itself.*

(2) *Each point set either is strongly α-separable, or it contains a limit point of itself which has an order not less than α.*

(3) *If M has a regular power β and $\alpha \leqq \beta$, then either M is semi-separable or β is the power of each of the following subsets of M:* (a) *that of all non-β-isolated points of M;* (b) *that of all complete limit points of M which belong to M.*

(4) *Each point set is almost perfectly α-compact in itself.*

(5) *Let $\alpha \leqq \beta$.* (a) *The power of the set of all isolated points of M is less than β;* (b) *the power of the set of all β-isolated points of M is not greater than β;* (c) *if $\beta <$ power of M, then M and the set of its non-β-isolated points have the same power.*[10]

PROOF. Clearly, the properties (2) to (5) imply (1). Suppose that E is a point set for which (2) does not hold. Then, for each point P of E there exists a subset $D(P)$ of E such that P is not an element of the closure of $E-D(P)$, $P \varepsilon D(P)$, and the power of $D(P)$ is less than α. Let T be a well-ordered sequence of the elements of E and δ

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[9] For example, $(E+F)'=E'+F'$; this is equivalent to assuming that Hausdorff's Axioms A and B hold in T. Cf. Fréchet, loc. cit., pp. 172, 173, and 181; and Appert, loc. cit., pp. 23–27. A cardinal α is regular if it is not the sum of fewer than α cardinlas, each of which is distinct from zero and is less than α; cf. W. Sierpiński, *Hypothèse du Continu*, Warsaw and Lwow, 1934, p. 152. With different hypotheses Appert obtains a result similar to part (5); loc. cit., p. 90.

[10] β is not necessarily regular.

be the smallest ordinal in the number class[11] $Z(\alpha)$. We shall define a well-ordered sequence $U=(q_1, q_2, \cdots, q_\omega, q_{\omega+1}, \cdots, q_\beta, \cdots)$ where $\beta<\delta$.

Proceed as follows: Let q_1 be the first element of T. Let β be an ordinal which is greater than unity and is less than δ; suppose that q_x has been defined for all ordinals $x<\beta$, and let U_β be the set of all such q_x's; let $S_\beta=\sum D(q_x)$, for $x<\beta$. Let q_β be the first point of T which does not belong to the closure of S_β.

We shall show that q_β exists for all ordinals $\beta<\delta$. For, if there exist β's for which this is not true, there exists a smallest one, say λ. Then each point of E belongs to the closure of S_λ. Then the power of U_λ and the powers of each of the sets $D(q_x)$ for $x<\lambda$ are each less than α. Since α is regular, the power of S_λ is less than α. Then, in contradiction to our supposition, E is strongly α-separable. Thus, λ does not exist, and the power of U is α.

By definition q_β is not a point or a limit point of $S_\beta \supset U_\beta$. Further, $S_{\beta+1}$, which contains $D(q_\beta)$, contains no point of $U-U_{\beta+1}$; thus q_β is not a limit point of the latter. Since the operation of derivation is distributive and $U=q_\beta+U_\beta+(U-U_{\beta+1})$, it follows that all points of U are isolated in U. This contradicts (1); thus (1) implies (2).

Let K be the set of all non-β-isolated points of M, and suppose that the power of K is less than β. Then the power of $M-K=L$ is β. Since L contains no limit point of itself of an order as great as β, it follows by (2) that there exists a subset H of L which is dense in L and has a power less than β. Then $H+K$ is dense in M and has a power less than β. Thus, M is semi-separable, and (2) implies (3a); (3b) is (3a) for the case $\beta=$ power of M.

Suppose that (4) does not hold. Then there exist two point sets, M and N, and two cardinals, β_1 and ϵ, such that $\alpha\leqq\beta_1=$ power of N, $\epsilon<\beta_1$, $M\supset N$, and N has no limit point in M of an order as great as ϵ. Let ϵ_1 be the smallest cardinal which is greater than ϵ, β be the larger of α and ϵ_1, and δ be the smallest ordinal in the number class $Z(\beta)$. Then β is regular and does not exceed β_1. Let F be a subset of N having the power β; since $N\supset F$, M contains no limit point of F of an order as great as ϵ. We shall define a well-ordered collection G of subsets of F such that the order type of G is δ: $G=(F_1, F_2, \cdots, F_\omega, \cdots, F_m, \cdots)$, where $m<\delta$.

Proceed as follows: Let T be a well-ordered sequence of all subsets of F, and F_0 be the null set. Suppose that m is an ordinal such that $m<\delta$ and F_x has been defined for all ordinals x which are less than m;

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[11] Cf., F. Hausdorff, loc. cit., pp. 124–125.

let $\Sigma_m = \sum F_x$, for $x < m$; let F_m be the first element of T which has a power less than β, which is a subset of $F - \Sigma_m$, and which is dense in $F - \Sigma_m$.

Suppose that the order type of G is λ, where $\lambda < \delta$. Since β is a regular cardinal, it follows that the power of Σ_λ is less than β, and that of $F - \Sigma_\lambda$ is β. Since F contains no limit point of itself of an order as great as ϵ, it follows from (2) that $F - \Sigma_\lambda$ is strongly β-separable. Then F_λ exists; and the supposition that λ exists involves a contradiction.

Let λ be the smallest ordinal in the number class $Z(\epsilon)$, $P \varepsilon F_\lambda$, and U be a neighborhood of P. Let x and y be ordinals such that $x < y \leqq \lambda$. Then $\Sigma_y \supset F_x$; since $F - \Sigma_y \supset F_y$, the product $F_x \cdot F_y$ is vacuous. Since $\Sigma_y \supset \Sigma_x$, $F - \Sigma_x \supset F - \Sigma_y \supset F_y$; then F_x is dense on F_y. Thus for $x < \lambda$ the product $U \cdot F_x$ is nonvacuous. Hence, the power of $U \cdot \Sigma_\lambda$ is at least as great as ϵ. Thus, we are involved in a contradiction, and (2) implies (4).

Property (1) implies (5a). Let M_b be the set of all β-isolated points of M, and suppose that its power exceeds β. Let β_1 be the smallest cardinal which is greater than β. Then β_1 is regular, and by (4) each subset of M_b having power β_1 contains a limit point of itself of order at least as great as β. Thus, we are involved in a contradiction; and (4) implies (5b), which implies (5c). Since (1) and (4) are equivalent, (4) implies (5).

THEOREM 2. *In a space V Fréchet in which the operation of derivation of point sets is distributive, the following are equivalent*:

(1) *Each uncountable set contains a limit point of itself.*

(2) *Each point set either contains a condensation point of itself, or it is separable.*

(3) *Each point set which has a regular power either is semi-separable, or it contains a complete limit point of itself.*

(4) *Each point set either is separable, or it has the same power as the set of all its condensation points which belong to it.*

(5) *Each point set which has a regular power either is semi-separable or it has the same power as the set of all its complete limit points which belong to it.*

(6) *Each point set is almost perfectly compact in itself.*

(7) *Part* (5) *of Theorem* 1 *holds for each transfinite cardinal* β *such that aleph-zero* $< \beta$.

PROOF. This may be established by letting the α of Theorem 1 be aleph-one. In particular, to establish (4) apply the following parts of Theorem 1: (a) Part (3b), if aleph-one = power of M; (b) part (5c), if aleph-one < power of M.

THEOREM 2A. *The following modification of Theorem* 1 *is permissible*: (I) *Allow* α *to be any regular cardinal which is greater than unity*; (II) *replace the property "M is strongly α-separable" by the following: "there exists N such that $\overline{N} \supset M \supset N$ and that the power of N is less than α."*

Note that the only finite cardinal which satisfies (I) is $\alpha = 2$. Except for obvious modifications, the proof is like that of Theorem 1. Note, however, the following: In considering the proof for (3) for the case power of $M = 2$, let $P \varepsilon M \cdot M'$; then $M - P$ is dense in M.

The following theorem gives relations between Appert's work and ours.

THEOREM 3. *Let* β *be transfinite,* α *be the cardinal which is next greatest to* β, *and* T *be a space V Fréchet. Each of the following implies property* (1) *of Theorem* 1.[12]

(1) *T is β-perfectly separable.*

(2) *Each point set in T is β-separable.*

(3) *Each point set in T possesses the Lindelöf property of order β.*

(4) *Each point set of T is β-condensed in itself.*[13]

Comment. If M is a point set in the space T of Theorem 1 and $\alpha \leqq \lambda <$ power of M, then by (4) of that theorem M contains a limit point of itself of an order at least as great as λ; that is, the upper bounds of the aggregates of the orders of the limit points which are involved in property (1) are the powers of the sets M concerned. Further, if (1) holds for a given cardinal, it and the properties equivalent to it hold for all greater cardinals. If α is an arbitrary cardinal which is greater than unity, whether finite or transfinite, the following example establishes the existence of a space T such that α is the smallest cardinal for which (1) holds relative to T.

Let α and β be cardinals such that $\alpha < \beta$; let $S_{\alpha\beta}$ be a space whose points are the elements of an aggregate, of power β, and in which the set U is a neighborhood of the point P if and only if the following conditions hold: (1) $P \varepsilon U$; (2) the power of the complement of U is less than α, and is zero if α is finite. The following are properties of this space: (A) if $\alpha \geqq$ aleph-one, no point set of power α or greater is strongly α-separable; (B) if $\alpha < \lambda$ and power $M = \lambda$, then M is strongly λ-separable; (C) if $\alpha \leqq$ power M, then each point of M is a complete limit point of M; thus property (1) of Theorem 1 holds; (D) if α is the smallest transfinite cardinal, the space is perfectly compact; (E) if α is transfinite, the space is a space H Fréchet; (F) the space satisfies the hypothesis of Theorem 1, for α any cardinal.[14] To establish (A) let power $M \geqq \alpha$, $M \supset N$, power $N < \alpha$, $P \varepsilon (M - N)$; then $P + (S_{\alpha\beta} - N)$ is a neighborhood of P, and P is not a limit point of N. To establish (B) let $M \supset N$, $\alpha <$ power M, $\alpha \leqq$ power $N <$ power M, $P \varepsilon (M - N)$, and U be a neighborhood of P; then power of $N \cdot (S_{\alpha\beta} - U) < \alpha$, and $N \cdot U$ is nonvacuous; thus P is a limit point of N and N is dense in M. A consideration of the property "almost perfect compactness" suggests the question: Does there exist a space in which each infinite point set contains a complete limit point of itself? F. B. Jones gave an answer in the affirmative by suggesting the example in (D). If α is finite, each nonvacuous point set in $S_{\alpha\beta}$ contains a complete limit point of itself.

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[12] The converse is not true. Note that in Theorem 1 we have the additional hypothesis that the operation of derivation of point sets be distributive.

[13] Property (1) means that T has a basis of a power that does not exceed β. Property (3) means that if a collection G of sets covers the point set M, then there exists H such that $G \supset H$, H covers M, and the power of H does not exceed β. The set M is β-condensed in itself provided that if α is the smallest cardinal which is greater than β, $M \supset K$ and power of $K = \alpha$, then there exists a point P of M such that if U is a neighborhood of P then α is the power of the product of K and the interior of U. Cf. Appert, loc. cit., pp. 82–88.

In Theorem 2, part (2), we consider two alternative properties, of which not more than one is required for each point set. For each of these properties Sierpiński[15] gave an example of a space in which each point set has the given property, but not the alternative property. By taking the sum of his two spaces we have one in which the property that holds for a point set varies with the set. It would be interesting to give an example of a space for which (2) holds and for which each point set that satisfies one of the alternative conditions but not the other has a subset relative to which the converse is true.

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[14] Cf. Appert, loc. cit., p. 25, and Fréchet, loc. cit., p. 186.

[15] Loc. cit., Fundamenta Mathematicae, vol. 2.

CESÀRO SUMMABILITY OF ORDINARY DOUBLE DIRICHLET SERIES

JOHN G. HERRIOT

1. **Introduction.** The purpose of this paper is to obtain some results in the Cesàro summability of ordinary double Dirichlet series similar to those obtained by H. Bohr[1] for the simple series. As is well known a double sequence $\{S_{mn}\}$ may tend to a finite limit as $m, n\to\infty$[2] without S_{mn} being a bounded function of m and n. In order to avoid difficulties in this respect and to obtain results analogous to those for simple series, the discussion will usually be restricted to bounded sequences.

Let $\sum_{m,n=1}^{\infty} u_{mn}$ be a double series of constant terms. Set

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\begin{aligned}
&S_{mn}^{00} = \sum_{i=1,j=1}^{m,n} u_{ij}, \qquad S_{mn}^{10} = \sum_{i=1}^{m} S_{in}^{00}, \qquad S_{mn}^{01} = \sum_{j=1}^{n} S_{mj}^{00}, \\
&S_{mn}^{rs} = \sum_{i=1}^{m} S_{in}^{r-1,s} = \sum_{j=1}^{n} S_{mj}^{r,s-1}, \qquad m, n, r, s = 1, 2, 3, \cdots .
\end{aligned} \tag{1}
$$

The double series is said to be summable (C, r, s) with summability value S if $S_{mn}^{rs}r!s!/m^r n^s \to S$ as $m, n\to\infty$. It is bounded (C, r, s) if $S_{mn}^{rs}r!s!/m^r n^s$ is bounded for all m and n. It is summable-bounded (C, r, s) if it is both summable (C, r, s) and bounded (C, r, s). In case the u_{mn} are functions of complex variables x and y similar definitions can be set up for uniform summability, uniform boundedness, and uniform summability-boundedness (C, r, s).

G. M. Merriman[3] has given the definition of summability (C, r, s) of a double series in a slightly different form. But our means differ from his only by a factor which is bounded and tends to one as $m, n\to\infty$. Consequently the two definitions are equivalent.

By means of Robison's[4] generalization of the Silverman-Toeplitz theorem, it is easily shown that summability-boundedness $(C, r-1, s)$ (or $(C, r, s-1)$) implies the same (C, r, s).

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[1] Bohr, H., *Bidrag til de Dirichlet'ske Raekkers Theori*, Dissertation, Copenhagen, 1910; *Über die Summabilität Dirichletscher Reihen*, Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, 1909, pp. 247–262; *Sur la série de Dirichlet*, Comptes Rendus de l'Académie des Sciences, Paris, vol. 148 (1909), pp. 75–80.

[2] Throughout this paper $m, n\to\infty$ means m and n tend to infinity simultaneously but independently.

[3] Merriman, G. M., *A set of necessary and sufficient conditions for the Cesàro summability of double series*, Annals of Mathematics, (2), vol. 29 (1928), pp. 343–354.

[4] Robison, G. M., *Divergent double sequences and series*, Transactions of this Society, vol. 28 (1926), pp. 50–73 (p. 53).

In §2 of the present paper we give some general lemmas on summable series which are mostly generalizations of theorems of Bohr. These are used in §3 to deduce a number of interesting theorems on summability of ordinary double Dirichlet series. We define associated abscissas of summability-bounded (C, r, s) and in §4 give some theorems showing relations between them.

2. **Preliminary lemmas.** The following lemmas will be useful.

LEMMA 1. *Let* $\sum_{m,n=1}^{\infty} u_{mn}(x, y)$ *be uniformly summable* (C, r, s) *for* x *and* y *in every* D_1' *and* D_2' *respectively for which* D_1' *and* D_2' *are closed and contained in the domains* D_1 *and* D_2 *respectively, and let* $u_{mn}(x, y)$ $(m, n=1, 2, 3, \cdots)$ *be regular analytic functions of* x *and* y *for* x *in* D_1 *and* y *in* D_2. *Then the series represents by its summability value* $S(x, y)$ *an analytic function regular for* x *in* D_1, y *in* D_2. *Moreover the series may be differentiated partially termwise arbitrarily often for* x *in* D_1, y *in* D_2. *The series*

$$\sum_{m=1,n=1}^{\infty,\infty} \frac{\partial^{p+q} u_{mn}(x, y)}{\partial x^p \partial y^q}$$

is summable (C, r, s) *for* x *in* D_1, y *in* D_2, *and indeed uniformly for* x *and* y *in every* D_1' *and* D_2' *respectively, to the function* $\partial^{p+q}S(x, y)/\partial x^p \partial y^q$ $(p, q=0, 1, 2, \cdots)$.

LEMMA 2. *Let* $F_{mn}(x, y)=\sum_{i=1}^{c} v_i(x)\gamma_{mni}(y)$, *where* $c=c(m)$ *is a function of* m *defined as a positive integer for positive integral* m, *and which increases monotonely with* m, *and let the following four conditions be satisfied*:

(A) $|v_i(x)| \leqq v_i$, *x in domain* D_1 $(i=1, 2, 3, \cdots)$;
(B) $\sum_{i=1}^{\infty} |v_i(x)|$ *converges uniformly in* D_1;
(C) $|\gamma_{mni}(y)| \leqq K$ *(constant)*, *y in domain* D_2 $(m, n, i=1, 2, 3, \cdots)$;
(D) $\lim_{m,n\to\infty} \gamma_{mni}(y)=\gamma_i(y)$ *uniformly in* D_2 $(i=1, 2, 3, \cdots)$.

Then $F_{mn}(x, y)$ *is uniformly bounded for* x *in* D_1, y *in* D_2 *for all* m *and* n, *and, as* $m, n\to\infty$, *tends uniformly to the sum of the uniformly absolutely convergent series*

$$\sum_{i=1}^{\infty} v_i(x)\gamma_i(y).$$

If the v_i and γ_{mni} are constants, a similar lemma can be stated omitting all references to uniformity and dropping (A).

LEMMA 3. *Let* $F_{mn}(x, y)=\sum_{i,j=1}^{c,d} v_i(x)w_j(y)\gamma_{mnij}$, *where* $c=c(m)$ *and*

$d=d(n)$ are functions of m and n, respectively, like $c(m)$ of Lemma 3, and let the following conditions be satisfied:

(A$_1$) $|v_i(x)| \leqq v_i$, *x in domain D_1* $(i=1, 2, 3, \cdots)$;
(A$_2$) $|w_j(y)| \leqq w_j$, *y in domain D_2* $(j=1, 2, 3, \cdots)$;
(B$_1$) $\sum_{i=1}^{\infty}|v_i(x)|$ *converges uniformly in* D_1;
(B$_2$) $\sum_{j=1}^{\infty}|w_j(y)|$ *converges uniformly in* D_2;
(C) $|\gamma_{mnij}| \leqq K$ *(constant)* $(m, n, i, j=1, 2, 3, \cdots)$;
(D) $\lim_{m,n\to\infty}\gamma_{mnij}=\gamma_{ij}$ $(i, j=1, 2, 3, \cdots)$.

Then $F_{mn}(x, y)$ is uniformly bounded for x in D_1, y in D_2 for all m and n, and, as $m, n\to\infty$, tends uniformly to the sum of the uniformly absolutely convergent series $\sum_{i,j=1}^{\infty}v_i(x)w_j(y)\gamma_{ij}$.

Again if the v_i and w_j are constants a similar lemma can be stated omitting all references to uniformity and dropping (A).

We may omit the proofs of these lemmas since they are analogous to those of Bohr.[5]

LEMMA 4. *Let $\sum_{m,n=1}^{\infty}u_{mn}$ be bounded (C, r, s) and let $\{\alpha_m(x)\}$ and $\{\beta_n(y)\}$ be sequences of functions of the complex variables x and y which satisfy the conditions:*

(A$_1$) $|\alpha_m(x)| \leqq K_m$, *x in domain D_1* $(m=1, 2, 3, \cdots)$;
(A$_2$) $|\beta_n(y)| \leqq K_n$, *y in domain D_2* $(n=1, 2, 3, \cdots)$;
(B$_1$) $\sum_{m=1}^{\infty}m^{p-1}|\Delta^p\alpha_m(x)|$ *converges uniformly*[6] *in* D_1 $(p=1, 2, 3, \cdots, r+1)$;
(B$_2$) $\sum_{n=1}^{\infty}n^{q-1}|\Delta^q\beta_n(y)|$ *converges uniformly in* D_2 $(q=1, 2, 3, \cdots, s+1)$;
(C$_1$) $\lim_{m\to\infty}\alpha_m(x)=0$, *$x$ in* D_1;
(C$_2$) $\lim_{n\to\infty}\beta_n(y)=0$, *$y$ in* D_2.

Then the series $\sum_{m,n=1}^{\infty}u_{mn}\alpha_m(x)\beta_n(y)$ is uniformly summable-bounded (C, r, s) for x in D_1, y in D_2.

If the α_m and β_n are constants, a similar lemma can be stated omitting all references to uniformity and dropping (A).

For simplicity of notation we shall indicate the proof of this simpler lemma. Let T_{mn}^{rs} be formed from $\sum_{m,n=1}^{\infty}u_{mn}\alpha_m\beta_n$, S_{mn}^{rs} from $\sum_{m,n=1}^{\infty}u_{mn}$. First we obtain the following relation between T_{mn}^{rs} and S_{mn}^{rs}:

$$(2) \qquad T_{mn}^{rs} = \sum_{p=0,q=0}^{r+1,s+1} C_{r+1,p}C_{s+1,q}P_{r,s,p,q,m,n},^{7}$$

where

---

[5] Bohr, loc. cit., Dissertation, pp. 53–60.

[6] $\Delta^p\alpha_m=\alpha_m-p\alpha_{m+1}+\frac{1}{2}p(p-1)\alpha_{m+2}-\cdots+(-1)^p\alpha_{m+p}$.

[7] $C_{r,p}$ denotes the binomial coefficient $r!/p!(r-p)!$.

$$
\begin{aligned}
&P_{r,s,0,0,m,n} = S^{rs}_{mn}\alpha_m\beta_n,\\
&P_{r,s,p,0,m,n} = \sum_{i=1}^{m-p} S^{rs}_{in}C_{m-i-1,p-1}\Delta^p\alpha_i\beta_n,\ p = 1, 2, 3, \cdots, r+1,\\
(3)\qquad &P_{r,s,0,q,m,n} = \sum_{j=1}^{n-q} S^{rs}_{mj}C_{n-j-1,q-1}\alpha_m\Delta^q\beta_j,\ q = 1, 2, 3, \cdots, s+1,\\
&P_{r,s,p,q,m,n} = \sum_{i=1,j=1}^{m-p,n-q} S^{rs}_{ij}C_{m-i-1,p-1}C_{n-j-1,q-1}\Delta^p\alpha_i\Delta^q\beta_j,\\
&\qquad p = 1, 2, 3, \cdots, r+1;\ q = 1, 2, 3, \cdots, s+1;
\end{aligned}
$$

if $m \leqq p$ or $n \leqq q$ or both, then $P_{r,s,p,q,m,n} = 0$.

The proof is by induction on r and s successively making use of the relation (for induction on r)

$$
(4)\qquad \sum_{i=p+1}^{m} P_{r,s,p,q,i,n} = P_{r+1,s,p,q,m,n} + P_{r+1,s,p+1,q,m,n}.
$$

Next we evaluate $\lim_{m,n\to\infty} T^{rs}_{mn}r!s!/m^r n^s$. By hypothesis

$$
\left| S^{rs}_{mn}r!s!/m^r n^s \right| \leqq K, \qquad m, n = 1, 2, 3, \cdots.
$$

Considering separately the various terms of (2) multiplied by $r!s!/m^r n^s$, we can show that all terms except that for which $p = r+1$, $q = s+1$ tend to zero and are bounded. For example, in case $p = 1, 2, 3, \cdots, r+1$, and $q = 0$, we use Lemma 2 (modified) choosing

$$
c = m - p, \quad v_i = i^{p-1}\Delta^p\alpha_i
$$

and taking up the remaining factors in γ_{mni}. Then $\gamma_{mni}\to 0$ as $m, n\to\infty$ and conditions (B), (C), and (D) of Lemma 2 are satisfied. In case $p = 1, 2, 3, \cdots, r+1$ and $q = 1, 2, 3, \cdots, s+1$, we apply Lemma 3 (modified). The term for which $p = r+1$, $q = s+1$ is thus seen to be bounded and to tend to a finite limit not necessarily zero as $m, n\to\infty$. This completes the proof of the (modified) lemma.

3. **Double Dirichlet series.** We now make use of the lemmas of §2 to prove some theorems concerning the summability of the ordinary double Dirichlet series

$$
(5)\qquad \sum_{m=1,n=1}^{\infty,\infty} a_{mn}/m^x n^y.
$$

THEOREM 1. *If the double Dirichlet series* (5) *is bounded* (C, r, s) *for* *the place* (x_0, y_0), *then it is summable-bounded* (C, r, s) *for every place* (x, y) *for which*[8] $R(x) > R(x_0)$ *and* $R(y) > R(y_0)$.[9]

PROOF. Let $x = x_0 + \delta$, $y = y_0 + \eta$, $R(\delta) > 0$, $R(\eta) > 0$. We apply Lemma 4 (modified) taking

$$u_{mn} = a_{mn}/m^{x_0}n^{y_0}, \quad \alpha_m = 1/m^\delta, \quad \beta_n = 1/n^\eta, \quad u_{mn}\alpha_m\beta_n = a_{mn}/m^x n^y.$$

Then $\lim_{m\to\infty} \alpha_m = \lim_{n\to\infty} \beta_n = 0$. Also, using the method of Bohr,[10]

$$\Delta^p \alpha_m = \delta(\delta+1) \cdots (\delta+p-1) \int_m^{m+1} da_1 \int_{a_1}^{a_1+1} da_2 \cdots$$
$$\cdot \int_{a_{p-2}}^{a_{p-1}+1} da_{p-1} \int_{a_{p-1}}^{a_{p-1}+1} \frac{da_p}{a_p^{p+\delta}} \cdot$$

Hence $m^{p-1}|\Delta^p \alpha_m| \leqq |\delta|\,|\delta+1| \cdots |\delta+p-1| \cdot 1/m^{1+R(\delta)}$ $(p = 1, 2, 3, \cdots, r+1)$, so that condition (B_1) of Lemma 4 is satisfied. Likewise (B_2) is satisfied. By the remark following Lemma 4, Theorem 1 follows.

Even if (5) is summable-bounded (C, r, s) for the place (x_0, y_0), it does not necessarily follow that it is summable-bounded (C, r, s) for every place (x_0, y) for which $R(y) > R(y_0)$, or for every place (x, y_0) for which $R(x) > R(x_0)$. For let $r = s = 0$ and let $a_{11} = -1$, $a_{1n} = 1/n(n-1)$ $(n = 2, 3, 4, \cdots)$, $a_{m1} = 2(-1)^m$ $(m = 2, 3, 4, \cdots)$, $a_{mn} = 2(-1)^{m-1}/n(n-1)$ $(m, n = 2, 3, 4, \cdots)$. Then $S_{mn}^{00} = (-1)^m/n$ $(m, n = 1, 2, 3, \cdots)$. Hence for $x = y = 0$ the series (5) is convergent-bounded with sum zero. But for $x = 0$, $y = 1$ the series fails to converge for

$$T_{mn}^{00} = \sum_{i=1, j=1}^{m,n} \frac{a_{ij}}{i^0 j^1} = (-1)^m \sum_{j=1}^{n-1} \frac{1}{j^2(j+1)} + \frac{(-1)^m}{n^2},$$

which tends to no limit as $m, n \to \infty$.

From Theorem 1 we deduce the existence of associated abscissas of summability-bounded (C, r, s), that is, numbers λ_r, μ_s such that (5) is summable-bounded (C, r, s) in the associated domains $R(x) > \lambda_r$, $R(y) > \mu_s$ and not summable-bounded (C, r, s) in the associated domains $R(x) < \lambda_r$, $R(y) < \mu_s$.

---

[8] $R(x)$ denotes the real part of x.

[9] This theorem and Theorems 3 and 7 bear some resemblance to but are not the same as Theorem IV of Merriman, *Concerning the summability of double series of a certain type*, Annals of Mathematics, (2), vol. 28 (1927), pp. 515–533. Cf. M. Gurney, *Cesàro summability of double series*, this Bulletin, vol. 38 (1932), pp. 825–827.

[10] Bohr, loc. cit., Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, pp. 248–249.

The question arises whether the series (5) has associated abscissas of summability (C, r, s) (the condition of boundedness not being imposed), that is, whether there exists a pair of numbers λ_r', μ_s' such that the series is summable (C, r, s) at all (x, y) for which $R(x) > \lambda_r'$, $R(y) > \mu_s'$, and is not summable (C, r, s) at any (x, y) for which $R(x) < \lambda_r'$, $R(y) < \mu_s'$. The answer is not in general in the affirmative. The following is an example of a series which is convergent but unbounded at $x = y = 0$, but at all other places it fails to converge and at *all* places it is not summable (C, r, s) for any positive integral r and s. Let $a_{1n} = -a_{2n} = n^n$ $(n = 1, 2, 3, \cdots)$, $a_{m1} = -a_{m2} = m^m$ $(m = 3, 4, 5, \cdots)$, $a_{mn} = 0$ $(m, n = 3, 4, 5, \cdots)$. Then

$$S_{mn}^{00} = (1 - 2^{-x}) \sum_{j=1}^{n} j^{j-y} + (1 - 2^{-y}) \sum_{i=3}^{m} i^{i-x}, \qquad m, n = 3, 4, 5, \cdots .$$

If $x = y = 0$, these S_{mn}^{00} are all 0, but otherwise the required limit does not exist. The evaluation of S_{mn}^{rs} is too involved to give here, but for all $r > 0$, $s > 0$, all (x, y), $S_{mn}^{rs} r!s!/m^r n^s$ tends to no finite limit as $m, n \to \infty$.

That such behavior is exceptional is shown by

THEOREM 2. *If the double Dirichlet series* (5) *is summable* (C, r, s) *in a domain* D *of the two complex variables* x *and* y, *it is bounded* (C, r, s) *at each point of this domain.*

Leja[11] has given a similar theorem concerning the convergence of general double Dirichlet series. Our theorem may be proved by a method analogous to his. The first step is to show that if the series (5) is summable (C, r, s) at the points (x_0, y), where x_0 is fixed and y runs over a plane domain d, then, for $j = 1, 2, 3, \cdots$, the series $\sum_{i=1}^{\infty} a_{ij}/i^{x_0}$ are bounded (C, r).

THEOREM 3. *A double Dirichlet series* (5) *having a pair of associated abscissas of summability-bounded* (C, r, s) λ_r, μ_s *is uniformly summable-bounded* (C, r, s) *for* x *and* y *in every pair of associated regions* D_1 *and* D_2 *defined by the relations* D_1: $R(x) \geqq \lambda_r + \epsilon$, $|x| \leqq E$; D_2: $R(y) \geqq \mu_s + \epsilon$, $|y| \leqq E$, ϵ *and* E *being arbitrary finite positive numbers.*

PROOF. Let $x_0 = \lambda_r + \epsilon/2$, $y_0 = \mu_s + \epsilon/2$, and let x be in D_1, y in D_2. The proof is now similar to that of Theorem 1 except that we apply Lemma 4 as stated.

THEOREM 4. *If* λ_r, μ_s *are a pair of associated abscissas of summabil-*

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[11] Leja, F., *Sur les séries de Dirichlet doubles*, Comptes-Rendus du 1mier Congrès des Mathématiciens des Pays Slaves, Warsaw, 1929, pp. 140–158.

-ity-bounded (C, r, s) *of the double Dirichlet series*

$$\sum_{m=2,n=2}^{\infty,\infty} a_{mn}/m^x n^y, \tag{6}$$

then λ_r, μ_s *are also a pair of associated abscissas of summability-bounded* (C, r, s) *of*

$$\sum_{m=2,n=2}^{\infty,\infty} a_{mn}(\log m)^\alpha (\log n)^\beta / m^x n^y = \sum_{m=2,n=2}^{\infty,\infty} b_{mn}/m^x n^y, \tag{7}$$

where α *and* β *are arbitrary complex numbers, and conversely.*

PROOF. Clearly it suffices to show that, if (6) is summable-bounded (C, r, s) for (x_0, y_0), then (7) is summable-bounded (C, r, s) at every (x, y) for which $R(x) > R(x_0)$ and $R(y) > R(y_0)$. To prove this we take $x = x_0 + \delta$, $y = y_0 + \eta$, $R(\delta) > 0$, $R(\eta) > 0$, and apply Lemma 4 (modified) with

$$u_{mn} = a_{mn}/m^{x_0} n^{y_0}, \qquad \alpha_m = (\log m)^\alpha / m^\delta, \qquad \beta_n = (\log n)^\beta / n^\eta,$$
$$u_{mn}\alpha_m\beta_n = b_{mn}/m^x n^y.$$

Then $\sum_{m,n=2}^{\infty} u_{mn}$ is summable-bounded (C, r, s) and condition (C) is satisfied. Again, using the method of Bohr[12]

$$\Delta^p \alpha_m = (-1)^p \int_m^{m+1} da_1 \int_{a_1}^{a_1+1} da_2 \cdots$$
$$\cdot \int_{a_{p-2}}^{a_{p-2}+1} da_{p-1} \int_{a_{p-1}}^{a_{p-1}+1} \frac{d^p}{da_p^p} \frac{(\log a_p)^\alpha}{a_p^\delta}\, da_p,$$
$$p = 1, 2, 3, \cdots, r+1.$$

Now the integrand in the last integral is easily shown to be the sum of a finite number of terms of the form a constant times $(\log a_p)^\gamma / a_p^{p+\delta}$. Also $(\log a_p)^\gamma / a_p^{R(\delta)/2}$ is bounded for $a_p \geqq 2$. Hence $m^{p-1} |\Delta^p \alpha_m|$ is not greater than a constant times $1/m^{1+R(\delta)/2}$, so that condition (B_1) and similarly (B_2) of Lemma 4 are satisfied and the desired result follows at once.

THEOREM 5. *A series* (5) *having a pair of associated abscissas of summability-bounded* (C, r, s) λ_r, μ_s *represents by its summability value an analytic function* $f(x, y)$ *regular in the associated half-planes* $R(x) > \lambda_r$, $R(y) > \mu_s$. *The series may be differentiated partially termwise arbitrarily often in these half-planes, that is, the double Dirichlet series*

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[12] Bohr, loc. cit., Dissertation, p. 83.

$$\sum_{m=1,n=1}^{\infty,\infty} a_{mn}(-\log m)^p(-\log n)^q/m^x n^y$$

is again summable-bounded (C, r, s) *in the same associated half-planes to the function* $\partial^{p+q}f(x, y)/\partial x^p\partial y^q$ $(p, q=0, 1, 2, \cdots)$.

This theorem follows immediately from Theorems 3 and 4 and Lemma 1.

A theorem similar to Theorem 5 can be stated and proved for termwise integration of (5) within its domain of summability-bounded.

THEOREM 6. *The analytic function defined by the summability value* (C, r, s) *of* (5) *forms the analytic continuation of the analytic function defined by the sum of the series in a pair of associated half-planes,* $R(x)>\lambda_0$, $R(y)>\mu_0$, *of convergence-bounded* (*provided that* λ_0, μ_0 *are not also associated abscissas of summability-bounded* (C, r, s)).

This is clear from the consistency of the Cesàro mean transformation.

THEOREM 7. *If the series* (5) *is bounded* $(C, r+1, s)$ *for* (x_0, y_0), *then it is summable-bounded* (C, r, s) *for every* (x, y) *for which* $R(x)>R(x_0)+1$ *and* $R(y)>R(y_0)$.

PROOF. Let $x=x_0+1+\delta$, $y=y_0+\eta$, $R(\delta)>0$, $R(\eta)>0$. Set

$$u_{mn} = a_{mn}/m^{x_0}n^{y_0}, \quad \alpha_m = 1/m^{1+\delta}, \quad \beta_n = 1/n^{\eta}, \quad u_{mn}\alpha_m\beta_n = a_{mn}/m^x n^y.$$

Let S^{rs}_{mn} be formed from $\sum_{m,n=1}^{\infty} u_{mn}$, T^{rs}_{mn} from (5). Then

$$\left| S^{r+1,s}_{mn}(r+1)!s!/m^{r+1}n^s \right| \leqq K, \qquad m, n = 1, 2, 3, \cdots.$$

Since $S^{rs}_{mn} = S^{r+1,s}_{mn} - S^{r+1,s}_{m-1,n}$, we have

$$\left| \frac{S^{rs}_{mn} r!s!}{m^{r+1}n^s} \right| \leqq \left| \frac{(|S^{r+1,s}_{mn}| + |S^{r+1,s}_{m-1,n}|)(r+1)!s!}{m^{r+1}n^s(r+1)} \right| \leqq 2K/(r+1),$$
$$m, n = 1, 2, 3, \cdots.$$

We have to prove that $T^{rs}_{mn} r!s!/m^r n^s$ is bounded for all m and n and tends to a finite limit as $m, n \to \infty$. The proof is now similar to that of Lemma 4 except that in applying Lemmas 2 and 3 we take $v_i = i^p\Delta^p\alpha_i$ instead of $i^{p-1}\Delta^p\alpha_i$.

COROLLARY. *If* λ_r, μ_s *are a pair of associated abscissas of summability-bounded* (C, r, s) *of* (5), *there exists a pair of associated abscissas of summability-bounded* $(C, r+1, s)$ λ_{r+1}, μ_s *such that* $0 \leqq \lambda_r - \lambda_{r+1} \leqq 1$.

4. **Associated abscissas of summability-bounded.** Leja[13] has given a number of theorems showing the relationship between the associated abscissas of convergence-bounded of general double Dirichlet series. We now generalize the content of these theorems for the case of ordinary double Dirichlet series to obtain relations between the associated abscissas of summability-bounded (C, r, s) of the series (5). For simplicity of notation we set

$$A = \limsup_{m+n\to\infty} \frac{\log \left| S^{rs}_{mn} r!s!/m^r n^s \right|}{\lambda_r \log m + \mu_s \log n},$$

where S^{rs}_{mn} is formed from $\sum_{m,n=1}^{\infty} a_{mn}$.

THEOREM 8. *If* $\lambda_r > 0$ *and* $\mu_s > 0$ *and if* $A \leqq 1$, *then the series* (5) *is summable-bounded* (C, r, s) *for every* (x, y) *for which* $R(x) > \lambda_r$ *and* $R(y) > \mu_s$.

PROOF. Let $x = \lambda_r + \delta$, $y = \mu_s + \eta$, $R(\delta) > 0$, $R(\eta) > 0$. Choose $\epsilon > 0$ but less than the minimum of $R(\delta)/2\lambda_r$ and $R(\eta)/2\mu_s$. Then there exists $K(\epsilon)$ such that

$$\left| S^{rs}_{mn} r!s!/m^r n^s \right| < K m^{\lambda_r(1+\epsilon)} n^{\mu_s(1+\epsilon)}, \qquad m, n = 1, 2, 3, \cdots .$$

Form T^{rs}_{mn} from (5) and set

$$u_{mn} = a_{mn}, \qquad \alpha_m = 1/m^x, \qquad \beta_n = 1/n^y, \qquad u_{mn}\alpha_m\beta_n = a_{mn}/m^x n^y.$$

It suffices to prove $T^{rs}_{mn} r!s!/m^r n^s$ is bounded for all m and n and tends to a finite limit as $m, n \to \infty$. The proof is now similar to that of Lemma 4 except that in applying Lemmas 2 and 3 we take $v_i = i^{p-1}\Delta^p \alpha_i \, i^{\lambda_r + R(\delta)/2}$ instead of $i^{p-1}\Delta^p\alpha_i$.

THEOREM 9. *If* $\lambda_r > 0$ *and* $\mu_s > 0$ *and if the series* (5) *is summable-bounded* (C, r, s) *for all* (x, y) *for which* $R(x) > \lambda_r$ *and* $R(y) > \mu_s$, *then* $A \leqq 1$.

PROOF. Let any $\epsilon > 0$ be assigned. Then (5) is summable-bounded (C, r, s) for $x = \lambda_r(1 + \epsilon/2)$, $y = \mu_s(1 + \epsilon/2)$. Set

$$u_{mn} = a_{mn}/m^x n^y, \qquad \alpha_m = m^x, \qquad \beta_n = n^y, \qquad u_{mn}\alpha_m\beta_n = a_{mn}.$$

Forming S^{rs}_{mn} from $\sum_{m,n=1}^{\infty} a_{mn}$ and T^{rs}_{mn} from $\sum_{m,n=1}^{\infty} u_{mn}$ and interchanging T^{rs}_{mn} and S^{rs}_{mn} in (2) we obtain finally

$$\left| \frac{S^{rs}_{mn} r!s!}{m^r n^s} \right| \leqq \frac{r!s!}{m^r n^s} \sum_{p=0,\,q=0}^{r+1,\,s+1} C_{r+1,p} C_{s+1,q} \left| P_{r,s,p,q,m,n} \right|, \tag{8}$$

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[13] Leja, F., loc. cit.

where now the $P_{r,s,p,q,m,n}$ involve the T_{ij}^{rs} in place of the S_{ij}^{rs}. Since $|T_{mn}^{rs}r!s!/m^rn^s| \leqq K$ (constant) ($m, n=1, 2, 3, \cdots$), it is easily shown that each term of the right member of (8) is dominated by $K'm^xn^y$.[14] Hence

$$|S_{mn}^{rs}r!s!/m^rn^s| \leqq K''m^xn^y = K''m^{\lambda_r(1+\epsilon/2)}n^{\mu_s(1+\epsilon/2)}.$$

Consequently there exists N such that for all m, n for which $m+n>N$

$$|S_{mn}^{rs}r!s!/m^rn^s| \leqq m^{\lambda_r(1+\epsilon)}n^{\mu_s(1+\epsilon)},$$

from which the theorem follows.

THEOREM 10. *A necessary and sufficient condition that* λ_r, μ_s (>0) *be a pair of associated abscissas of summability-bounded* (C, r, s) *of the series* (5) *is that* $A=1$.

This theorem is an immediate consequence of Theorems 8 and 9.

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[14] The proof of this is analogous to that of Bohr, loc. cit., Dissertation, pp. 92–93.

A CHARACTERIZATION OF EUCLIDEAN SPACES

R. S. PHILLIPS

The purpose of this paper is to give an elementary proof of the fact that a Banach space in which there exist projection transformations of norm one on every two-dimensional linear subspace is a euclidean space. S. Kakutani [1] has pointed out that a modification of a proof due to Blaschke [2] will prove this theorem. F. Bohnenblust has been able to establish this theorem for the complex case by still another method.[1]

A Banach space is a linear, normed, complete space [3, chap. 5]. A euclidean space of dimension α, where α is any cardinal number, is defined to be the Banach space of sequences x_ν of real numbers where ν ranges over a class of cardinal number α, and $\sum x_\nu^2$ is finite and equal to the square of the norm [4]. We consider only spaces having at least three linearly independent elements.

P. Jordan and J. von Neumann have shown [5] that a Banach space which is euclidean in every two-dimensional linear subspace is itself a euclidean space. It is thus sufficient to show that the "unit sphere" S for any three-dimensional linear subspace is an ellipsoid.

Because of the norm properties, S is a convex body symmetric about the origin o, and contains o as an interior point. Let γ be a plane containing o and let C_γ be the curve of intersection of γ and the boundary S' of S. The existence for each γ of a projection operation of norm one, whose direction of projection is that of the unit vector v_γ, implies that the cylinder generated by lines of direction v_γ tracing C_γ contains S. Our theorem is therefore an immediate consequence of the following lemma on convex bodies (which need not be symmetric about o).

LEMMA.[2] *If S is a convex body such that for every γ there exists a cylinder generated by C_γ containing S, then S is an ellipsoid.*

We topologize the planes γ by representing each by its direction cosines as a point on the unit sphere and using the usual topology of the unit sphere.

The proof of the lemma is divided into two parts. We first show that v_γ is uniquely determined by γ, that v_γ is a continuous function of γ, and that S' has a tangent plane at each of its points. It is then easy to demonstrate that the curves of intersection for any set of parallel planes are similar. Finally we prove that such a convex body is an ellipsoid.

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[1] F. Bohnenblust's result is not yet published.

[2] W. Blaschke has proved a similar theorem under the assumption that there exists a tangent plane at each point of S' [2].

We state without proof the following elementary propositions about boundaries of convex bodies. S' and C_γ are homeomorphs of the two-sphere and the one-sphere respectively. At each point of C_γ there exist two one sided tangents. The right (or left) sided tangents at points q approach the right (or left) sided tangents at p as q approaches p from the right (or left). Furthermore C_γ has a tangent at all except a denumerable set of its points. Given any one sided tangent t at p to C_γ, there exists a plane of support at p containing[3] t. Finally given a tangent t at p to C_γ there exist two half tangent planes to S' at p each containing t; that is, for any $\beta \neq \gamma$ containing the line $\overline{op}$, each one sided half tangent to C_β at p is contained in one of two half-planes, whose common bound is[4] t.

Suppose C_γ has a tangent t at p and that P_1, P_2 are the half tangent planes at p to S'. P_1 and P_2 may or may not be distinct. Let $d(x)$ be the distance from $x \varepsilon S'$ to the closest of P_1 and P_2, and let $r(x)$ be the distance from $x \varepsilon S'$ to p. The convexity of S then implies that $d(x) \leqq e(r) \cdot r(x)$ where $e(r) \rightarrow 0$ with r. For a $\beta \neq \gamma$ containing the line $\overline{op}$, we consider any $x \varepsilon C_\beta$ and denote by P_x the plane P_1 or P_2 from which $d(x)$ is measured. Then in order that the line through x in the direction of v_β does not pierce S', it is necessary that the component of v_β perpendicular to P_x be less than $2 \cdot e(2r(x))/\sin(\gamma, \beta)$ which approaches zero as $x \rightarrow p$. Hence $p + v_\beta$ is contained in both P_1 and P_2.

It is now easy to show that v_γ is uniquely determined by γ. Suppose two such directions of projection existed and let p be a point of tangency on C_γ with the tangent t. Then there exist two distinct half tangent planes containing t as their common line. The above argument shows that for any β containing $\overline{op}$, $p + v_\beta = t$. But this is impossible for a convex body with interior points.

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[3] Given a one-sided tangent t at p to C_γ, there exists a plane of support at p containing t. C_γ is convex. Therefore C_γ lies entirely on one side of any plane containing t. If the theorem were false, then we could find a plane containing t and interior points of S on both sides of the line through t. The convex extension of some neighborhoods about these points belongs to S and intersects γ on both sides of the line through t, which is impossible.

[4] Given a tangent t at p to C_γ, there exist two half tangent planes at S' at p each containing t. If the contrary were true, there would exist a β containing the line $\overline{op}$ such that a half tangent t' to C_β at p did not determine a plane of support with t at p. The plane through t, t' therefore contains an interior point of S. Again the convex extension of some neighborhood of such a point and C_γ belongs to S and intersects β on both sides of the line through t', which is impossible.

If $v_{\gamma_n} \to v_0$ as $\gamma_n \to \gamma$, then v_0 defines a direction of projection for γ. If the contrary were true, there would exist $p \varepsilon C_\gamma$ and a real number λ such that $q = p + \lambda v_0$ is an interior point of S. Let β be the plane determined by o, p, q and p_n the point of $C_{\gamma_n} \cdot C_\beta$ closest to p. Then, as $n \to \infty$, $p_n \to p$ and the points not interior to S, $p_n + \lambda v_{\gamma_n} \to q$, which is impossible. By the uniqueness, $v_0 = v_\gamma$. Since the v_γ form a compact set, it follows that v_γ is a continuous function of γ.

We next show that S' has a tangent plane at each of its points. Let $p \varepsilon S'$, γ contain $\overline{op}$, and let t be, say, the right sided tangent at p to C_γ. Then there exist points $p_n \varepsilon C_\gamma$ which approach p from the right and which have tangents t_n to C_γ. Let us choose a subsequence for which a set of half tangent planes $P_{n'}$ converge. For convenience we renumber this subsequence $1, 2, \cdots, n, \cdots$. Suppose $P_n \to P$. Then P contains t. Finally for any $\beta \neq \gamma$ containing $\overline{op}$, let us choose a sequence β_n containing $\overline{op_n}$ such that $\beta_n \to \beta$. As above, $p_n + v_{\beta_n}$ lies in P_n and since $v_{\beta_n} \to v_\beta$, $p + v_\beta$ lies in P. As v_β does not lie in β, it follows that the v_β for β containing $\overline{op}$ determine P. But t was an arbitrary one-sided tangent at p. Hence P contains all one sided tangents to curves C_β (β containing $\overline{op}$) and is therefore the tangent plane to S' at p.

We now define any directed line through o to be the z axis. The x-y plane is then the plane containing o which is parallel to the tangent plane to S' at the intersection p of S' and the z axis. In a system of cylindrical coordinates, let γ_θ be the plane $\theta = \text{const}$. Then v_{γ_θ} lies in the x-y plane. The curve of intersection C_z of S' with the plane $z = \text{const.}$ is defined by the differential equation

$$dr/d\theta = rF(\theta), \qquad r(0) = f(z)$$

where $F(\theta)$ is a continuous function independent of z. S' is therefore expressible in the form $r = f(z) \cdot g(\theta)$. Clearly the C_{γ_θ} differ only by a linear transformation.

We next prove that C_γ is an ellipse.[5] For this we need to know that there exists a linear orientation-preserving transformation sending C_{γ_0} into itself and p into any other point q of C_{γ_0}.

Let r be the point of tangency of a plane parallel to the x-z plane having a positive y component. Suppose γ_1 is the plane defined by p, o, r. We have shown that C_{γ_0} goes into C_{γ_1} by a linear transformation which leaves invariant points of the z axis. We can repeat the above construction about the line $\overline{or}$. Hence if γ_2 is the plane defined by q, o, r, then C_{γ_1} goes into C_{γ_2} and p goes into q by a linear transformation leaving the x-z plane and points of $\overline{or}$ invariant. Repeating the above construction about the line $\overline{oq}$, C_{γ_2} goes into C_{γ_0} by a linear transformation which leaves points of $\overline{oq}$ invariant. The product of these transformations is the desired linear transformation.

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[5] The remainder of the proof is similar to an argument used by Garrett Birkhoff, Duke Mathematical Journal, vol. 1 (1935), pp. 169–172, Theorem 1.

The set C_{γ_0} is compact and bounded away from o. Therefore the group of all orientation-preserving linear transformations of C_{γ_0} into itself is bounded and hence equivalent, after a linear transformation, to a subgroup G of the orthogonal group [6, p. 465, Theorem 19]. Since G is transitive on lines through o, G must be the entire orthogonal group. The set of points invariant under G is the circle. Therefore a suitable linear transformation sends C_{γ_0} into the circle. It follows that all C_γ through p are ellipses. p was chosen arbitrarily. All C_γ are therefore ellipses. If we now take a particular C_γ and choose its major axis to be the z axis of our construction, S' will be generated by this ellipse tracing an ellipse in the x-y plane and rotating about the z axis. S is therefore an ellipsoid.

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THE INSTITUTE FOR ADVANCED STUDY

METRIC SEPARABILITY AND OUTER INTEGRALS[1]

JOHN F. RANDOLPH

R. L. Jeffery [1] investigated an upper integral for functions (from the line to real numbers) not necessarily measurable. Let f be bounded, $\alpha < f(x) < \beta$ for $x \varepsilon A$, let $e_i = E_x[a_i \leqq f(x) < a_{i+1}]$ where $\alpha = a_0 < a_1 < \cdots < a_n = \beta$ and consider $\sum a_i m^* e_i$. If as n increases and max $(a_{i+1} - a_i)$ approaches zero, the limit of this sum exists and is independent of the subdivisions, this limit is the upper integral of f over A, $\int_A^* f(x)dx$.

Two point sets with finite outer measure are metrically separable if the outer measure of their sum is the sum of their outer measures. A function is metrically separable on a set A if for each constant λ the part of A on which the function is greater than λ is metrically separable from the rest of A. Jeffery proved that metric separability may be made the basis of a comprehensive theory of integration which includes Young, Pierpont, and Lebesgue integration.

All measurable functions are metrically separable, but a function defined over a non-measurable set (and so necessarily non-measurable) may still be metrically separable. However, if f is metrically separable and possesses an outer integral on a set A, there exists a measurable set $B \supset A$ and a function ϕ, measurable on B and equal to f on A, such that $\int_B \phi = \int_A^* f$. If f is metrically separable and summable on A_1 and on A_2, it need not be metrically separable on $A_1 + A_2$, but is summable on this set. Jeffery's methods of proving these results are not applicable, but the same results hold, as this paper shows, for functions from the plane to real numbers if Carathéodory outer linear measure and integration with respect to this measure are used.[2]

1. Equivalence. Let A be a set with finite outer linear (Carathéodory or Gillespie) measure and $\Gamma[A]$ the set of points of the complement of A where the superior outer density of A is positive. Then, [2], $\overline{A} = A + \Gamma[A]$ is a *massgleiche Hülle* of A, that is, is linearly measurable with linear measure equal to the outer linear measure of A. Thus A is linearly measurable if and only if $\Gamma[A]$ has linear measure zero, that is, $L^*\Gamma[A] = 0$.

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[1] Presented to the Society, in part December 30, 1936, under the title *Metric separability and the Hildebrandt integral*, and in part October 28, 1939, under the title *Metric separability*.

[2] The same methods also prove these results if outer Gillespie linear measure (see [3]) is used.

If L^*A_1 and L^*A_2 are finite, then A_1 and A_2 are metrically separable if and only if

$$LA_1\bar{A}_2 = 0 \qquad (\text{or } LA_2\bar{A}_1 = 0).$$

For $L^*(A_1+A_2) \leqq L^*(A_1+\bar{A}_2) \leqq L(\overline{A_1+A_2}) = L^*(A_1+A_2)$ so the equality holds, and then since $\bar{A}_2$ is measurable

$$L^*(A_1 + \bar{A}_2) = L^*A_1 + L\bar{A}_2 - L^*A_1\bar{A}_2 = L^*A_1 + L^*A_2 - L^*A_1\bar{A}_2,$$

so

$$L^*(A_1 + A_2) = L^*A_1 + L^*A_2$$

if and only if $L^*A_1\bar{A}_2 = 0$. In particular if A_1 and A_2 have no points in common, they are metrically separable if and only if

$$L^*A_1\Gamma[A_2] = 0 \qquad (\text{or } L^*A_2\Gamma[A_1] = 0).$$

Also if $A = A_1 + \cdots + A_m$, L^*A_i finite, A_i and A_j metrically separable, then $L^*A = L^*A_1 + \cdots + L^*A_m$ and

$$A + \Gamma[A] = A_1 + \Gamma[A_1] + \cdots + A_m + \Gamma[A_m], \tag{1}$$

both equations following quite simply.

We now prove a theorem for bounded functions on sets of finite outer linear measure that may be extended in the usual way to functions summable, in the sense of outer integrals.

THEOREM. *If A is a plane set with L^*A finite and f is a bounded ($\alpha < f(p) < \beta$, $p \in A$) function metrically separable on A (so $\int_A^* f$ exists) and $B = A + \Gamma[A]$, then there is a function ϕ which is measurable on B, $p \in A$ implies $\phi(p) = f(p)$, and*

$$\int_B \phi(p)dL(B) = \int_A^* f(p)dL^*(A).$$

If $c < d$ are any two numbers, let $A(c, d) = E_p[c \leqq f(p) < d;\ p \in A]$. For each $n = 1, 2, \cdots$ let $\alpha = a_{n,0}, a_{n,1}, \cdots, a_{n,2^n} = \beta$ be a subdivision such that

$$a_{n,i} - a_{n,i-1} = \beta - \alpha/2^n, \qquad i = 1, 2, \cdots, 2^n,$$

and define

$$A_i = A(a_{n,2^n-i};\ a_{n,2^n-i+1}), \qquad i = 1, 2, \cdots, 2^n.$$

The union of these sets is A and A_i and A_j are metrically separable $i \neq j$ (since f is metrically separable on A); so from (1)

$$B = \sum_{i=1}^{2^n} A_i + \Gamma[A_i].$$

We now define for each $n=1, 2, 3, \cdots$

$$\phi_n(p) = \begin{cases} f(p) \quad \text{if} \quad p \,\varepsilon\, A, \\ a_{n,2^n-i} \text{ if } p \,\varepsilon\, \{\Gamma[A_i] - (\Gamma[A(a_{n,2^n-i+1})\beta] + A)\}, \; i = 1, 2, \cdots. \end{cases}$$

Thus $\phi_n(p)$ is defined for each point p of B. Since $\phi_n(p) \leqq \phi_{n+1}(p) < \beta$ the limit $\phi(p) = \lim_{n\to\infty} \phi_n(p)$ exists. Clearly $\phi(p) = f(p)$ if $p \,\varepsilon\, A$.

Also ϕ is a measurable function. First $E_p[\phi_n(p) \geqq a_{n,r}] = \overline{A}(a_{n,r}; \beta) - A(\alpha; a_{n,r})$. However, since f is metrically separable on A, the sets $A(a_{n,r}; \beta)$ and $A(\alpha; a_{n,r})$ are metrically separable and have no points in common

$$L\{\Gamma[A(a_{n,r}; \beta)]A(\alpha; a_{n,r})\} = 0,$$

that is, the measurable set $\overline{A}(a_{n,r}; \beta)$ represents almost all points where $\phi_n(p) \geqq a_{n,r}$. Now if λ is any number $\alpha \leqq \lambda \leqq \beta$, let $a_{n_1,r_1}, a_{n_2,r_2}, \cdots$ where $n_1 \leqq n_2 \leqq \cdots$ be an increasing sequence of points of subdivisions approaching λ. Then the measurable set

$$\prod_{i=1}^{\infty} \overline{A}(a_{n_i,r_i}; \beta)$$

differs from $E_p[\phi(p) \geqq \lambda, \; p \,\varepsilon\, \beta]$ by a set of measure zero. Thus ϕ is a measurable function.

To prove the integral equality let $\alpha = a_0 < a_1 < \cdots < a_n = \beta$ be a subdivision and define

$$s_i = \underset{p}{E}[p \,\varepsilon\, A, \; a_i \leqq f(p) < a_{i+1}], \qquad S_i = \underset{p}{E}[p \,\varepsilon\, B, \; a_i \leqq \phi(p) < a_{i+1}].$$

Then S_i is measurable, $S_i \supset s_i$ and

$$L^*A = L^*s_1 + \cdots + L^*s_m \leqq LS_1 + \cdots + LS_m = LB = L^*A,$$

so $L^*s_i = LS_i$, $i = 1, 2, \cdots, m$, and thus by multiplying by a_i, summing and taking limits we see the desired equality

$$\int_A^* f = \int_B \phi$$

for bounded functions over sets of finite outer linear measure.

2. **Additivity.** If G_1 and G_2, each with finite outer measure, are metrically separable and f is a function metrically separable on each, f is metrically separable on G_1+G_2. However, if G_1 and G_2 are not

metrically separable, then f may not be metrically separable on their union, for example, if G_1 and G_2 are complements on the unit interval each with outer measure unity and $f(x) = 1$ if $x \varepsilon G_1$, $f(x) = 2$ if $x \varepsilon G_2$. If however, f is metrically separable and the outer integral of f exists on each set G_1 and G_2, then the outer integral exists on $G_1 + G_2$. For let $G_1 = G_{11} + G_{12}$ where $G_{12} = G_1(G_2 + \Gamma[G_2])$ and $G_{11} = G_1 - G_{12}$ and in like manner take $G_2 = G_{21} + G_{22}$. Then G_{11} and G_{12} are metrically separable as are G_{21} and G_{22} as well as G_{11} and G_{21}, while no subset of positive outer measure of G_{12} is metrically separable from G_{22} and no subset of G_{22} is metrically separable from G_{12}. We may then apply the following theorem (proved for positive bounded functions, but extendible in the usual way) to obtain the property of additivity over sets of the existence of outer integrals.

THEOREM. *Let A_1 and A_2 be two point sets of the plane with outer linear measure finite such that no part of either with positive outer linear measure is metrically separable from the other. Then if f is a positive and bounded $(0 < f(p) < \beta;\ p \varepsilon A_1 + A_2)$ function metrically separable on A_1 and on A_2, the outer integral of f exists on $A_1 + A_2$.*

For each $n = 1, 2, 3, \cdots$ let $0 = a_{n,0}, a_{n,1}, \cdots, a_{n,2^n} = \beta$ be a subdivision of the interval $(0, \beta)$ into 2^n equal parts. Define

$$s_{n1i} = \underset{p}{E}[p \varepsilon A_1,\ a_{n,i-1} \leqq f(p) < a_{n,i}],$$
$$s_{n2i} = \underset{p}{E}[p \varepsilon A_2,\ a_{n,i-1} \leqq f(p) < a_{n,i}], \qquad i = 1, 2, \cdots, 2^n,$$

and

$$e_n = \sum_{i=1}^{2^n} \overline{s_{n1i}}\ \overline{s_{n2i}}$$

where $\overline{s_{n1i}} = s_{n1i} + \Gamma[s_{n1i}]$. Then e_n is a measurable subset of the measurable set $\overline{A_1} + \overline{A_2}$. Also let

$$E_n = (\overline{A_1} + \overline{A_2}) - e_n.$$

Since $s_{n+1,1,2i-1}$ and $s_{n+1,1,2i}$ are metrically separable and their union is s_{n1j} we have

$$\bar{s}_{n+1,1,2i-1} + \bar{s}_{n+1,1,2i} = \bar{s}_{n,1,i}$$

and

$$\bar{s}_{n+1,2,2i-1} + \bar{s}_{n+1,2,2i} = \bar{s}_{n,2,i}, \qquad i = 1, 2, \cdots, 2^n.$$

If then $p \varepsilon e_{n+1}$ there will be a j such that p will be either in

$\mathfrak{s}_{n+1,1,2i-1}\mathfrak{s}_{n+1,2,2i-1}$ or $\mathfrak{s}_{n+1,1,2i}\mathfrak{s}_{n+1,2,2i}$ so in either case will belong to $\mathfrak{s}_{n,1,i}\mathfrak{s}_{n,2,i}$ and so to e_n, that is, $e_{n+1} \subset e_n$. Thus $E_{n+1} \supset E_n$ and the sets

$$e = e_1 \cdot e_2 \cdots, \qquad E = E_1 + E_2 + \cdots$$

are clearly metrically separable.

We first prove that the outer integral of f exists over $e(A_1+A_2)$.

Let $b_{ni} = E_p[p \,\varepsilon\, e(A_1+A_2),\ a_{n,i-1} \leqq f(p) < a_{n,i}]$, $n = 1, 2, 3, \cdots$, $i = 1, 2, \cdots, 2^n$. Then $b_{n,i} = e(s_{n,1,i}+s_{n,2,i})$. But $e_n\mathfrak{s}_{n1i}$ and $e\bar{s}_{n2i}$ differ from each other only in a set of measure zero so $e\bar{s}_{n1i}$ and $e\bar{s}_{n2i}$ differ at most by a set of measure zero, so

$$Le(\mathfrak{s}_{n1i} + \mathfrak{s}_{n2i}) = Le\mathfrak{s}_{n1i} = Le\mathfrak{s}_{n2i},$$

so

$$L^*e(s_{n1i} + s_{n2i}) = L^*es_{n1i} = L^*es_{n2i}$$

or

$$L^*b_{ni} = L^*es_{ni} = L^*es_{n2i}.$$

Thus

$$\sum_{i=1}^{2^n} a_{ni}L^*b_{ni} = \sum_{i=1}^{2^n} a_{ni}L^*es_{n1i} = \sum_{i=1}^{2^n} a_{ni}L^*es_{n2i},$$

and consequently the integral of f exists over $e(A_1+A_2)$ and

$$\int_{e(A_1+A_2)}^{*} f(p)dL(p) = \int_{eA_1}^{*} f(p)dL(p) = \int_{eA_2}^{*} f(p)dL(p).$$

We now show that f has an outer integral on $E(A_1+A_2)$ which moreover is the sum of the outer integrals over EA_1 and EA_2.

Let $e_n' = \sum_{i=1}^{2^n} E\mathfrak{s}_{n1i}\mathfrak{s}_{n2i}$ and let $E_n' = E(\bar{A}_1+\bar{A}_2) - e_n'$. Now $e_n' \supset e_{n+1}'$ so $E_n' \subset E_{n+1}'$. Also $\lim_{n\to\infty} L^*e_n' = 0$ for otherwise we would have $L^*Ee > 0$. Therefore $\lim_{n\to\infty} L^*E_n' = E$ and thus $\lim_{n\to\infty} L^*E(A_1+A_2) = L^*E(A_1+A_2)$, $\lim_{n\to\infty} L^*E_n'A_1 = L_1EA_1$ and $\lim_{n\to\infty} L^*E_n'A_2 = EA_2$.

Now for $n' > n$ let $h_{n'ji} = E_p[p \,\varepsilon\, E_n'A_j,\ a_{n',i-1} \leqq f < a_{n',i}]$, $j = 1, 2$; $i = 1, 2, \cdots, 2^{n'}$. Then $h_{n'1i}$ and $h_{n'2j}$ are metrically separable, so we have

$$L^*(h_{n'1i} + h_{n'2i}) = L^*h_{n'1i} + L^*h_{n'2i}$$

and thus

$$\int_{E_n'(A_1+A_2)}^{*} f = \int_{E_n'A_1}^{*} f + \int_{E_n'A_2}^{*} f.$$

Also since

$$\int^*_{B\ 'A_1} f + \int^*_{(B-B_n')A_1} f = \int^*_{BA_1} f,$$

we have

$$\lim_{n\to\infty} \int^*_{B_n'A_1} f = \int^*_{BA_1} f, \qquad \lim_{n\to\infty} \int^*_{B_n'A_2} f = \int^*_{BA_2} f.$$

Then since

$$\int^*_{B_n'(A_1+A_2)} f \leqq \int^*_{B_n'+1(A_1+A_2)} f \leqq \int^*_{BA_1} f + \int^*_{BA_2} f,$$

we see that

$$\lim_{n\to\infty} \int^*_{B_n'(A_1+A_2)} f \text{ exists}$$

and is less than or equal to

$$\int^*_{BA_1} f + \int^*_{BA_2} f.$$

We may further assert that the equality holds. For $\epsilon > 0$ given there is an N such that $n \geqq N$ implies

$$\int^*_{B_n'A_j} f > \int^*_{BA_j} f - \frac{\epsilon}{2}, \qquad j = 1, 2,$$

or

$$\int^*_{B_n'(A_1+A_2)} f = \int^*_{B_n'A_1} f + \int^*_{B_n'A_2} f > \int^*_{BA_1} f + \int^*_{BA_2} f - \epsilon.$$

Thus

$$\int^*_{A_1+B_1} f = \int^*_{B(A_1+A_2)} f + \int^*_{e(A_1+A_2)} f = \int^*_{BA_1} f + \int^*_{BA_2} f + \int^*_{e(A_1+A_2)} f.$$

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CORNELL UNIVERSITY

A PROPERTY OF A SIMPLY ORDERED SET[1]

K. W. FOLLEY

Sierpinski[2] has shown that the set of real numbers of the interval (0, 1) may be decomposed into c disjoint subsets, each of power less than c, and such that the sum of every c of these subsets has at least one point in common with every perfect subset of the interval.

The object of the present paper is to show that the same method of proof may be used to prove an analogous theorem concerning a more general type of set.

DEFINITIONS. *A simply ordered set M is a set such that if any two of its elements are given it is known which one precedes.*

A subset of M is said to be cofinal (coinitial) with M if no element of M follows (precedes) all the elements of the subset.

An η_α subset of M is one which is neither cofinal nor coinitial with any subset of M of power less than $\aleph_\alpha$ and which contains no pair of neighboring subsets both of which have power less than $\aleph_\alpha$.

Various properties of simply ordered sets M containing everywhere dense η_α subsets, including a discussion of the perfect subsets of M, were discussed by the writer in a previous paper.[3]

THEOREM 1. *Let M be a simply ordered set containing an everywhere dense η_α subset N. There exists a decomposition of M into $2^{\aleph_\alpha}$ disjoint subsets, each of power less than $2^{\aleph_\alpha}$, and such that the sum of every $2^{\aleph_\alpha}$ of these subsets has at least one point in common with every perfect subset of M.*

PROOF. Let ϕ be the smallest ordinal number of power $2^{\aleph_\alpha}$. A transfinite sequence of type ϕ formed of all the points of M exists, namely,

$$(1) \qquad m_1, m_2, m_3, \cdots, m_\xi, \cdots, \qquad \xi < \phi.$$

The perfect subsets of M having power $2^{\aleph_\alpha}$ may be arranged in the form of a transfinite sequence of type ϕ as follows:

$$(2) \qquad M_1, M_2, M_3, \cdots, M_\xi, \cdots, \qquad \xi < \phi.$$

Let us now define a transfinite sequence $\{H_\xi\}_{\xi<\phi}$ of subsets of M: H_1 is formed of the single element m_1.

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[1] Presented to the Society, November 22, 1940.

[2] Fundamenta Mathematicae, vol. 24 (1935), pp. 8–11.

[3] Proceedings of the Royal Society of Canada, Section III, vol. 22 (1928), pp. 225–239.

Now, let β be any ordinal number between 1 and ϕ. Suppose we have defined all sets H_ξ, where $\xi<\beta$, and suppose further that the power of H_ξ is not greater than $\bar{\xi}$. Let $S_\beta=\sum_{\xi<\beta}H_\xi$. Then S_β is evidently of power not greater than $\bar{\beta}^2<2^{\aleph_\alpha}$.

We shall define a transfinite sequence $\{p^\beta_\xi\}$ as follows. M_1 being of power $2^{\aleph_\alpha}$, and S_β being of power less than $2^{\aleph_\alpha}$, M_1-S_β contains points. Let p^β_1 be the first element of (1) which belongs to M_1-S_β. Now let η be any ordinal number between 1 and β, and let us suppose we have already defined the points p^β_ξ, where $\xi<\eta$. Let all these points form the set T^β_η of power not greater than $\bar{\eta}\leqq\bar{\beta}<2^{\aleph_\alpha}$. The set $M_\eta-(S_\beta+T_\eta)$ contains points. Let p^β_η be the first point of (1) which belongs to the set $E_\eta-(S_\beta+T_\eta)$.

Define H_β as the set of all points p^β_ξ, where $\xi<\beta$; it is a set of power not greater than $\bar{\beta}<2^{\aleph_\alpha}$.

The sets H_ξ, with $\xi<\phi$, are thus defined by transfinite induction. They are evidently disjoint subsets of M, each of power less than $2^{\aleph_\alpha}$. The set $R=M-\sum_{\xi<\phi}H_\xi$ is of power not greater than $2^{\aleph_\alpha}$. Hence the elements of R may be arranged in the form of a transfinite sequence of type $\psi\leqq\phi$ as follows:

$$q_1,\ q_2,\ q_3,\ \cdots,\ q_\xi,\ \cdots,\qquad \xi<\psi.$$

Let $N_\xi=H_\xi+q_\xi$ for $\xi<\psi$, and if $\psi<\phi$, $M_\xi=H_\xi$ for $\psi\leqq\xi<\phi$.

The sets N_ξ are disjoint subsets of M, each of power less than $2^{\aleph_\alpha}$. Moreover $M=\sum_{\xi<\phi}N_\xi$.

Now, let F be the sum of any $2^{\aleph_\alpha}$ of the sets N_ξ, $\xi<\phi$. If μ is any ordinal number less than ϕ, there is an ordinal number β such that $\mu<\beta<\phi$, and such that N_β belongs to F. Since $\beta>\mu$ we have p^β_μ belonging to H_μ and therefore to N_μ. Besides p^β_μ belongs to M_μ. Hence p^β_μ is a point of $M_\mu\cdot H_\beta$ and so of $F\cdot M_\mu$.

It follows that F has at least one point in common with every subset M_ξ of (2), which was to be proved.

THEOREM 2. *The generalized hypothesis of the continuum* ($2^{\aleph_\alpha}=\aleph_{\alpha+1}$) *is equivalent to the following statement:*

The set M may be decomposed into disjoint subsets, each of power not greater than $\aleph_\alpha$, such that the sum of any class of more than $\aleph_\alpha$ of them has at least one point in common with every perfect subset of M.

PROOF. (a) If the generalized hypothesis of the continuum is assumed, Theorem 1 becomes the second part of Theorem 2.

(b) Suppose that the second part of Theorem 2 is true. There are $2^{\aleph_\alpha}$ perfect subsets of[4] M. Each disjoint subset of M having a power not greater than $\aleph_\alpha$, it follows that a class of $\aleph_{\alpha+1}$ of them has power $\aleph_{\alpha+1}$. Such a class has at least one point in common with every perfect subset of M. Thus, there results the inequality

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[4] See K. W. Folley, loc. cit., p. 232.

$$\aleph_{\alpha+1} \leqq 2^{\aleph_\alpha}.$$

The inequality in the opposite sense being well known, the generalized hypothesis of the continuum follows.

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UNIQUENESS THEOREMS FOR RATIONAL FUNCTIONS[1]

FRITZ HERZOG

In his book on the theory of meromorphic functions,[2] R. Nevanlinna proved a number of "uniqueness theorems." The most important of them[3] states that if two functions $w=f(x)$ and $w=g(x)$, meromorphic in the whole x-plane, assume *five* values of w (finite or infinite) at the same points x they must be identical. If we understand by the *distribution* of a function $w=\phi(x)$ with respect to a given value of w simply the set of all points x where $\phi(x)$ assumes that value w, *regardless of multiplicity*, we may state the above theorem in the following way: Two meromorphic functions which have identical distributions with respect to five values of the dependent variable must be identical. In proving this theorem, Nevanlinna explicitly assumes the functions to be transcendental (i.e., not rational).[3] It is trivial, however, that the theorem would apply to two rational functions $w=f(x)$ and $w=g(x)$ as well, which can be easily seen by considering the transcendental functions $w=f(e^x)$ and $w=g(e^x)$.

The example of the functions $w=e^x$ and $w=e^{-x}$, which have identical distributions with respect to the four values $w=1, -1, 0, \infty$, shows that five is the smallest number for which the above-mentioned uniqueness theorem holds true. It will be shown in this paper that such is not the case for rational functions for which five may, indeed, be replaced by *four*. (See Theorem I.)

The question arises as to what can be said about two rational functions that have identical distributions with respect to only *three* values of the dependent variable.[4] It is the main object of this paper to study this problem.

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[1] Presented to the Society, February 24, 1940.

[2] Rolf Nevanlinna, *Le Théorème de Picard-Borel et la Théorie des Fonctions Méromorphes*, Paris, 1929.

[3] See loc. cit., p. 109.

It will be found that there exist pairs of rational functions which have identical distributions with respect to three values without being identical (Examples I, II, III). Furthermore, under certain restrictive conditions concerning, for instance, the degrees of the functions, uniqueness theorems for rational functions with identical distributions with respect to three values will be derived. (See Theorems II and III.)

Throughout the paper we shall use x or z for the independent complex variable and w for the dependent complex variable. Unless otherwise stated, x is to be restricted to finite values while z and w may assume also the value ∞.

As far as the variables z and w are concerned, we may subject them to linear substitutions without changing the character of the problem under consideration. This is due to the fact that a substitution $u=(au_1+b)/(cu_1+d)$ with $ad-bc\neq 0$ gives a one-to-one correspondence between the u and u_1 planes (both $u=\infty$ and $u_1=\infty$ being included). We shall sometimes make use of such linear substitutions without further explanation.

We finally remark that all functions mentioned in the paper are understood to be not constant, unless otherwise stated.

We consider two rational functions $w=f(x)$ and $w=g(x)$ which have identical distributions with respect to four values of w and shall show that they must be identical. In accordance with what was said above, x is restricted to finite values, that is, the two functions are assumed to have identical distributions with respect to the four values of w, only as far as finite values of x are concerned, while the point $x=\infty$ is disregarded (except at certain points in the proof). We thus obtain a more general theorem.

THEOREM I. *If $w=f(x)$ and $w=g(x)$ are two rational functions and if w_1, w_2, w_3, w_4 are four different values of w that are assumed by the two functions at the same points x, then $f(x)\equiv g(x)$.*

PROOF. Let m and n be the degrees of $f(x)$ and $g(x)$, respectively; we may assume that $m\geqq n$. Let $w_0=f(\infty)$ where w_0 may be either different from w_1, w_2, w_3, and w_4 or equal to one of them. In the latter case we may assume that $w_0=w_1$.

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[4] Cf. Nevanlinna's remark, loc. cit., p. 109: "... une fonction rationnelle est univoquement déterminée par *trois* distributions." In view of the results of this paper, this remark must be understood in such a way that identical distribution is to imply also equal multiplicities. Neither in Nevanlinna's uniqueness theorem nor in this paper is this the case.

Now let r_k be the actual number of finite w_k-points of $f(x)$ and $g(x)$ ($k=1, 2, 3, 4$), that is, their number not counting multiplicity. Let m_0 be the multiplicity of w_0 at $x=\infty$ and m_k be the number of finite w_k-points of $f(x)$ ($k=1, 2, 3, 4$), counting multiplicity. Then we have $m_1=m$ or $m_0+m_1=m$, according to whether $w_0\neq w_1$ or $w_0=w_1$. Hence, in any case,

$$m_0+m_1\geqq m. \tag{1}$$

Also, in any case,

$$m_2=m_3=m_4=m. \tag{2}$$

We may assume, without loss of generality, that w_0, w_1, w_2, w_3, and w_4 are all finite because a suitable linear substitution of w will, if necessary, transform the w_k into finite values.

Now let $f=F_1/F_2$ and $g=G_1/G_2$ be representations of f and g as reduced fractions of polynomials. We consider the functions

$$f'=(F_2F_1'-F_1F_2')/F_2^2, \tag{3}$$

$$f-g=(F_1G_2-F_2G_1)/F_2G_2. \tag{4}$$

(3) is not identically zero, it is of degree at most $2m$, it vanishes at $x=\infty$ with a multiplicity m_0+1 (which can be seen easily from the expansion of $f(x)$ in a power series about $x=\infty$), and it has m_k-r_k zeros, counting multiplicity, at the r_k finite w_k-points of $f(x)$ ($k=1, 2, 3, 4$). These facts yield

$$(m_0+1)+(m_1-r_1)+(m_2-r_2)+(m_3-r_3)+(m_4-r_4)\leqq 2m. \tag{5}$$

Using (1) and (2), we obtain from (5)

$$4m+1-(r_1+r_2+r_3+r_4)\leqq 2m$$

or

$$r_1+r_2+r_3+r_4\geqq 2m+1. \tag{6}$$

On the other hand, (4) is of degree at most $m+n\leqq 2m$, it vanishes at the r_k finite points x where $f(x)=g(x)=w_k$ ($k=1, 2, 3, 4$), and hence, by (6), it has at least $r_1+r_2+r_3+r_4\geqq 2m+1$ zeros. Therefore, (4) must vanish identically, which proves the theorem.

That Theorem I does not hold when only three identical distributions are assumed can be seen from very simple examples, such as $w=f(x)=4x/(x+1)^2$ and $w=g(x)=2x/(x+1)$ which have identical distributions with respect to the three values $w=0$, ∞, 1, provided we consider only finite x. The value $w=0$, in particular, is assumed by $f(x)$ and $g(x)$ at $x=0$. If, however, x were allowed to assume also the value ∞, the distributions of $f(x)$ and $g(x)$ with respect to $w=0$ would not be identical any more since $f(\infty)=0$ but $g(\infty)\neq 0$.

We shall, therefore, from now on allow the independent variable to assume also the value ∞ and study the problem of two rational functions $w=f(z)$ and $w=g(z)$ which have identical distributions with respect to three values of w.

We first consider the special case of polynomials. Since a polynomial assumes the value $w=\infty$ at $z=\infty$ only, the question is whether two polynomials with identical distributions with respect to two (finite) values of w must be identical. The answer is given by the following theorem.

THEOREM II. *If two polynomials $w=F(z)$ and $w=G(z)$ assume two finite values w_1 and w_2 $(w_1\neq w_2)$ at the same points z, then $F(z)\equiv G(z)$.*

PROOF. Let m and n be the degrees of $F(z)$ and $G(z)$, respectively, let $m\geqq n$ and let r_k be the actual number of w_k-points of $F(z)$ and $G(z)$ $(k=1, 2)$, not counting multiplicity. Then $F'(z)$ is not identically zero, it is of degree $m-1$ and it has $m-r_k$ zeros, counting multiplicity, at the r_k points where $F(z)=w_k$ $(k=1, 2)$. This gives $(m-r_1)+(m-r_2)\leqq m-1$ or

$$r_1+r_2\geqq m+1. \tag{7}$$

On the other hand, $F(z)-G(z)$ is of degree at most m and it vanishes at the r_k points where $F(z)=G(z)=w_k$ $(k=1, 2)$. Hence, by (7), it has at least $r_1+r_2\geqq m+1$ zeros and must, therefore, vanish identically. This proves the theorem.

By subjecting both z and w to linear substitutions the two polynomials become two rational functions which have the property that there is one value of w which is assumed by both functions at one and the same point z only. Thus, according to what was said in the introduction, Theorem II yields immediately the following corollary.

COROLLARY. *If two rational functions $w=f(z)$ and $w=g(z)$ assume three different values w_1, w_2, w_3 at the same points z and if one of these three values is assumed at one point z only, then $f(z)\equiv g(z)$.*

The corollary just proved is one of several similar statements that will be derived in the following. These statements give sufficient conditions under which two rational functions with identical distributions with respect to three values must be identical. It will simplify the language if we put these statements in the form of necessary conditions which two different rational functions have to satisfy if they have identical distributions with respect to three values of the dependent variable.

Thus we consider two rational functions $w=f(z)$ and $w=g(z)$ which assume three different values w_1, w_2, w_3 at the same points z, which we shall call z_{ks} ($k=1, 2, 3$; $s=1, 2, \cdots, r_k$; all z_{ks} different). Furthermore, we assume that the degrees of $f(z)$ and $g(z)$ are m and n, respectively, and that $m \geqq n$. Finally, let m_{ks} and n_{ks} be the multiplicities with which $w=f(z)$ and $w=g(z)$, respectively, assume the value $w=w_k$ at $z=z_{ks}$ ($m_{ks} \geqq 1$, $n_{ks} \geqq 1$). It will be convenient to enter these data in the following table.

Value of w	*Distribution of f(z) and g(z) with respect to w_k*	*Multiplicities at z_{ks} of*	
		$f(z)$	$g(z)$
w_1	$z_{11}, z_{12}, \cdots, z_{1r_1}$	$m_{11}, m_{12}, \cdots, m_{1r_1}$	$n_{11}, n_{12}, \cdots, n_{1r_1}$
w_2	$z_{21}, z_{22}, \cdots, z_{2r_2}$	$m_{21}, m_{22}, \cdots, m_{2r_2}$	$n_{21}, n_{22}, \cdots, n_{2r_2}$
w_3	$z_{31}, z_{32}, \cdots, z_{3r_3}$	$m_{31}, m_{32}, \cdots, m_{3r_3}$	$n_{31}, n_{32}, \cdots, n_{3r_3}$

We note that for $k=1, 2, 3$

$$m_{k1} + m_{k2} + \cdots + m_{kr_k} = m, \tag{8}$$

$$n_{k1} + n_{k2} + \cdots + n_{kr_k} = n. \tag{9}$$

THEOREM III. *If $w=f(z)$ and $w=g(z)$ are two rational functions with identical distributions with respect to three values of w and if w_k, r_k, z_{ks}, m, n, m_{ks}, n_{ks} are as defined above, then the following are necessary conditions for $f(z) \not\equiv g(z)$:*

$$\sum_{k=1}^{3} \sum_{s=1}^{r_k} \min (m_{ks}, n_{ks}) \leqq m + n, \tag{a}$$

$$r_1 + r_2 + r_3 \geqq m + 2, \tag{b}$$

$$2 \leqq r_k \leqq n, \qquad \textit{for } k = 1, 2, 3, \tag{c}$$

$$m + n \geqq 6, \tag{d}$$

$$m \leqq 3n - 2, \tag{e}$$

$$(m_{k1}, m_{k2}, \cdots, m_{kr_k}) = (n_{k1}, n_{k2}, \cdots, n_{kr_k}) \qquad \textit{for at most one } k.^5 \tag{f}$$

PROOF. For the proof of conditions (a) to (e) we assume that the three w_k as well as all the z_{ks} are finite numbers.

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[5] This equation is to be considered an identity of two vectors, that is, $m_{k1}=n_{k1}$, $m_{k2}=n_{k2}$, $\cdots$, $m_{kr}=n_{kr}$. .

(a) The function $f(z)-g(z)$ (see (4) above) is not identically zero, it is of degree at most $m+n$, it has a zero at $z=z_{ks}$ the multiplicity of which is at least min (m_{ks}, n_{ks}), and it has, therefore, at least $\sum_{k=1}^{3}\sum_{s=1}^{r_k}$ min (m_{ks}, n_{ks}) zeros, counting multiplicity. Thus (a) must hold.

(b) The function $f'(z)$ (see (3) above) is not identically zero and the numerator $F_2F_1' - F_1F_2'$ is of degree at most $2m-2$. At the points z_{ks} $(s=1, 2, \cdots, r_k)$ $f'(z)$ has $m-r_k$ zeros, counting multiplicities $(k=1, 2, 3)$, hence $f'(z)$ has at least $(m-r_1)+(m-r_2)+(m-r_3)$ finite zeros which must be zeros of the numerator $F_2F_1' - F_1F_2'$. This gives

$$3m - (r_1 + r_2 + r_3) \leqq 2m - 2,$$

from which (b) follows.

(c) The inequality $r_k \geqq 2$ (for $k=1, 2, 3$) is easily seen to be identical with the corollary, proved above. That $r_k \leqq n$ (for $k=1, 2, 3$) follows at once from (9) and $n_{ks} \geqq 1$.

(d) From inequality (a) and the fact that min $(m_{ks}, n_{ks}) \geqq 1$ (for $k=1, 2, 3$ and $s=1, 2, \cdots, r_k$) we conclude

$$r_1 + r_2 + r_3 \leqq m + n. \tag{10}$$

(10) together with $r_k \geqq 2$ from (c) yields (d).

(e) From (b) we conclude

$$m \leqq r_1 + r_2 + r_3 - 2. \tag{11}$$

(11) together with $r_k \leqq n$ from (c) yields (e).

(f) Suppose we should have $(m_{k1}, m_{k2}, \cdots, m_{kr_k}) = (n_{k1}, n_{k2}, \cdots, n_{kr_k})$ for two values of k, say $k=1$ and $k=2$. We may then assume that $w_1=0$ and $w_2=\infty$. Thus $f(z)$ and $g(z)$ have the same zeros and the same poles, including their multiplicities, and hence $f(z)\equiv cg(z)$ with $c=$const. Since $f(z_{31})=g(z_{31})=w_3$ $(w_3\neq 0, \neq \infty)$ we have $c=1$ which contradicts $f(z)\not\equiv g(z)$. This proves condition (f).

We remark that the inequalities (a) to (e) in the form in which they are written above cannot be improved. Indeed, in Example I below both sides of these inequalities are actually equal. As to condition (f), the relation $(m_{k1}, m_{k2}, \cdots, m_{kr_k}) = (n_{k1}, n_{k2}, \cdots, n_{kr_k})$ may actually be satisfied for one k, as shown by Example III.

We shall now give three examples of pairs of rational functions with identical distributions with respect to three values of the dependent variable. We shall briefly describe the procedure for finding such examples. For notations in what follows, we refer to the above table.

We select degrees m and n with $m \geqq n$ and satisfying the conditions of Theorem III. Then we select positive integers r_k, m_{ks} and n_{ks}, compatible with (8) and (9) as well as with all the conditions of Theorem III. By the principle of linear substitutions of w and z we may then assign any three arbitrary values to the three w_k, say $w_1=0$, $w_2=\infty$, $w_3=1$, and three arbitrary values to three arbitrarily selected ones among the z_{ks}, say $z_{11}=1$, $z_{21}=\infty$, $z_{22}=0$. (Observe that, by condition (c), $r_2 \geq 2$.) With the other z_{ks} as unknowns and with two unknown constants a and b we have then in view of the distributions with respect to $w_1=0$ and $w_2=\infty$

$$(12)\quad f(z) = \left[(z-1)^{m_{11}}\prod_{s=2}^{r_1}(z-z_{1s})^{m_{1s}}\right]\Big/\left[az^{m_{22}}\prod_{s=3}^{r_2}(z-z_{2s})^{m_{2s}}\right],$$

$$(13)\quad g(z) = \left[(z-1)^{n_{11}}\prod_{s=2}^{r_1}(z-z_{1s})^{n_{1s}}\right]\Big/\left[bz^{n_{22}}\prod_{s=3}^{r_2}(z-z_{2s})^{n_{2s}}\right]$$

where the product in the denominator to the right of (12) and (13) is to be unity if $r_2=2$. Now the distribution with respect to $w_3=1$ gives

$$(14)\quad (z-1)^{m_{11}}\prod_{s=2}^{r_1}(z-z_{1s})^{m_{1s}} - az^{m_{22}}\prod_{s=3}^{r_2}(z-z_{2s})^{m_{2s}} = \prod_{s=1}^{r_3}(z-z_{3s})^{m_{3s}},$$

$$(15)\quad (z-1)^{n_{11}}\prod_{s=2}^{r_1}(z-z_{1s})^{n_{1s}} - bz^{n_{22}}\prod_{s=3}^{r_2}(z-z_{2s})^{n_{2s}} = \prod_{s=1}^{r_3}(z-z_{3s})^{n_{3s}}.$$

If now the coefficients of equal powers of z on both sides of (14) and (15) are put equal to one another we obtain $m+n$ equations for the $r_1+r_2+r_3-1$ unknowns z_{ks}, a and b.[6] Each solution of these equations in which the z_{ks} are different from each other and from 0 and 1 and in which a and b are different from 0 corresponds to a pair of functions of the desired type.

By condition (c) of Theorem III, the smallest value for n is 2. If $n=2$ we have, by conditions (d) and (e), $m=4$. The above procedure then leads to the following example:

EXAMPLE I. $f(z)=(z-1)^3(z+3)/16z$, $g(z)=(z-1)(z+3)/4z$.

w_k	z_{ks}	m_{ks}	n_{ks}
0	$1, -3$	3, 1	1, 1
∞	$\infty, 0$	3, 1	1, 1
1	$-1, 3$	3, 1	1, 1

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[6] From (10) it follows that we have at least one more equation than unknowns.

We remark without proof that the above pair is essentially the only pair with $m=4$ and $n=2$. That is to say that first no other combinations of m_{ks} and n_{ks} are possible (except permutations of the above) and that secondly all other pairs of degrees $m=4$ and $n=2$ are obtained from the above by a suitable linear substitution of w and z.

The next lowest degrees, according to condition (d), are $m=3, n=3$. These values lead to the following example.

EXAMPLE II.

$$f(z) = (z-1)^2(z+2)/3(\rho^2-1)z, \quad g(z) = z-1)(z+2)^2/3(1-\rho)z^2,$$
$$\rho = -\tfrac{1}{2}+\tfrac{1}{2}i3^{1/2}.$$

w_k	z_{ks}	m_{ks}	n_{ks}
0	$1, -2$	2, 1	1, 2
∞	$\infty, 0$	2, 1	1, 2
1	$\rho, -2\rho$	2, 1	1, 2

Example II is again essentially the only pair of functions with $m=3$, $n=3$ in the same sense as Example I above.

The case $m=4$, $n=4$ leads to two interesting examples. One of them is the following.

EXAMPLE III. $f(z)=(z-1)^3(z+1)/4z$, $g(z)=(z-1)(z+1)^3/4z^3$.

w_k	z_{ks}	m_{ks}	n_{ks}
0	$1, -1$	3, 1	1, 3
∞	$\infty, 0$	3, 1	1, 3
1	$i, -i, 1+2^{1/2}, 1-2^{1/2}$	1, 1, 1, 1	1, 1, 1, 1

In this example the relation of condition (f) of Theorem III is actually satisfied for one k, namely, $k=3$.

Another example with $m=4$, $n=4$ can be obtained from Example I by replacing z by $-3/z$ in the function $f(z)$. It is easily seen that $f(-3/z)$ and $f(z)$ form a pair of functions of the desired type. Thus the three functions[7] $f(z), f(-3/z)$ and $g(z)$ of Example I form a *triplet* of rational functions with identical distributions with respect to three values of the dependent variable.

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[7] Note that $g(-3/z)\equiv g(z)$.

The last mentioned example raises the question as to whether there could be an infinite number of rational functions which have given distributions with respect to three given values. This question is answered as follows.

THEOREM IV. *There can be only a finite number of rational functions which have given distributions with respect to three given values of the dependent variable.*

PROOF. Let z_{ks} be the given distributions with respect to the given values w_k (see table preceding Theorem III). Let $w=f(z)$ be any rational function having these distributions, let m be its degree and m_{ks} be its multiplicity at $z=z_{ks}$. Then, by the proof of condition (b) in Theorem III, m cannot exceed $r_1+r_2+r_3-2$. This leaves only a finite number of values for m, hence by (8) there is only a finite number of possible combinations of m_{ks}. Each such combination can, according to condition (f) of Theorem III, lead to at most one function $f(z)$. This proves the theorem.

It may be noted in conclusion that any pair of functions $w=f(z)$ and $w=g(z)$ of the above-mentioned type leads to infinitely many more pairs of the same kind, other than those obtained by linear substitutions of z and w. For, if $\phi(z)$ is any rational function whatever, then $w=f(\phi(z))$ and $w=g(\phi(z))$ have again identical distributions with respect to the same three values of w as the original functions $w=f(z)$ and $w=g(z)$. This shows, in particular, that the degrees m and n of such pairs of functions may exceed any given number.

CORNELL UNIVERSITY

CONTINUA OF FINITE DEGREE AND CERTAIN PRODUCT SETS[1]

O. G. HARROLD, JR.[2]

The continua of finite degree have been studied and identified with certain well known classes of continua in a paper by G. T. Whyburn.[3] The author has shown that the continua of finite degree are identical with the continua homeomorphic to a continuum of finite length.[4] The object of the present note is to obtain other internal characterizations of these continua.

The symbol M will represent a (compact) continuum. The continuum M is said to be of finite degree at the point $p \in M$ provided that to each $\epsilon > 0$ there corresponds an *uncountable* family of neighborhoods (U) of p such that (a) the diameter of each neighborhood is less than ϵ, (b) each $F(U)$ is finite, where $F(U)$ is the boundary of U, and (c) for any pair of neighborhoods U and V either $\overline{U} \subset V$ or $\overline{V} \subset U$. If every point is of finite degree, the continuum M is said to be of finite degree. The characterization which we find most useful below is that a continuum M is of finite degree if and only if every subcontinuum contains uncountably many local separating points of M.

It will be shown that the classes of continua defined by each of the following properties are identical with the continua of finite degree.

PROPERTY N^0. *M is locally connected and to each pair of closed, disjoint subsets A and B in M there corresponds a finite collection of disjoint, perfect sets $H^1, H^2, \cdots, H^k$ such that any continuum K in M intersecting both A and B contains some H^i.*

PROPERTY Q. *If K and K_i, $(i=1, 2, \cdots)$, are nondegenerate continua in M with* $\lim K_i = K$, *there exists an integer n such that $\prod_n^\infty K_i$ is an uncountable set.*

It will be noted that the Property N^0 is highly analogous to Property N which characterizes the locally connected continua such that no true cyclic element has a continuum of condensation.[5]

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[1] Presented to the Society, December 29, 1939.

[2] National Research Fellow.

[3] G. T. Whyburn, *Concerning continua of finite degree and local separating points*, American Journal of Mathematics, vol. 57, pp. 11–16.

[4] See abstract 45-9-321, this Bulletin.

[5] This concept is due to R. L. Moore. See his *Fundamental Point Set Theorems*, Rice Institute Pamphlets, vol. 23, no. 1, 1936.

For this characterization see G. T. Whyburn, *On continua of condensation*, American Journal of Mathematics, vol. 58, pp. 705–708.

THEOREM A. *A continuum is of finite degree if and only if it has Property* N^0.

PROOF. A continuum of finite degree has the property that any pair of closed, disjoint subsets A and B in M can be separated by a finite number of points $q^1, q^2, \cdots, q^n$ which are local separating points of degree two.[3] Suppose M is of finite degree. Set $d=\min \rho(q^i, q^j)$, $i\neq j$, $\rho(q^i, A+B)$. Since q^i is of degree two, there exists an uncountable family of neighborhoods $[U(q^i)]$ such that (a) $\delta(U)<\frac{1}{2}d$, (b) $F(U)$ consists of at most two points, and (c) for any pair of distinct elements U and V of the family either $\overline{U}\subset V$ or $\overline{V}\subset U$. Set

$$Y^i = \overline{\sum F(U(q^i))},$$

i fixed. Let H^i be a perfect subset of Y^i. There is thus determined a finite collection of disjoint, perfect sets $H^1, H^2, \cdots, H^n$. Let K be any continuum in M intersecting both A and B. Since M is of finite degree (thus hereditarily locally connected), K may be taken as an arc. Since K contains some q^i, it contains the corresponding H^i.

Conversely, suppose M has Property N^0. To prove M is of finite degree it suffices to show that every nondegenerate subcontinuum in M contains uncountably many local separating points of M. Let K be a continuum in M containing the distinct points x and y. Set $A=x$, $B=y$. Then by Property N^0 there exists disjoint, perfect sets $H^1, H^2, \cdots, H^n$ such that $K\supset H^1$ (say). If $K\cdot L$ is uncountable, where L is the set of local separating points of M, our end is attained. If $K\cdot L$ is countable, there is a point $z^1\in H^1(M-L)$. Set $d=1/2 \min \rho(x, H^i), \rho(y, H^i), \rho(H^i, H^j), i\neq j$. Let V be a region in M containing z^1 which is of diameter less than d. Set $d^1=\rho[z^1, F(V)]$. Let W be the $\frac{1}{2}d^1$ neighborhood of z^1, that is, the set of all points b such that $\rho(z^1, b)\leq\frac{1}{2}d^1$. Let X^1 and Y^1 be the components of $K-KW$ containing x and y, respectively. Since X^1 and Y^1 each contain a point of V and $z^1\in M-L$, there exists an arc s^1 in $V-z^1$ joining X^1 and Y^1. Hence $K^1=X^1+s^1+Y^1$ is a continuum from x to y not containing H^1. If K^1 contains no other H^i, we have a contradiction to Property N^0. If $K^1\supset H^2$, then $H^2\subset X^1+Y^1$ on account of the manner of selection of s^1. Since $K\cdot L$ is countable, $(X^1+Y^1)L$ is countable, hence there exists a point $z^2\in H^2(M-L)$. Using the same d as before, a continuum K^2 is constructed (with K^1 replacing K) such that K^2 contains neither H^1 nor H^2 and K^2 intersects both A and B. Performing these steps (with z^2 replacing z^1) will clearly give a continuum $K^2=X^2+s^2+Y^2$ not containing H^2, and since $s^2H^1=0$ and $X^2+Y^2\subset K^1$, we have K^2 contains neither H^1 nor H^2. After a finite number of such steps a continuum K^j is obtained which intersects both A and B and contains no H^i. This completes the proof that having Property N^0 is equivalent to being of finite degree.

THEOREM B. *A continuum is of finite degree if and only if it has Property* Q.

PROOF. Let M be of finite degree. Since this implies that M is hereditarily locally connected, it may be assumed without loss of generality that the continua K_i such that $\lim K_i = K$ are arcs. Let the end-points of K_i be a_i and b_i. Suppose $\lim a_i = x$ and $\lim b_i = y$. Since K by assumption is a nondegenerate continuum, we may take $x \neq y$. In a continuum of finite degree the points of degree two are dense on every subcontinuum, hence let z be a point of degree two in $K-(x+y)$. Let X, Y and Z be neighborhoods of x, y and z respectively, such that $(\overline{X}\cdot\overline{Z}+\overline{X}\cdot\overline{Y}+\overline{Y}\cdot\overline{Z})=0$. To Z there corresponds an uncountable family of neighborhoods (U) such that $\overline{U}\subset Z$ and $F(U)$ consists of at most two points. There is a definite $V=V(z)$ such that $\overline{V}\subset U$ for uncountably many U. Since $z\in\lim K_i$, there is an integer n such that arcs K_i, $i\geqq n$, intersect V, hence if we take $m\geqq n$ so large that $a_i\in X$, $b_i\in Y$ for $i\geqq m$, then K_i must contain two boundary points of each U for which $V\subset U\subset Z$. Thus $\prod_m^\infty K_i$ is uncountable.

To show that Property Q implies finite degree we show that infinite degree implies Property non-Q. It is clear that a continuum containing a convergence continuum has Property non-Q, hence we need consider only hereditarily locally connected continua. Let T be an arc in the continuum M which contains only a countable number of the local separating points of M. For each positive integer n let W_n be the set of all points of M at a distance not greater than $1/n$ from T. Let M_n be the component of W_n which contains T. Let L_n be the set of all points which separate the end-points a and b of T in M_n. Then in M_n there exist arcs s_n and t_n from a to b such that[6] $s_n t_n = a+L_n+b$. Set $K_{2n-1}=s_n$ and $K_{2n}=t_n$. Then $\lim K_n = T$. But $\prod_n^\infty K_i$ is at most countable, no matter what positive integer n is. For K_{2n-1} and K_{2n} have only a countable number of common points, namely, $a+L_n+b$. Hence M does not have Property Q.

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[6] G. T. Whyburn, *Some properties of continuous curves*, this Bulletin, vol. 33 (1927), pp. 305–308. See Theorem 3.

NOTE ON SOME ELEMENTARY PROPERTIES OF POLYNOMIALS

P. ERDÖS

In a previous paper T. Grünwald[1] and I proved that if $f(x)$ is a polynomial of degree $n \geqq 2$ and satisfies the following conditions:

$$\text{(1)} \qquad \begin{gathered} \text{all roots of } f(x) \text{ are real}, \quad f(-1) = f(+1) = 0, \\ f(x) \neq 0 \quad \text{for} \quad -1 < x < 1, \quad \max_{-1<x<1} f(x) = 1, \end{gathered}$$

then

$$\text{(2)} \qquad \int_{-1}^{+1} f(x) \leqq \frac{4}{3}.$$

Equality occurs only for $f(x) = 1 - x^2$.

This result can be generalized as follows: Suppose $f(x)$ satisfies (1) and let $f(a) = f(b) = d \leqq 1$, $-1 < a < b < 1$; then

$$\text{(3)} \qquad b - a \leqq 2(1-d)^{1/2}.$$

Again equality occurs only for $f(x) = 1 - x^2$. It is clear that (2) follows from (3) by integration with respect to d.

Proof. Instead of (3) we prove the following slightly more general result: Let $f(x)$ satisfy (1), and determine the greatest positive constant c_f such that

$$f(a)f(a + c_f) = d^2, \quad -1 < a < a + c_f < 1;$$

then

$$\text{(4)} \qquad c_f \leqq 2(1-d)^{1/2}.$$

Equality holds only for $f(x) = 1 - x^2$, $a = -(1-d)^{1/2}$.

Suppose there exists a polynomial of degree $n > 2$ satisfying (1) with $c_f \geqq 2(1-d)^{1/2}$; then we will prove that there exists a polynomial of degree $n-1$ with $c_f > 2(1-d)^{1/2}$; and this proves (4) since it is easy to prove that (4) is satisfied for polynomials of second degree, that is, for $1 - x^2$.

Denote the roots of $f(x)$ by $x_1 = -1$, $x_2 = 1$, $x_3, \cdots, x_n$ and suppose first that for $i > 2$ the x_i are not all of the same sign. Let x_n be the largest positive root and x_{n-1} the smallest negative root, and denote by y the root of $f'(x)$ in $(-1, +1)$. Consider the polynomial of degree n

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[1] Annals of Mathematics, (2), vol. 40 (1939), pp. 537–548.

$$\phi(x) = c\frac{f(x)(x-y)^2}{(x-x_n)(x-x_{n-1})},$$

where we choose c so that $\phi(x) \geqq 0$ for $-1 \leqq x \leqq 1$. Then it is easy to see that for large x, $\phi(x)$ and $f(x)$ have opposite signs. Thus their leading coefficients have opposite signs. Hence it is possible to choose c such that the polynomial $F(x) = f(x) + \phi(x)$ is of degree $n-1$. Since $n-2$ of its roots are real it can have only real roots, and since $F'(y) = 0$, $F(y) = 1$, it follows that $\max_{-1 \leqq x \leqq 1} F(x) = 1$. Thus $F(x)$ satisfies (1) (obviously $F(x) \neq 0$ for $-1 < x < 1$) and $F(x) \geqq f(x)$ in $-1, +1$, equality occurring only for -1, y, $+1$. Thus $c_F > c_f$. Hence we may suppose that for $i > 2$ all the x_i are of the same sign; without loss of generality we may suppose them negative. Suppose that

$$f(a)f(b) = d^2, \qquad b - a = c_f.$$

We can suppose that $-1 < a < y < b < 1$. We now prove that

$$b - y < y - a. \tag{5}$$

For if not then

$$|f'(b)| > |f'(a)|, \qquad f(b) < f(a), \tag{6}$$

that is,

$$|f'(b)| = \left|(b-y)\prod_{i=1}^{n-2}(b - y_i)\right|, \qquad |f'(a)| = \left|(y-a)\prod_{i=1}^{n-2}(y_i - a)\right|,$$

$$y > y_i, \quad i = 1, 2, \cdots, n-2,$$

where $b - y \geqq y - a$ and all other factors in $|f'(b)|$ are greater then the corresponding factors in $f'(a)$. This proves the first inequality of (6). To prove the second inequality we remark that from what has just been said it follows that for $u_1 - y = y - u_2$, $-1 < u_2 < y < u_1 < 1$, we have

$$|f'(u_1)| > |f'(u_2)|,$$

and since $b - z \geqq y - a$ the second inequality follows by integration.

By simple calculation it follows from (6) that

$$f(b-\epsilon)f(a-\epsilon) > f(a)f(b) = d^2, \qquad \epsilon > 0 \text{ sufficiently small.}$$

Thus $b - a < c_f$. This contradiction proves (5).

Let x_n be the root of $f(x)$ with greatest absolute value. Consider the new polynomial

$$f_1(x) = c\frac{x - x_n'}{x - x_n} f(x), \qquad x_n' = x_n - \delta,\ \delta > 0,$$

where c is chosen in such a way that $\max_{-1\leq x\leq 1} f_1(x)=1$. Then we prove

$$c_{f_1} > c_f. \tag{7}$$

To show (7) it will suffice to show that c_f is an increasing function of $|x_n|$. Choose δ so small that if we denote by $y^{(1)}$ the root of $f_1'(x)$ in $(-1, +1)$ we have $b-y_1<y_1-a$ (it is clear that $y_1<y$).

Put now

$$c = \left| \frac{1}{f(y_1)} \frac{x_n - y_1}{x_n - \delta - y_1} \right|.$$

(Evidently $cf_1(x)$ satisfies (1).)

Now

$$c^2 f_1(a) f_1(b) = c^2 \frac{a - x_n + \delta}{a - x_n} \frac{b - x_n + \delta}{b - x_n} f(a)f(b)$$

$$> \left(1 + \frac{\delta}{a - x_n}\right)\left(1 + \frac{\delta}{b - x_n}\right)\left(\frac{1}{1 + \delta/y_1 - x_n}\right)^2 f(a)f(b)$$

(that is, $f(y_1)<1$). But from (5) we have

$$\delta\left(\frac{1}{a - x_n} + \frac{1}{b - x_n}\right) > \frac{2\delta}{\left(\frac{a+b}{2} - x_n\right)} > \frac{2\delta}{y_1 - x_n}$$

and

$$\frac{\delta^2}{(a - x_n)(b - x_n)} > \frac{\delta^2}{\left(\frac{a+b}{2} - x_n\right)^2} > \frac{\delta^2}{(y_1 - x_n)^2}.$$

Thus

$$c^2 f_1(a) f_1(b) > f(a)f(b) = d^2.$$

Hence (7) is proved.

If $|x_n|$ tends to infinity $f(x)$ tends to $F(x)=f(x)/(x-x_n)$, which is of degree $n-1$. From (7) it follows that $c_F>c_f$, which proves the theorem.

Let $f(x)$ be a polynomial of degree n all the roots of which are in the interval $(-1,+1)$; and further let $\max_{-1\leq x\leq 1}|f(x)|=1$. For whi polynomial is

$$\int_{-1}^{+1} |f(x)|$$

maximal? I was not able to answer this question but it seems very likely that the maximum is reached for $f(x)=T_n(x/c)$, where $c=1/x_n$, and x_n is the greatest root of $T_n(x)$ (the nth Tchebicheff polynomial). Hence $T_n(1/c)=T_n(-1/c)=0$ and all other roots of $T_n(x/c)$ are in $(-1, +1)$. It is easy to see that $T_n(x)$ satisfies the following condition: Let x_i and x_{i+1} be two consecutive roots of $T_n(x)$; then

$$\frac{1}{x_{i+1}-x_i}\int_{x_i}^{x_{i+1}} |T_n(x)| = d_n,$$

where d_n is independent of i, and $\lim d_n=2/\pi$.

This fact suggests the following conjecture which is a generalization of the previous one: Let $f(x)$ be a polynomial of degree n all the roots of which are in $(-1, +1)$, such that $\max_{-1\leqq x\leqq 1} |f(x)|=1$ and let x_i and x_{i+1} be two consecutive roots of $f(x)$; then

$$\int_{x_i}^{x_{i+1}} |f(x)| \leqq d_n(x_{i+1}-x_i).$$

Equality holds only for $T_n(cx)$.

It seems very likely that the following result holds: Let $\phi(\theta)$ be a trigonometric polynomial all the roots of which are real, further let $\max_{0\leqq\theta\leqq 2\pi} |\phi(\theta)|=1$. Then

$$\int_0^{2\pi} |\phi(\theta)| \leqq 4.$$

Let $f(x)$ be a polynomial of degree n with leading coefficient 1 and all roots in $(-1, +1)$; then the sum of the intervals in $(-1, +1)$ for which $|f(x)|\geqq 1$ does not exceed 1. The proof is quite simple. Evidently

$$f(x)f(-x) = \prod_{i=1}^{n}(x_i^2 - x^2) \leqq 1 \qquad \text{for } |x| \leqq 1,$$

equality occurring only for $x=0$, $|x_i|=1$. Thus one of the numbers $f(x)$ or $f(-x)$ is less than 1, which establishes the result. It is also easy to see that if the sum of the intervals in question is exactly 1 then $f(x)=(1\pm x)^n$. It would not be difficult to prove the following slightly more general result: Let $f(x)$ have leading coefficient 1 and all ots in $(-1, +1)$; then if $-1<a<0<b<1$ at least one of the numrs $|f(a)|$ or $|f(b)|$ is less than 1. These problems become very much ore difficult if instead of the interval $-1, +1$ we consider the unit circle. The question would be to determine the polynomial (or polynomials) of degree not greater than n with leading coefficient 1 and all roots in the unit circle such that the area of the set of points for which $|f(x)| \geqq 1$ shall be as big as possible. A first guess would be $f(x) = (x-a)^n$, $|a| = 1$, but it can be shown that for sufficiently large n this is not the case. The complete solution of this problem seems difficult.

Mr. Eröd[2] proved that there exists a constant c independent of n such that for a polynomial of degree n satisfying the above conditions the area of the set of points for which $|f(x)| \leqq 1$ is not less than c. The best value of c is not known.

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[2] Oral communication.

ON UNCONDITIONAL CONVERGENCE IN NORMED VECTOR SPACES

T. H. HILDEBRANDT

Suppose X is a complete normed vector or Banach space of elements x. Orlicz[1] has given the following two definitions of unconditional convergence of an infinite series $\sum_n x_n$ of elements from X and proved their equivalence:

A. $\sum_n x_n$ is unconditionally convergent if and only if any rearrangement of the series is convergent.

B. $\sum x_n$ is unconditionally convergent if and only if $\sum_k x_{n_k}$ converges, where $\{x_{n_k}\}$ is any subsequence of $\{x_n\}$.

Pettis[2] has shown that either of these conditions is equivalent to the statement:

C. Every subseries of $\sum_n x_n$ is weakly convergent to an element of X, that is, $\{x_{n_k}\}$ implies the existence of an element x_σ such that, for every $\bar{x}$ of the conjugate space $\bar{X}$, $\sum_k \bar{x}(x_{n_k}) = \bar{x}(x_\sigma)$.

In proving this equivalence, he shows that condition C implies the following:

D. $\lim_n \sum_{m=n}^{\infty} \bar{x}(x_m) = 0$ uniformly for $\|\bar{x}\| = 1$.

E. H. Moore[3] has shown that for real, complex or quaternionic numbers, absolute and therefore unconditional convergence is equivalent to the following definition of convergence:

Let σ be any finite subset $n_1, \cdots, n_k$ of the positive integers, and denote $\sum_{i=1}^{k} x_{n_i}$ by $\sum_\sigma x_n$. Then

E. $\sum x_n$ converges in the σ-sense, if $\lim_\sigma \sum_\sigma x_n$ exists, where the limit is the Moore-Smith limit, and $\sigma_1 \geqq \sigma_2$ means that σ_1 contains all of the numbers in σ_2.[4]

Obviously the Moore-Smith limit can be extended to normed vector spaces, and the fundamental properties carry over. It is the purpose of this note to show that convergence in the σ-sense is equivalent to each of the conditions A, B, and D, that is, A, B, D and E are equivalent definitions of unconditional convergence, to which the

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[1] *Ueber unbedingte Konvergenz in Funktionenräumen*, Studia Mathematica, vol. 4 (1933), pp. 33–38.

[2] *Integration in vector spaces*, Transactions of this Society, vol. 44 (1938), pp. 281–282.

[3] *General Analysis*, Memoirs of the American Philosophical Society, vol. 1, part 2, 1939, p. 63.

[4] See Alaoglu, Annals of Mathematics, (2), vol. 41 (1940), p. 259, where a similar definition for weak unconditional convergence is given.

Pettis result adds C as a fifth equivalent definition. We consider definition E the most elegant of the definitions of unconditional convergence.[5]

A *is equivalent to* E. Assume that every rearrangement of $\sum x_n$ is convergent and suppose if possible $\lim_\sigma \sum_\sigma x_n$ is not equal to $x = \sum_n x_n$, summed in its natural order. Then there exists an $e > 0$, such that for every σ there exists a $\sigma' \geqq \sigma$ such that $\|x - \sum_{\sigma'} x_n\| \geqq e$. Let n_0 be chosen so that for $n \geqq n_0$ we have $\|x - \sum_1^n x_m\| < e/2$. Let σ_1 be the set $1, 2, \cdots, n_0$, and σ_1' chosen so that $\|x - \sum_{\sigma_1'} x_n\| \geqq e$. Let σ_2 include all of the integers less than or equal to any integer in σ_1'. Repeating the process produces a σ_1', a σ_3, and so on, and defines a rearrangement of $\sum x_n$, namely, $\sigma_1, \sigma_1' - \sigma_1, \sigma_2 - \sigma_1', \cdots$ (where $\sigma - \sigma'$ means the elements of σ not in σ'), which is not a convergent series since

$$\left\| \sum_{\sigma_m} x_n - \sum_{\sigma_m'} x_n \right\| = \left\| \sum_{\sigma_m' - \sigma_m} x_n \right\| \geqq e/2.$$

Since $\lim_\sigma \sum_\sigma x_n$ is unique when it exists, it follows at once that every rearrangement of $\sum_n x_n$ converges to the same limit.[6]

Conversely, suppose $\sum_\sigma x_n$ approaches a limit x, that is, for every $e > 0$, there exists a σ_e such that if $\sigma \geqq \sigma_e$ then $\|\sum_\sigma x_n - x\| \leqq e$. Let $\{x_{n_k}\}$ be any rearrangement of $\{x_n\}$. Then we need only to choose k_e so that the set of integers $n_1, \cdots, n_{k_e}$ includes all of the integers in σ_e to be assured that for $k' \geqq k_e$ it is true that

$$\left\| \sum_{k=1}^{k'} x_{n_k} - x \right\| \leqq e.$$

B *is equivalent to* E. Suppose every subseries $\sum_k x_{n_k}$ converges but $\lim_\sigma \sum_\sigma x_n$ does not exist, that is, there exists an $e > 0$ such that for every σ there exist $\sigma', \sigma'' \geqq \sigma$ such that $\|\sum_{\sigma'} x_n - \sum_{\sigma''} x_n\| > e$. If $\sigma' + \sigma''$ denotes the set including all elements of σ' and σ'', then either $\|\sum_{\sigma'} x_n - \sum_{\sigma'+\sigma''} x_n\| \geqq e/2$ or $\|\sum_{\sigma''} x_n - \sum_{\sigma'+\sigma''} x_n\| \geqq e/2$, that is, we can assume $\sigma' \geqq \sigma'' \geqq \sigma$. We obtain a nonconvergent subseries as follows: Take $\sigma_1 = 1$. This gives rise to $\sigma_1' \geqq \sigma_1'' \geqq \sigma_1$ so that $\|\sum_{\sigma_1'} x_n - \sum_{\sigma_1''} x_n\| > e$. Let $n_1, \cdots, n_{k_1}$ be the elements of $\sigma_1' - \sigma_1''$. Take $\sigma_2 = \sigma_1'$, and $n_{k_1+1}, \cdots, n_{k_2}$ to be the elements of $\sigma_2' - \sigma_2''$. Proceeding in this manner we get a series $x_{n_1} + x_{n_2} + \cdots + x_{n_{k_1}} + \cdots + x_{n_{k_2}} + \cdots$ such that $\|\sum_{k_m+1}^{k_{m+1}} x_{n_k}\| > e$, that is, one which is not convergent.

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[5] In view of the results of Orlicz and Pettis, it would be sufficient to prove that D implies E implies A. For the sake of completeness and elegance we have preferred to prove each equivalence separately.

[6] See Orlicz, Studia Mathematica, vol. 1 (1929), p. 242.

Conversely if $\lim_\sigma \sum_\sigma x_n$ exists and $\{x_{n_k}\}$ is any subsequence of x_n, then if k_e is chosen so that n_{k_e} is larger than any integer appearing in the σ_e involved in the definition of the limit, it will be certain that for $k' \geqq k_e$ and any l

$$\left\| \sum_{k=k'}^{k'+l} x_{n_k} \right\| \leqq 2e,$$

that is, $\sum_k x_{n_k}$ is convergent.

D *is equivalent to* E. Suppose $\lim_n \sum_{m=n}^{\infty} |\bar{x}(x_m)| = 0$ uniformly for $\|\bar{x}\| = 1$. We demonstrate that $\lim_{\sigma_1,\sigma_2} \|\sum_{\sigma_1} x_n - \sum_{\sigma_2} x_n\| = 0$. This is obviously equivalent to showing that $\lim_{\sigma_1 \geqq \sigma_2} \|\sum_{\sigma_1} x_n - \sum_{\sigma_2} x_n\| = 0$, since

$$\left\| \sum_{\sigma_1} x_n - \sum_{\sigma_2} x_n \right\| \leqq \left\| \sum_{\sigma_1} x_n - \sum_{\sigma_1+\sigma_2} x_n \right\| + \left\| \sum_{\sigma_2} x_n - \sum_{\sigma_1+\sigma_2} x_n \right\|.$$

Let n_e be such that for $n \geqq n_e$ we have $\sum_n^\infty |\bar{x}(x_m)| \leqq e$ if $\|\bar{x}\| = 1$, and take $\sigma_e = 1, 2, \cdots, n_e$ with $\sigma_1 \geqq \sigma_2 \geqq \sigma_e$. Then $\sum_{\sigma_1-\sigma_2} |\bar{x}(x_n)| \leqq e$ and so $|\bar{x}(\sum_{\sigma_1-\sigma_2} x_n)| \leqq e$. Since this is uniform for $\|\bar{x}\| = 1$, it follows that

$$\left\| \sum_{\sigma_1-\sigma_2} x_n \right\| = \text{l.u.b.} \left[\left| \bar{x}\left(\sum_{\sigma_1-\sigma_2} x_n \right) \right| \text{ for } \|\bar{x}\| = 1 \right] \leqq e.$$

Conversely suppose $\lim_\sigma \sum_\sigma x_n$ exists. Then for $\sigma_1 \geqq \sigma_2 \geqq \sigma_e$ we have $\|\sum_{\sigma_1-\sigma_2} x_n\| \leqq e$. Take n_e greater than the largest integer in σ_e. Let $m \geqq n_e$ and $\|\bar{x}\| = 1$. Denote by $\sigma_{1\bar{x}}$ those of the integers $m, m+1, \cdots, m+k$, for which $\bar{x}(x_n) \geqq 0$, and by $\sigma_{2\bar{x}}$ the integers for which $\bar{x}(x_n) < 0$. Then

$$\begin{aligned} \sum_m^{m+k} |\bar{x}(x_n)| &= \sum_{\sigma_{1\bar{x}}} |\bar{x}(x_n)| + \sum_{\sigma_{2\bar{x}}} |\bar{x}(x_n)| \\ &= \left| \bar{x}\left(\sum_{\sigma_{1\bar{x}}} (x_n) \right) \right| + \left| \bar{x}\left(\sum_{\sigma_{2\bar{x}}} x_n \right) \right| \leqq \left\| \sum_{\sigma_{1\bar{x}}} x_n \right\| + \left\| \sum_{\sigma_{2\bar{x}}} x_n \right\| \leqq 2e; \end{aligned}$$

that is, $\lim_n \sum_n^\infty |\bar{x}(x_n)| = 0$, uniformly for $\|\bar{x}\| = 1$. A similar procedure would take care of the case in which $\bar{x}(x)$ were complex valued.

We note that condition D is equivalent to the statement that the linear operations (functionals) $\bar{x}$ for which $\|\bar{x}\| = 1$ map the sequence x_n on a compact subset of l, the space of absolutely convergent series, that is, any unconditionally convergent series can be interpreted as a completely continuous transformation on the adjoint space $\overline{X}$ to the space l.[7]

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[7] See Dunford, Transactions of this Society, vol. 44 (1938), p. 322.

Further it can be deduced from condition B that if $x(\sigma)=\sum_k x_{n_k}$ where $\sigma=n_1, n_2, \cdots$ then $x(\sigma)$ is a completely additive set function on subsets of the integers to the space X.

E. H. Moore, in the reference mentioned above, considers the case in which the set of integers $1, 2, \cdots, n, \cdots$ is replaced by a general set: $\mathfrak{P}$, and then defines a general sum $\sum x(p)$ by an obvious generalization. He shows that if $\sum x(p)$ exists, then $x(p)$ is zero except at a denumerable set of elements $p_1, \cdots, p_n, \cdots$ and $\sum_n |x(p_n)|$ exists. This result is extensible to the case where x is on $\mathfrak{P}$ to a linear normed complete space in the form:

If x is on $\mathfrak{P}$ to X and if $\sum x(p)$ exists in the sense that $\lim_\sigma \sum_\sigma x(p)$ exists, where the σ are finite subsets of $\mathfrak{P}$, then $x(p)$ differs from zero at most at a denumerable set of elements $p_1, \cdots, p_n, \cdots$ and $\sum x(p_n)$ is unconditionally convergent.

The first part of this theorem depends on the fact, easily derived by a slight change in the proof of "E implies D" above, that if $\sum_p x(p)$ exists then $\lim_\sigma \sum_\sigma |\bar{x}(x_p)|$ exists uniformly for $\|\bar{x}\|=1$, that is, for every $e>0$ there exists a σ_e such that if $\sigma_1 \geqq \sigma_2 \geqq \sigma_e$ and $\|\bar{x}\|=1$ then $\sum_{\sigma_1-\sigma_2} |\bar{x}(x_p)| \leqq e$. Let $\mathfrak{P}_0$ be the sum of the sets σ_e for $e=1/n$. This set will be denumerable. If p is not of $\mathfrak{P}_0$ then $|\bar{x}(x_p)| \leqq 1/n$ for all n and $\|\bar{x}\|=1$, that is, $x_p=0$. The second part of the theorem is obvious.

UNIVERSITY OF MICHIGAN

A NOTE ON THE LIMIT OF ORBITS[1]

G. E. SCHWEIGERT

Theorem A, stated below, is established in a recent paper[2] by D. W. Hall and the author. Various factors favoring a change in the point of view have enabled the writer to see a simpler proposition which contains Theorem A as a corollary. The purpose of this note is to prove this newer theorem and to point to its implications in the particular case shown by the example at the end.

A single valued continuous transformation $T(M)=M$ of a compact metric space onto itself is said to be *pointwise periodic* provided that for each point x in M there exists a positive integer n such that $T^n(x)=x$. Such a transformation must be one-to-one, hence a homeomorphism. The least positive integer n such that $T^n(x)=x$ is said to be the period of a point x in M and is denoted by $p(T, x)$. By the orbit of a point under T, or more briefly by a *point-orbit*, we shall mean the finite set consisting of a point x of M and all the images $T(x)$, $T^2(x)\equiv TT(x)$, and so on, under T. Similarly if X is a component of an invariant set L, that is, a set L for which $T(L)=L$, then the finite sum of disjoint components $X+T(X)+T^2(X)+\cdots+T^k(X)$ is said to be the *component-orbit* of X relative to L under T. This extention of the notion of a point-orbit[3] is possible since the property of being a component (maximal connected set) in L is invariant under T. In all that follows the brief term component-orbit will be used instead of the original precise form in which an invariant set is chosen first. Furthermore if G is such a component-orbit sets X or K mentioned in close proximity to G may be understood without confusion to be components such as X in the definition above. In other words it is best to think of each G as a function of X or K as if it were written $G(X)$ or $G(K)$, so that these sets may be used freely without accounting for their presence each time. The lower case letters such as x and p will denote points, and point-orbits will bear a subscript, for example, G_x.

These two types of orbit are obviously related in such a fashion that: (a) The number of components in a n orbit G equals at most the minimum value of $p(T, x)$ for $x \in X$; (b) If G has k components $T^i(X)$ and if G_x is a point-orbit associated with $x \in X$, then the number of points in $G_x \cdot T^i(X)$ is the same for all values of i in the range 0 to $k-1$. Furthermore the set $G_x \cdot T^i(X)$ is a point-orbit under T^k for each fixed i. It is to be understood that $T^0(X) \equiv X$ as in case for $i=0$ above.

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[1] Presented in part to the Society, April 15, 1939.

[2] *Properties of invariant sets under pointwise periodic homeomorphisms*, Duke Mathematical Journal, vol. 4 (1938), pp. 719–724.

[3] See L. Whyburn, *Rotation groups about a set of fixed points*, Fundamenta Mathematicae, vol. 28 (1937), p. 124. This is perhaps the first and certainly a most consistent and fruitful use of this extended notion.

THEOREM A. *If M is a compact metric space, $T(M)=M$ a pointwise periodic transformation, $\{G_{x_k}\}$ a convergent sequence of point-orbits under T with limit set L, and if there is in L a connected set B such that $T(B)=B$, then L is connected.*

The following lemma is stated without proof since it is practically the same as Theorem A and follows from the exact argument used in the proof as published.

LEMMA 1. *If M is a compact metric space, $T(M)=M$ a pointwise periodic transformation, $\{G_{x_k}\}$ a convergent sequence of point-orbits under T with limit set L, and if G_x denotes any point-orbit in L, then L has the property that for every separation*[4] *$L=L^1+L^2$ the sets $G_x \cdot L^1$ and $G_x \cdot L^2$ are non-vacuous.*

We are now in a position to prove an extended form of this result.

LEMMA 2. *If M is a compact metric space, $T(M)=M$ a pointwise periodic transformation, $\{G_{x_k}\}$ a convergent sequence of point-orbits under T with limit set L, and if X is a component of L, then $L=G$ where G is the component-orbit of X under T.*

PROOF. *Case* (i): *There exists an isolated component K in L,* that is, K is a component which is both open and closed in L.

Let G denote the component-orbit of K and consider the equations

$$L = G + (L - G), \qquad G = K + T(K) + \cdots + T^m(K),$$

m a positive integer.

Now T is a homeomorphism on L, and K being isolated in L implies that $T(K)$ and $T^2(K)$, and so on, are also isolated in L so that G is both open and closed in L. This shows that the set $L-G$ must be vacuous, for otherwise $L=G+(L-G)$ is a separation of L contrary to Lemma 1. If $L-G$ is vacuous then $L=G$ and Lemma 2 is true.

Case (ii): *L contains no isolated components.*

We will show that this case is impossible in view of Lemma 1. Let

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[4] We say that $M=R+S$ is a separation of M provided neither R nor S is vacuous and $R\bar{S}=0=\bar{R}S$, where $\bar{N}$ is the closure of a set N.

X be any component of L and let m be a positive integer chosen so that X, $T(X)$, $\cdots$, $T^m(X)$ represent the $(m+1)$ distinct components of the component-orbit G. Now L is not connected, for otherwise L is itself an isolated component in L, and we may therefore choose a separation of L; say $L=L(1)+L(2)$. Neither $L(1)$ nor $L(2)$ is an isolated component in L and hence we may choose separations $L(1)=L(1, 1)+L(1, 2)$ and $L(2)=L(2, 1)+L(2, 2)$ so that $L=L(1, 1)+L(1, 2)+L(2, 1)+L(2, 2)$. Each of the summands $L(i_1, i_2)$ when $i_1=1, 2$ and $i_2=1, 2$ is an isolated set in L, hence fails to be connected, and the process continues. In general

$$L = \sum_{i_1,i_2,\cdots,i_j} L(i_1, i_2, \cdots, i_j), \qquad \text{where } i_k = 1, 2.$$

while k has the range $1, 2, \cdots, j$; thus L is expressed as the sum of 2^j disjoint sets each isolated in L. Let a fixed value of $j=j^*$ be chosen so that 2^{j^*} is greater than $m+1$. Obviously each component of G must intersect some set $L(i_1, i_2, \cdots, i_j)$ and must be contained in that set. Since there are more sets in L than there are components in G there exists a set $L^* \equiv L(I_1, I_2, \cdots, I_j)$ such that $L^*G=0$. However $L=L^*+(L-L^*)$ is a separation of L and no point-orbit in G intersects L^*. This is impossible because of Lemma 1. Thus Case (ii) never holds and Lemma 2 is true.

The next and final lemma will be recognized as one having the same conclusion as Lemma 1 established however from fewer assumptions.

LEMMA 1*. *If M is a compact metric space, $T(M)=M$ a pointwise periodic transformation, $\{G_i\}$ a convergent sequence of closed component-orbits under T with limit set L, and if G_x denotes any point-orbit in L, then L has the property that for every separation*

$$L = L^1 + L^2$$

the sets $G_x \cdot L^1$ and $G_x \cdot L^2$ are non-vacuous.

PROOF. Assume contrary to the lemma that $L=L^1+L^2$ is a separation of L and there exists a point x in L such that the point-orbit G_x is contained in one side, say L^1, of the separation. In these circumstances it is possible to choose a subsequence $\{G_{i_k}\}$ (of the given sequence of component-orbits) which in turn allows us to choose points $x_k \in G_{i_k}$ such that the *point*-orbits $\{G_{x_k}\}$ *converge* to a limit set, say L^*, and the points $\{x_k\}$ converge to the point x, which defines G_x. Since the point-orbits converge, we may use Lemma 2 which informs us that the limit set L^* is a component-orbit. Moreover each of the *finite number* of components of L^* may be taken as one side of a separation of L^* and this means, in view of Lemma 1 (or Lemma 2) and the fact that x is in L^*, that each component Y of L^* is intersected by G_x. Now G_x is contained in L^1 hence Y is contained in L^1 and in turn L^* is contained in L^1.

On the other hand, using the fact that we have a separation of L and L is closed there exists a neighborhood V containing L^2 having the properties that the boundary $F(V)$ is disjoint with L and V is disjoint with L^1.

From this it follows that: (α) *the neighborhood V has at most a finite number of points in common with the compact closed set* $\sum G_{x_k}+L^*$. To show this, consider any infinite set in the product. This set determines a limit point z which belongs to L^* and hence to L^1. Such a point z is in $\bar{V}=V+F(V)$ and not in $F(V)$ (since z is in L), hence z is in V. This means z is in V and L^1 both which is impossible.

Let p be any point of L^2 and consider V as a neighborhood of p. It follows that there exists an integer K_1 such that $k>K_1$ implies $G_{i_k}\cdot V$ non-vacuous since $\{G_{i_k}\}$ converges to L. Hence there exists a component Y_k of G_{i_k} having the property that $Y_k\cdot V$ is non-vacuous for each $k>K_1$.

If we recall the introductory remarks concerning the relation between component-orbits and the point-orbits which they contain we see that the inequality $G_{x_k}\cdot\Gamma_k\neq 0$ holds for every component Γ_k in the orbit G_{i_k} since $x_k\in G_{i_k}$. Hence we have $G_{x_k}Y_k\neq 0$ for the particular (when k is fixed) component selected in the paragraph above. This latter statement concerning Y_k together with the fact (statement (α) in italics above) that $V\cdot(\sum G_{x_k}+L^*)$ is at most finite tell us there exists an integer K_2 such that for $k>K_2$ the component Y_k intersects the complement of V.

Combining the two properties associated with K_1 and K_2 it is clear that for $k>\max(K_1, K_2)$ each Y_k intersects V and the complement of V. Hence for each k there exists a point $y_k\in Y_k$ such that $y_k\in F(V)$. This leads immediately to the conclusion that there exists a point y common to L and $F(V)$. By the choice of V this cannot be true and the proof is complete. With the aid of Lemma 1* we now prove the new theorem which yields the same conclusion as Lemma 2.

THEOREM. *If M is a compact metric space, $T(M)=M$ a pointwise periodic transformation and $\{G_i\}$ a convergent sequence of closed component-orbits under T with limit set L, then L is likewise a closed component-orbit.*

PROOF. *Case* (i): *There exists a component X of L such that X is isolated in L.*

In this case $L=G+(L-G)$, where G is the component orbit of X under T, is either a separation of L or the left member is empty. This is true because G is both open and closed in L. In view of Lemma 1* the left side is empty; hence $L=G$ as stated by the theorem.

Case (ii): *No component of L is isolated in L.*

Since L is not connected let $L=L(1)+L(2)$ be some separation of L. By the reasoning applied in Case (i) there exists no isolated component in either $L(1)$ or $L(2)$. Now let $L(1)=L(1, 1)+L(1, 2)$ and $L(2)=L(2, 1)+L(2, 2)$ be separations of these sets and express L as the sum of the four sets with double indices. Proceeding this way the proof is the same as that in Case (ii) of Lemma 2.

REMARKS. Let $\{G_{x_k}\}$ be a sequence of point-orbits such that the number of points in G_{x_k} is bounded independent of k. It is a well known fact that if $\{G_{x_k}\}$ converges to a limit set L then L is also a point-orbit having the same bound. The theorem of this note and the simple result just mentioned are obviously of the same type in that each characterizes the limit set by the exact property it assumes for each element of a given sequence. This analogy is supported by the fact that if there is a common bound for the number of components in the orbits of the sequence the conclusion of the theorem can be augmented to include this bound for the component number of the limit set. On the other hand any improvement in the conclusion of Lemma 2 which might be suggested by this esthetic point of view (because the elements of the sequence have a minimal character) seems doomed to failure. These four possibilities form a complete set of the bounded, non-bounded, minimal and non-minimal types of sequence.

These remarks suggest that we examine certain theorems looking toward a possible analogy with *periodic* homeomorphisms—that is, the type of homeomorphism for which there exists a positive integer N such that every point-orbit contains at most N points. In this connection let us suppose that M is a compact metric space and K is any subset of M such that $\overline{K}=M$. If $T(K)=K$ is periodic on K and $S(M)=M$ is any homeomorphic extension of T, *then S is periodic on M*. This conclusion follows by an elementary demonstration. Homeomorphic extension is used here as meaning $S(M)=M$ is a homeomorphism such that $S(x)=T(x)$ for $x\in K$. The author feels that it is not only natural to ask about such extensions for the pointwise periodic type of homeomorphism, but feels called upon to apologize for the relatively minor position the question must occupy in a note such as this. The example given below showing that the most obvious analogue of the extension theorem above does not hold came up in a discussion group and it was decided on an intuitive basis that the extension would be pointwise periodic. Thus one may see in this example something of the inspiration for the theory of this note, a concrete expression of that theory, and at the same time a refutation of the intuitive decision.

EXAMPLE. Let a_1b_1 denote an arc with end points a_1 and b_1. Similarly let $a_{11}b_{11}$ and $a_{12}b_{12}$ be arcs with end points a_{11}, b_{11} and a_{12}, b_{12} respectively. If we identify the 3 points $b_1=a_{11}=a_{12}$ and identify no other points, we obtain a triod; that is, a Y shaped figure in which three otherwise distinct arcs have a common end point. The end points of this triod are a_1, b_{11} and b_{12}. Choose four more arcs calling them $a_{11i}b_{11i}$ and $a_{12i}b_{12i}$ where $i=1, 2$. Identify the end points $b_{11}=a_{111}=a_{121}$ and also the end points $b_{12}=a_{121}=a_{122}$ making however no other identification. This process can be continued indefinitely, the general rule being to identify the a-end points of exactly two new arcs with a particular b-end point of the dendrite obtained in the previous stage. (Note that a_1 remains as an end point at all times.) The completed figure is to be a dendrite such that all end points other than a_1 lie on the Cantor discontinuum (middle third set) of some linear interval. The discontinuum is chosen in advance, the arcs added at each stage all have the same diameter, say δ_n, where δ_n approaches zero as n increases, and the arcs as added "reach" toward, but never intersect the discontinuum. This is a well known construction usually used to illustrate a dendrite with uncountably many end points.

We now denote the complete dendrite by D, the Cantor set of end points by C, and let $K=D-C$. A homeomorphism $T(K)=K$ is defined as follows: T leaves each point of the arc a_1b_1 fixed. The arc $a_{11}b_{11}$ is sent into the arc $a_{12}b_{12}$ so that $T(b_{11})=b_{12}$ and $T(b_{12})=b_{11}$ and in general $p(T, x)=2$ for $x\in a_{11}b_{11}-a_{11}$. Note that $a_{11}=a_{12}=b_1$ is fixed. We will call this the second stage. The four arcs $a_{1jk}b_{1jk}$ for $j=1, 2$ and $k=1, 2$ are permuted by T so that $p(T, x)=4$ for $x\in a_{1jk}b_{1jk}-a_{1jk}$ while $T(b_{11})=b_{12}=T(a_{111})=a_{121}=a_{122}$ has period 2 as was defined previously. This is the third stage. In general the arcs added at the nth stage suffer a permutation among themselves of the period 2^{n-1} and the transformation agrees with the part described for the $(n-1)$th stage in the natural way indicated by the common points.

It is clearly true that $T(K)=K$ so defined and the associated inverse $T^{-1}(K)=K$ homeomorphism are each uniformly continuous on K. If necessary D can be constructed in the unit square using straight line segments. The high degree of symmetry allows T to act as a "rigid motion" from which the uniform continuity is apparent. It follows that $T(K)=K$ may be extended to a homeomorphism $S(D)=D$

defined over all of D. However S is not pointwise periodic for

$$G_{b_1} = b_1, \qquad G_{b_{11}} = b_{11} + b_{12},$$

and so on, is a sequence of point-orbits which may be considered as component-orbits relative to themselves. This sequence converges to C and C is not the orbit of one of its components; that is, not the orbit of one of its points since it contains uncountably many points. The local connectivity, symmetry, and so on, of this example indicate that any conditions on $\overline{K}=M=D$ that are sufficient to make S inherit the pointwise periodic property must be highly restrictive.

THE UNIVERSITY OF VIRGINIA AND
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CORRECTION TO "CLOSURE OF PRODUCTS OF FUNCTIONS"[1]

D. G. BOURGIN

Page 811, line 7. *Replace* $F(t)$ *by* $c(t)F(t)$.

Page 811, line 10. *Replace* corresponding *by* largest.

Add to postulate (a) $T(E)$ admits no finite basis.

Page 813, lines 13 and 14. *Replace* (b') *and* (c') *by* (b'') *and* (c'').

In footnote 13 *replace* E *by* I_2 *and* The sets · · · also *by* The sequence $\{t^r s^p\}$ · · · also.

Page 814, line 25. *Replace* the set $F(t)$, $\phi_\sigma(t)$, $\sigma=1, \cdots, n$, is linearly independent *by* $F(t)$ is not linearly dependent on $\phi_\sigma(t)$, $\sigma=1, \cdots, n$.

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[1] This Bulletin, vol. 46 (1940), pp. 807–815.